Shouting at a Staircase:

Acoustic Scattering off a Plane with a Step

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10th May 2019

Introduction

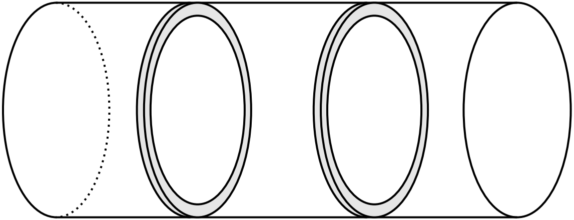
- 1st-year PhD student at DAMTP in the Waves group, supervised by David Abrahams (Isaac Newton Institute) and Nigel Peake (DAMTP)
- PhD in area of Structural Acoustics
  - What is the internal and external response of a structure under heavy fluid loading?
- Use both solid and fluid mechanics in work — fluid mechanics for the fluid loading and solid mechanics for response of structure

Motivation

- My supervisors told me to do it!
  (arguably no further motivation is needed)
- To optimise design of submarine hulls to reduce sound emitted by internal sources (*e.g.* vibrations from engine)
  - Defense purposes
  - Ecological purposes
  - Environmental purposes
- Also interesting from mathematical point of view — essentially singular perturbation problem in acoustics
- Analytical approach is useful as problem is difficult for computational approach — need very fine-grained mesh to simulate at sufficient detail but this requires heavy computational resources

Motivation

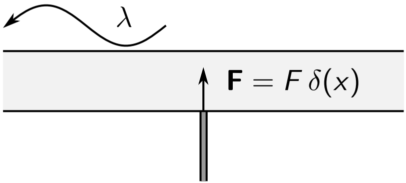
- Question: What is the optimal design and placement of the bulkheads to minimise vibrations and hence noise?
- Initially model hull of submarine as a cylinder (neglecting complex end effects) with bulkheads as reinforcing ribs



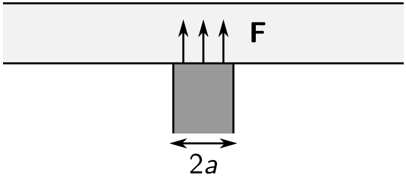
- Could also treat surface with noise-reducing coating (*e.g.* via tiles of layered metamaterials)

Motivation

- Further simplify by assuming wavelength of wave is very large compared to width of constraint ($a/\lambda \ll 1 \implies ka \ll 1$)



- Simplify even further by considering the ribs to be sufficiently far from each other so that we can consider each rib separately — this is a classical elastostatic 'punch' problem



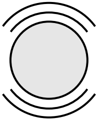
Motivation

- These define an ‘outer’ and ‘inner’ problem respectively, necessitating use of Matched Asymptotic Expansions:
  - Outer problem is on the scale of the wave
  - Inner problem is on the scale of the rib
- Solve each problem separately and then determine unknown constants in each problem by matching
- Hence form a composite solution that is valid for both the inner and outer problem
- A problem of this type was tackled by Abrahams *et al.*[1]
- Aim of PhD is to extend results to different geometries and in-plane vibrations
- Started by considering scattering off plane with step — topic of this talk!

[1] P. A. Cotterill *et al. The time-harmonic antiplane elastic response of a constrained layer*, Journal of Sound and Vibration, **348**, pp. 167-184 (2015)

Matched Asymptotic Expansions Example

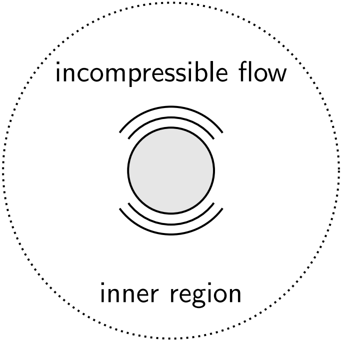
- Easiest to illustrate concept of MAEs with a simple example
- Consider a small sphere oscillating in still fluid; what is the acoustic field generated by the oscillations?



