Part III Solitons, Instantons, and Geometry, Sheet Three.

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1. **Action from Yang–Mills.** Derive the $SU(2)$ Yang–Mills theory on $\mathbb{R}^4$ form the action. Let $A_a(x)$ be a solution to these equations. Show that, for any nonzero constant c, the potential $\widetilde{A}_a(x) = cA_a(cx)$ is also a solution and that it has the same action.

2. $\mathfrak{so}(4)$**, and self–duality.** Let T_1, T_2, T_3 form a basis of $\mathfrak{su}(2)$ such that

$$[T_\alpha, T_\beta] = -\varepsilon_{\alpha\beta\gamma}T_\gamma, \quad \alpha, \beta, \gamma = 1, 2, 3,$$

and let the symbols $\sigma_{ab} = -\sigma_{ba}$ where $a, b = 1, \dots, 4$ be defined by

$$\sigma_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}T_\gamma, \quad \sigma_{\alpha 4} = T_\alpha.$$

Show that

$$\sigma_{ab} = \frac{1}{2}\varepsilon_{ab}{}^{cd}\sigma_{cd}, \quad \text{and} \quad \sigma_{ab}\sigma_{ac} = -\frac{3}{4}\mathbf{1}\delta_{bc} - \sigma_{bc}.$$

Identify $\Lambda^2(\mathbb{R}^4)$ with the Lie algebra $\mathfrak{so}(4)$ and deduce that $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

3. **An explicit instanton.** Let $V = 1 + r^{-2}$, where $r^2 := \delta_{ab}x^a x^b$. Show that the one–form

$$A = \sigma_{ab}\, \frac{1}{V}\frac{\partial V}{\partial x^b}dx^a \tag{1}$$

is a solution of the anti–self–dual Yang–Mills equations on $\mathbb{R}^4$.

The one–form A is singular at $r = 0$. What can you say about the behaviour of the field strength F at $r = 0$?

4. **... and its Chern number.** Find, by explicit integration, the Chern number of the solution (1).

5. **Hodge operator in various signatures.** Let F be a two–form on $\mathbb{R}^4$. Show, from the definition of the Hodge operator, that

   (a) $**F = \pm F$ depending on the signature.

   (b) $*F \wedge *F = F \wedge F$.

Show that in the $U(1)$ theory $F \to *F$ interchanges the electric and magnetic fields with factors of ± 1 or $\pm i$ and determine the different cases in the corresponding signatures.

Let F be a non–zero real self–dual two–form on $\mathbb{R}^4$ such that $F \wedge F = 0$. What is the signature of the underlying metric?

6. **Self–duality in null coordinates.** Show that the two–forms

$$\omega_1 = dw \wedge dz, \qquad \omega_2 = dw \wedge d\tilde{w} - dz \wedge d\tilde{z}, \qquad \omega_3 = d\tilde{w} \wedge d\tilde{z}$$

span the space of SD two–forms in $\mathbb{C}^4$, where

$$ds^2 = 2(dz d\tilde{z} - dw d\tilde{w}), \qquad \text{vol} = dw \wedge d\tilde{w} \wedge dz \wedge d\tilde{z}.$$

Show that a two form F is ASD iff $F \wedge \omega_i = 0$.

7. K**–equation from the Lax pair.** Use the Lax pair formulation of ASDYM to

   (a) Deduce the existence of a gauge such that $A = A_{\tilde{w}} d\tilde{w} + A_{\tilde{z}} d\tilde{z}$

   (b) Deduce the existence of a $\mathfrak{g}$ valued function $K = K(w, z, \tilde{w}, \tilde{z})$ such that $A_{\tilde{w}} = \partial_z K, A_{\tilde{z}} = \partial_w K$

   (c) Reduce the ASDYM to a single second order PDE

$$\partial_z \partial_{\tilde{z}} K - \partial_w \partial_{\tilde{w}} K + [\partial_w K, \partial_z K] = 0.$$

What is the residual gauge freedom in K?

8. **Chern–Simons three–form.** Let A be a 1–form gauge potential on $\mathbb{R}^n$ with values in $\mathfrak{su}(2)$, and let F be its curvature. Verify that $Tr(A), Tr(A \wedge A), Tr(A \wedge A \wedge A \wedge A)$ and $Tr(F)$ all vanish.

   Verify that $C_2 = dY_3$, where C_2 and Y_3 are the second Chern form, and the Chern–Simons three–form respectively.

9. **ASD Yang–Mills as gradient flow.** Let $A = A_i dx^i, i = 1, 2, 3$ be a gauge potential on $\mathbb{R}^3$ with values in the Lie algebra $\mathfrak{g}$. Find the Euler–Lagrange equations arising from varying the Chern–Simons functional

$$W[A] = \int_{\mathbb{R}^3} \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

with respect to A.

Now consider a one parameter family of $\mathfrak{g}$–valued one–forms $A = A(t)$ on $\mathbb{R}^3$, and define a one–form on $\mathbb{R}^4$ by $\mathcal{A} = A + \phi dt$, where the function $\phi = \phi(x^i, t)$ takes its values in $\mathfrak{su}(2)$. Show that, in a gauge where $\phi = 0$, the anti–self–dual Yang–Mills equations on $\mathcal{A}$ take the gradient flow form

$$\frac{dA_i(t)}{dt} = \frac{\delta W[A]}{\delta A_i}.$$

10. **Left and right invariant vector fields on a Lie group.** Let $G \subset GL(n+1, \mathbb{R})$ be a Lie group consisting of matrices of the form

$$g = \begin{pmatrix} x^0 & x^1 & x^2 & \dots & x^n \\ 0 & 1 & 0 & \dots & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

where $x^0 \in \mathbb{R}^+, x^i \in \mathbb{R}$ for $i = 1, \dots n$.

- By considering g near the identity matrix find the structure constants of the Lie algebra $\mathfrak{g}$ of G.
- Construct the left invariant one-forms $\sigma^a, a = 0, 1, \dots, n$, and the left invariant vector fields of G. Show that

$$d\sigma^a = \sum_{b,c=0}^{n} \kappa^a{}_{bc} \sigma^b \wedge \sigma^c$$

for some $\kappa^a{}_{bc}$ which should be determined

11. **Connection and curvature of a principal bundle.** Consider a connection $\omega = \gamma^{-1} A \gamma + \gamma^{-1} d\gamma$ on a principal G–bundle $P \to B$, where A is a one–form on B and $\gamma^{-1} d\gamma$ is the Maurer–Cartan form on G.

    (a) Show that the transformation of the fibres $\gamma' = g\gamma$, where $g \in G$ depends on the coordinates on B, does not change ω if A transforms like a gauge potential.

(b) Let $\Omega = d\omega + \omega \wedge \omega$. Show that $\Omega = \gamma^{-1} F \gamma$ for some F which should be found.

(c) Let $D_a, a = 1, \ldots, \dim(B)$ be linearly independent vector fields on P such that

$$D_a \lrcorner\, \omega = 0.$$

Show that $D_a = \partial_a - A_a^\alpha R_\alpha$, where $\partial_a = \partial/\partial x^a$ are vector fields on B and R_α are right–invariant vector fields on G. Demonstrate that

$$[D_a, D_b] = -F_{ab}^\alpha R_\alpha.$$