A Brief and Gentle Introduction to the Wiener–Hopf Technique

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Lecture 1

The motivation for the Wiener–Hopf method was Equation (4) of Milne's paper in 1926 [5].

$$\boxed{B(\tau) = \frac{1}{2}\int_0^\infty B(t)\mathrm{Ei}(|t-\tau|)\,\mathrm{d}t, \quad \tau > 0,}$$

where $\mathrm{Ei}(z)$ is the exponential integral and is defined as follows:

$$\boxed{\mathrm{Ei}(z) = \int_z^\infty \frac{\mathrm{e}^{-y}}{y}\,\mathrm{d}y = \int_1^\infty \frac{\mathrm{e}^{-zy}}{y}\,\mathrm{d}y.}$$

(Note that in Abramowitz & Stegun and generally today this is written as $E_1(z)$.) This was the focus of Wiener and Hopf's 1931 paper: 'Über eine klasse singulärer integralgleichungen' [8]. Hopf asked in 'Mathematical Problems of Radiative Equilibrium' [3] if it was useful to rigorously solve equations which are of an approximate nature, as Milne's is. 'The answer to this is not always in the negative.' Indeed, their efforts at obtaining an explicit analytical solution to this particular equation has led to a valuable tool, which has spawned many thousands of papers covering many areas of mathematics, science and engineering.

Hence consider

$$f(x) = \int_0^\infty K(x-y)f(y)\,\mathrm{d}y, \quad x > 0,$$

where the kernel $K(x)$ is taken to be

$$K(x) = E_1(|x|) = \frac{1}{2}\int_{|x|}^\infty \frac{\mathrm{e}^{-y}}{y}\,\mathrm{d}y.$$

This is a homogeneous second-kind integral equation with a semi-infinite domain and a difference kernel (which appears everywhere in physics). It is also singular, *i.e.* $K(x) \sim -\log x - \gamma_{\mathrm{e}} + x$ as $x \to 0$. We make some simplifications:

- **Simplification 1:** Change $K(x) = \frac{1}{2}\mathrm{e}^{-|x|}$ — this is nonsingular.
- **Simplification 2:** Add forcing and tweak the form:

  $$5\mathrm{e}^{-|x|/2} = f(x) + 3\int_0^\infty K(x-y)f(y)\,\mathrm{d}y, \tag{1}$$

  where $K(x) = \frac{1}{2}\mathrm{e}^{-|x|}$ as in the first simplification. This is now inhomogeneous with a semi-infinite domain.
- **Simplification 3:** Change it to an infinite domain.

  $$5\mathrm{e}^{-|x|/2} = f(x) + \frac{3}{2}\int_{-\infty}^\infty K(x-y)f(y)\,\mathrm{d}y, \quad -\infty < x < \infty. \tag{2}$$

The solution to Equation (2) can be written in terms of Fourier transforms which are defined as follows:

$$F(\alpha) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x, \qquad f(x) = \frac{1}{2\pi}\int_{\mathcal{C}} F(\alpha)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha,$$

with $\mathcal{C}$ being the integration contour as defined in Figure 1.

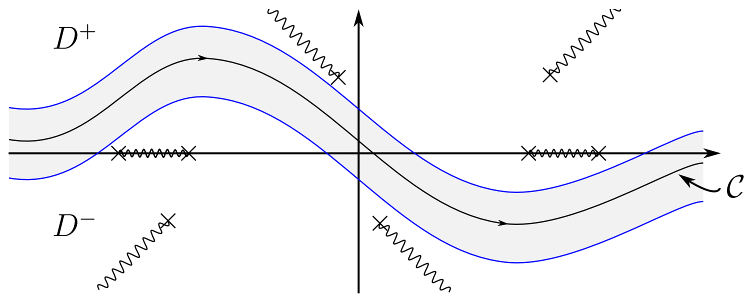


Figure 1: The singularities on the negative real line are taken to lie in the lower half-plane (LHP), whilst the singularities on the positive real line lie in the upper half-plane (UHP). Thus the integration contour is indented, if necessary, as shown, and lies within a strip D of non-zero width. The region above and including D is denoted D^+, and the domain below and including D is D^-.

Apply Fourier transforms to the left-hand side of Equation (2):

$$5\int_{-\infty}^{\infty} \mathrm{e}^{-|x|/2}\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x = 5\int_{-\infty}^{0} \mathrm{e}^{x(\mathrm{i}\alpha+1/2)}\,\mathrm{d}x + 5\int_{0}^{\infty} \mathrm{e}^{x(\mathrm{i}\alpha-1/2)}\,\mathrm{d}x = \frac{5}{\alpha^2+\frac{1}{4}}.$$

Hence the Fourier transform of Equation (2) is

$$\frac{5}{\alpha^2+\frac{1}{4}} = F(\alpha) + \frac{3}{2}\int_{-\infty}^{\infty} f(y)\left\{\int_{-\infty}^{\infty} \mathrm{e}^{-|x-y|}\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x\right\}\mathrm{d}y,$$

assuming it is OK to interchange the order of integration.