- Find the acoustic field ϕ by solving the Helmholtz equation $(\nabla^2 + k^2)\phi = 0$, where k is the acoustic wavenumber
- Have two boundary conditions:
  - Sommerfeld condition at infinity: no waves incoming from infinity
  - No-flux boundary condition: velocity of the fluid is equal to velocity of sphere on boundary of sphere

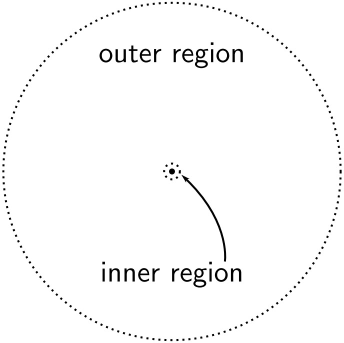
Inner Problem Example

- On scale of sphere, it only ‘sees’ incompressible flow within wavelength of oscillations, so only need to solve Laplace's equation $\nabla^2\phi = 0$ in this ‘inner’ region — hence terminology
- We still have the no-flux boundary condition, but now cannot apply the Sommerfeld condition
- Hence have a second-order PDE with one boundary condition — this leaves one constant undetermined in solution!



Outer Problem Example

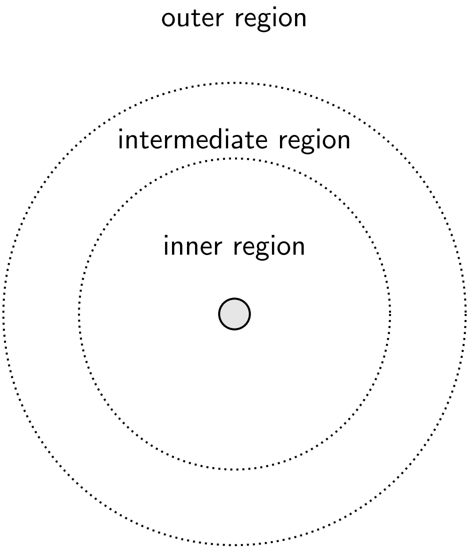
- Far away from sphere, see it as a point at the origin with singular behaviour
- Now need to solve full Helmholtz equation with Sommerfeld boundary condition, but this time we cannot apply no-flux
- Again end up with second-order PDE with one boundary condition and so have one undetermined constant



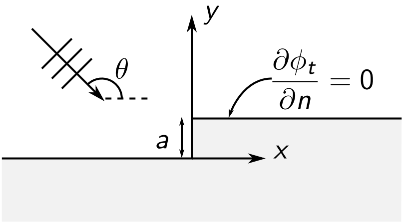
Matching Process

- We have two solutions with two undetermined constants: one for the inner problem and one for the outer problem
- How do we find these constants?
- Logic: there must be some kind of intermediate region in between the inner and outer regions
- In this intermediate region, both solutions are valid and so can ‘match’ solutions using a matching process to find the constants
- There are multiple matching processes, but the ‘Asymptotic Matching Principle’ pioneered by Van Dyke and improved by Crighton & Leppington is the most straightforward to use
- Add the inner and outer solutions to obtain the composite solution which is uniformly valid throughout whole space of problem

Intermediate Region



Problem



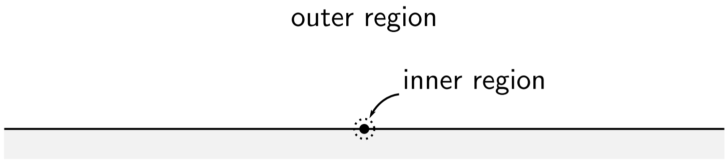
- Given $\phi_t = \phi_i + \phi$, where ϕ_i is known time-harmonic incident wave plus a reflected time-harmonic wave so that ϕ is then purely scattering potential — think of it as a correction to ϕ_i for small step of non-dimensional size $\epsilon \equiv ka$
- Wave equation for ϕ therefore simplifies to Helmholtz equation: $(\nabla^2 + k^2)\phi = 0$
- Boundary conditions: Sommerfeld radiation condition and treat boundary surface as acoustically-hard surface
- Question: What is the far-field behaviour of ϕ and what is order of solution in terms of ϵ?

