Part III Solitons, Instantons, and Geometry, Sheet One

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1. **Sine-Gordon kink.** Starting with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\phi_t^2 - \phi_x^2) - (1 - \cos\beta\phi)$$

derive the Sine–Gordon equation. Find a kink solution, and use the Bogomolny bound to find its energy. How many types of kinks are there?

2. **Unstable kinks.** The Lagrangian density for a complex scalar field ϕ in $1+1$ is

$$\mathcal{L} = \frac{1}{2}|\phi_t|^2 - \frac{1}{2}|\phi_x|^2 - \frac{1}{2}\lambda^2(a^2 - |\phi|^2)^2, \qquad a \in \mathbb{R}.$$

Find the field equations, and verify that the real kink $\phi_0(x) = a\tanh(\lambda a x)$ is a solution. Now consider a small pure imaginary perturbation $\phi(x,t) = \phi_0(x) + i\eta(x,t)$ with η real and find the linear equation satisfied by η.

By considering $\eta = \mathrm{sech}(\alpha x)e^{\omega t}$ show that the kink is unstable.

3. **Particle Interpretation**. Suppose that $U(\phi) \geq 0$ and that $U = 0$ at a non–empty and discrete set of values of ϕ. Show that the Bogomolny equations $\phi_x = \pm\partial_\phi W, \quad \phi_t = 0$ imply the static field equation

$$\phi_{xx} = \partial_\phi U, \quad \text{where} \quad U(\phi) = \frac{1}{2}(\partial_\phi W)^2.$$

Explain how the static field equation can be interpreted as the equation for the particle motion in the inverted potential $-U$, where the 'position' ϕ is regarded as a function of 'time' x.

Assuming that the vacua of U are quadratic minima, find the generic form of $\phi(x)$ as it approaches the minma. Suppose that $U = 0$ at $\phi_1 < \phi_2 < \phi_3$. Use the Newtonian interpretation above to show that there is no static kink connecting ϕ_1 to ϕ_3.

4. **Moduli space approximation.** Consider the static kink solution

$$\phi = \tanh(x - c),$$

where the constant c is the location of the kink in the ϕ^4 theory resulting from the potential $U = (1-\phi^2)^2/2$.

Use the Lorentz boost to construct an exact time–dependent solution to the field equation.

The *moduli space approximation* replaces the constant parameter c by a function $c(t)$. This leads to an approximate solution to the field equations if $\dot{c}$ is small. Substitute $\phi = \tanh(x - c(t))$ into the kinetic energy of the kink, and show that the resulting expression is of the form

$$T = \frac{1}{2} M \dot{c}^2$$

where the constant 'mass' M should be determined. Deduce that the original ϕ^4 Lagrangian reduces to a Lagrangian of a particle in 1D moving in the constant potential. Solve the resulting Euler–Lagrange equations and compare your answer with the exact solution you obtained from the Lorentz boost.

The resulting second order field equation for $c(t)$ can be viewed as the geodesic equation in the moduli space $\mathbb{R}$, with a flat Riemannian metric Mdc^2. In other soliton models (e. g. non–abelian monopoles or gauged vortices) the moduli space of static solutions is higher–dimensional, and inherits a curved Riemannian metric from the kinetic term in the Lagrangian. The geodesics of this metric approximate non–relativistic soliton dynamics.

5. **Sine Gordon on a wormhole.** One way to evade the Derrick's non–existence scaling argument in higher dimensions, is to consider solitons on curved backgrounds.

   Consider a scalar field equation

   $$\Box\phi + \sin\phi = 0,$$

   where $\Box$ is the wave operator on the (3+1) dimensional wormhole space–time with the metric

   $$ds^2 = dt^2 - dr^2 - (r^2 + a^2)d\omega^2$$

   with $(t, r) \in \mathbb{R}^2$, $d\omega^2$ the round metric on the unit two–sphere, and a is a positive constant.

Assume that $\phi = \phi(t, r)$, and show that the field equation becomes

$$\phi_{tt} = \phi_{rr} + \frac{2r}{r^2 + a^2}\phi_r - \sin\phi. \tag{1}$$

Show that the energy functional

$$E = \int_{-\infty}^{\infty} \left(\frac{1}{2}{\phi_t}^2 + \frac{1}{2}{\phi_r}^2 + 2\sin^2(\phi/2)\right)(r^2 + a^2)dr$$

is conserved.

Finiteness of the energy requires that at both asymptotically flat ends of the wormhole one has

$$\phi(t, -\infty) = 2n_-\pi, \quad \phi(t, \infty) = 2n_+\pi$$

where n_-, n_+ are integers, and w.l.g. we can chose $n_- = 0$.