Strategy

- Key assumption: $ka \equiv \epsilon \ll 1$; *i.e.* small step
- Presence of two distinct length scales (a and $1/k$) implies singular perturbation problem
- Hence need to solve two problems: 'outer' and 'inner problem' — one for each length scale
- This is where method of Matched Asymptotic Expansions comes into use
- Will have undetermined constants in solutions to both problems due to impossibility of application of condition
  - $\partial\phi/\partial x = 0$ for outer problem (cannot meaningfully apply boundary condition to vanishingly thin edge)
  - Sommerfeld radiation condition for inner problem (cannot go to ∞)

Outer Problem

- For the outer problem, the plane appears uniformly flat with some sort of point inhomogeneity at the origin



- After scaling the problem appropriately for the outer problem, need to solve $(\nabla^2 + 1)\phi = 0$ with Sommerfeld boundary condition and acoustically-hard surface for flat plane

Solution to Outer Problem

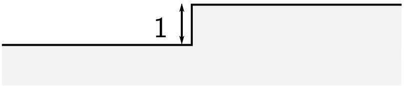
- Do so via Fourier transforms: end up with a fairly mathematical expression

$$\phi = -\epsilon \frac{\mathrm{i}\sin^2\theta}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\alpha kx-(\alpha^2-1)^{1/2}ky}}{(\alpha^2-1)^{1/2}(\alpha+\cos\theta)}\,\mathrm{d}\alpha + \epsilon a_1 H_0^{(1)}(kr)$$

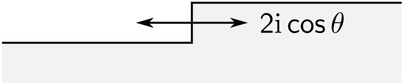
- Here the Fourier integral is a representation of the small phase shift due to the step
- $H_0^{(1)}(r)$ is a Hankel function and represents outgoing cylindrical waves due to the point inhomogeneity which looks like a monopole source
- The undetermined constant here is a_1 which will be found later via the matching process

Inner Problem

- This time the inner problem looks like the step with height 1



- After scaling for the inner problem, we get $(\nabla^2 + \epsilon^2)\phi = 0$ which can be approximated as $\nabla^2\phi = 0$ since ϵ^2 is much smaller than the ∇^2 term — like the inner problem in previous example
- We find there is a forcing of $2\mathrm{i}\cos\theta$ on the edge of the step which is the mechanism for generation of the cylindrical waves



Solution to Inner Problem

- Solve via conformal mapping methods† or otherwise
- The solution ends up being

$$\phi = \frac{2\mathrm{i}}{\pi} \cos\theta \log\left(\frac{\pi}{2}\frac{|x|}{\epsilon}\right) + B_1$$

- Here B_1 is the undetermined constant, to be determined by matching
- Using the Asymptotic Matching Principle gives, after a **lot** of handwaving,

$$a_1 = \cos\theta, \quad B_1 = \cos\theta\left(1 + \frac{2\mathrm{i}}{\pi}\left\{\gamma + \log\left(\frac{\epsilon}{\pi}\right)\right\}\right)$$

- But we only care about the far-field behaviour, so a_1 is what we are really interested in

† If we have time, the precise method will be discussed at the end

Far-Field Behaviour of Acoustic Field

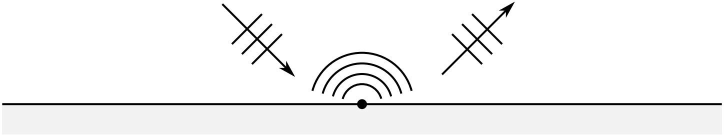
- After much work, have found that

$$\phi_t = \mathrm{e}^{-\mathrm{i}k(x\cos\theta + y\sin\theta)} + \mathrm{e}^{-\mathrm{i}k(x\cos\theta - y\sin\theta)} + \epsilon\left(-\frac{\mathrm{i}\sin^2\theta}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\alpha kx-(\alpha^2-1)^{1/2}ky}}{(\alpha^2-1)^{1/2}(\alpha+\cos\theta)}\,\mathrm{d}\alpha + \cos\theta H_0^{(1)}(kr)\right)$$

- Could carry on to next order, but not much point since they would be much smaller than the leading-order and ϵ terms
- What does this tell us?