Use the interpretation of (1) with $\phi = \phi(r)$ as the equation of motion of a particle moving in a potential $-2\sin^2(\phi/2)$ with 'time–dependent' friction coefficient to show the existence of static kink solutions with arbitrary kink number $N = n_+$.

According to the Soliton Resolution Conjecture for any smooth and finite energy generic initial data, the solution eventually resolves into a superposition of a radiative component plus a static kink solution for some N. This conjecture is applicable to other soliton models, and remains open in general.

6. **Bäcklund transformations.** The Sine–Gordon equation is

$$\phi_{xx} - \phi_{tt} = \sin(\phi), \qquad \phi = \phi(x, t).$$

Set $\tau = (x + t)/2, \rho = (x - t)/2$ and consider the Bäcklund transformations

$$\partial_\rho(\phi_1 - \phi_0) = 2b\sin\left(\frac{\phi_1 + \phi_0}{2}\right), \qquad \partial_\tau(\phi_1 + \phi_0) = 2b^{-1}\sin\left(\frac{\phi_1 - \phi_0}{2}\right),$$

where $b =$ const and ϕ_0, ϕ_1 are functions of (τ, ρ). Show that ϕ_1 is a solution to the Sine–Gordon equation if ϕ_0 is.

Take $\phi_0 = 0$ and construct the 1-soliton (kink) solution ϕ_1.

Draw the graph of $\phi_1(x,t)$ for a fixed value of t. What happens when t varies?

The Sine–Gordon equation is (unlike the ϕ^4 model) completely integrable and multiple applications of the Bäcklund transformations allow constructions of explicit time–dependent soliton solutions with different topological charges. One such solution representing a kink and anti-kink pair approaching eachother with topological charge 0 is

$$\phi(x,t) = 4\arctan\left(\frac{v\cosh\left(x/\sqrt{1-v^2}\right)}{\sinh\left(vt/\sqrt{1-v^2}\right)}\right).$$

7. **Sigma model lumps from holomorphic maps.** Let $\phi : \mathbb{R}^{2,1} \to S^2$. Set
$$\phi^1 + i\phi^2 = \frac{2u}{1+|u|^2}, \qquad \phi^3 = \frac{1-|u|^2}{1+|u|^2},$$
and deduce that the Bogomolny equations
$$\partial_i\phi^a = \pm\varepsilon_{ij}\varepsilon^{abc}\phi^b\partial_j\phi^c, \qquad \phi_t = 0$$
imply that u is holomorphic or antiholomorphic in $z = x_1 + ix_2$.

Find the expression for the total energy
$$E[\phi] = \frac{1}{2}\int \partial_j\phi^a\partial_j\phi^a d^2x$$
in terms of u.

By counting the pre–images or otherwise find the topological degree of ϕ corresponding to $u(z) = u_0 + u_1 z + \ldots + u_k z^k$, where $u_0, \ldots, u_k$ are constants with $u_k \neq 0$.

It is a non–trivial result in analysis (established by Karen Uhlenbeck in 1989) that all finite energy solutions to the 2nd order Sigma model equations are solutions to the Bogomolny equations you have just constructed.

8. **Topological degree of maps between spheres.** Let $z \in \mathbb{C}$. Restrict the holomorphic map $z \to z^k$ to the unit circle $|z| = 1$, and compute its degree.

Let $(\theta \in [0, \pi], \phi \in [0, 2\pi]$ be the polar coordinates on S^2, and let $f : S^2 \to S^2$ be such that $f(\theta, \phi) = (\theta, k\phi)$. Compute the degree of f.

Consider the volume form on S^2 to prove the general degree formula for $f : S^2 \to S^2$ given in lectures.

Starting from the degree k–maps from S^1 to S^1 constructed above you can construct degree k maps from S^2 to S^2 (also constructed above). An inductive application of the topological suspension then shows that there there exists maps of all degrees from S^n to S^n.

9. **Topological degree and Lie groups.** Consider the map $g : S^3 \to SU(2)$ defined by

$$g(x_1, x_2, x_3, x_4) = x_4 + i(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3),$$

where σ_i are Pauli matrices and $x_1^2 + x_2^2 + x_3^3 + x_4^2 = 1$ and find its degree. By calculating $Tr((dg\, g^{-1})^3)$ at the point on S^3 where $x_4 = 1$, or otherwise deduce that the formula

$$\deg(g) = \frac{1}{24\pi^2} \int_{S^3} Tr((dg\, g^{-1})^3)$$

is correctly normalised.

Watch out for this formula when we discuss the instanton number in the Yang–Mills theory. It will also appear at the end of the course, when we classify principal $SU(2)$ bundles over S^4.

References

[1] Dunajski, M. (2009) *Solitons, Instantons, and Twistors*, Oxford University Press.

[2] Manton, N. and Sutcliffe, P. (2004) *Topological Solitons*, CUP.