Far-Field Behaviour of Acoustic Field

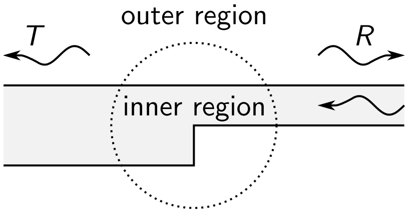
- To zeroth-order, the far-field acoustic field is of a plane wave reflecting off a flat surface
- The first-order $\mathcal{O}(\epsilon)$ corrections are outgoing cylindrical waves represented by Hankel functions emanating from the inhomogeneity at the origin



- Recent work showed that if the step is instead a sloping ramp, the a_1 term is still the same and so the far-field behaviour is unaffected by the precise profile of the inhomogeneity

Future Work

- Aim to extend analysis to flexural structural waves within elastic structure as in figure to find reflection and transmission coefficients R and T respectively



- Outer region appears as uniform plates joined together but cannot determine R and T from this
- Inner region appears like step problem but now with additional boundary condition on top side of structure; need to analyse to match to outer problem for R and T

Summary

- PhD in Structural Acoustics with eventual goal of minimising sound from underwater vehicles, with work of interest to Thales
- Described how MAEs might be used for problems with different length scales
- Showed how to apply MAEs to simple problem of scattering off plane with step
- Future work: extend ideas of scattering off step to new and exciting geometries as well as in-plane vibrations for elastic structures
- Questions?

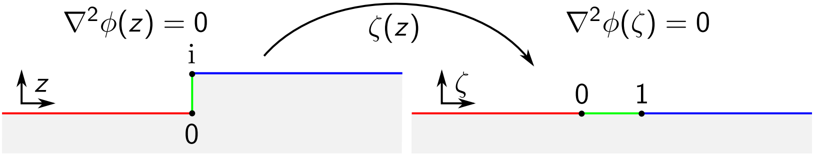
Bonus: Crash Course in Schwarz–Christoffel Mappings

- Recall that we needed to solve Laplace's equation ($\nabla^2\phi = 0$) in the inner problem
- We can do this via Schwarz–Christoffel (SC) mappings which are a particular type of conformal mapping
- SC mappings allow us to map complicated polygonal geometries (and their boundary conditions) to nice geometries, (*e.g.* upper half plane or disc)
- Can solve Laplace's equation in nice geometry and then map solution back to original geometry via inverse mapping

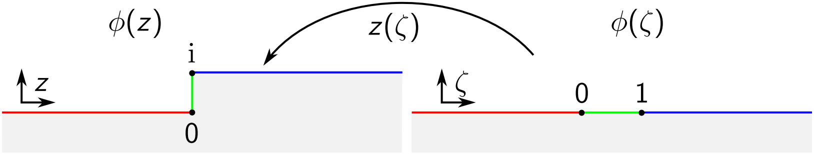
Bonus: Crash Course in Schwarz–Christoffel Mappings

In our case:

- Step 1: Switch to complex coordinates and then map plane with step (z-plane) to upper half plane (ζ-plane)[2]



- Step 2: Solve $\nabla^2\phi(\zeta) = 0$ in upper half plane via Fourier transforms
- Step 3: Map solution $\phi(\zeta)$ back to plane with step



[2]We also choose ∞ in the z-plane to be mapped to ∞ in the ζ-plane