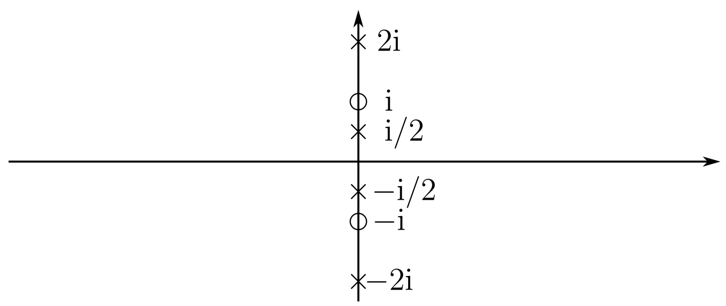


Figure 2: The poles and zeros of $F(\alpha)$ (see Equation (3)) in the complex α-plane.

Let $x - y = u$, then

$$
\begin{aligned}
\frac{5}{\alpha^2 + \frac{1}{4}} &= F(\alpha) + \frac{3}{2}\int_{-\infty}^{\infty} f(y)\mathrm{e}^{\mathrm{i}\alpha y}\,\mathrm{d}y \int_{-\infty}^{\infty} \mathrm{e}^{-|u|}\mathrm{e}^{\mathrm{i}\alpha u}\,\mathrm{d}u,\\
&= F(\alpha) + \frac{3}{\alpha^2+1}F(\alpha),\\
&= \left(\frac{\alpha^2+4}{\alpha^2+1}\right)F(\alpha).
\end{aligned}
$$

Solving for $F(\alpha)$ gives

$$
F(\alpha) = \frac{5(\alpha^2+1)}{(\alpha^2+\frac{1}{4})(\alpha^2+4)}. \tag{3}
$$

Hence the function $f(x)$ is the inverse transform of this:

$$
f(x) = \frac{5}{2\pi}\int_{-\infty}^{\infty} \frac{(\alpha^2+1)\mathrm{e}^{-\mathrm{i}\alpha x}}{(\alpha^2+\frac{1}{4})(\alpha^2+4)}\,\mathrm{d}\alpha.
$$

This is trivial to solve. For $x > 0$, deform the contour into the LHP, collecting residues at $\alpha = -\mathrm{i}/2$ and $-2\mathrm{i}$ as in Figure 2. For $x > 0$, we could deform the contour into the UHP and collect residues as before, but we note that the solution is even in x by inspection. After a little algebra, we find

$$
\boxed{f(x) = \mathrm{e}^{-2|x|} + \mathrm{e}^{-|x|/2}.}
$$

Observations:

- The difference kernel gives a convolution operator for the Fourier transform.
- It is easy for an infinite domain, but what about a semi-infinite domain?

  We can write (using additive decomposition)

$$
F(\alpha) = \underbrace{\frac{(-\frac{5}{2} - 2\mathrm{i}\alpha)}{(\alpha-2\mathrm{i})(\alpha-\frac{\mathrm{i}}{2})}}_{\text{singularities in UHP}} + \underbrace{\frac{(-\frac{5}{2} + 2\mathrm{i}\alpha)}{(\alpha+2\mathrm{i})(\alpha+\frac{\mathrm{i}}{2})}}_{\text{singularities in LHP}}.
$$

  So we may write these two terms as

$$
F(\alpha) = F^-(\alpha) \qquad + \qquad F^+(\alpha).
$$

  For example,

$$
f^+(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F^+(\alpha)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}x = \begin{cases} \mathrm{e}^{-2x} + \mathrm{e}^{-x/2} & x > 0,\\ 0 & x < 0. \end{cases}
$$

In other words, the half-range Fourier transform,

$$\int_0^\infty \left(\mathrm{e}^{-2x} + \mathrm{e}^{-x/2}\right) \mathrm{e}^{\mathrm{i}\alpha x}\, \mathrm{d}x = F^+(\alpha),$$

has no singularities in the UHP, and indeed, the half-range Fourier transform of any function which is identically zero for $x < 0$ has singularities only in the LHP (*i.e.* below the strip D). Similarly, the half-range Fourier transform of a function which is identically zero for $x > 0$ has singularities only in the UHP (*i.e.* above the strip D). In summary, we use the superscript $\oplus$ on a function $F^+(\alpha)$ to indicate analyticity in the region D^+, and D^- denotes the region of analyticity of $\ominus$ functions.

Lecture 2

There has been debate about whether Wiener–Hopf equations had been studied, and solved, prior to Wiener and Hopf. Carleman ($\sim$ 1920s) studied singular integral equations with a difference kernel and finite limits [2, 1]. Toeplitz (1911) studied systems of equations $\mathsf{A}\mathbf{x} = \mathbf{b}$ with a special matrix A, of either infinite or semi-infinite extent [7].

Discrete Problems

It is perhaps easiest to summarise the Wiener–Hopf procedure for discrete systems. For these, instead of employing the Fourier transform, we use the z-transform instead. Consider an algebraic system of equations containing a Toeplitz matrix:

$$\begin{pmatrix} a_0 & a_1 & & \\ a_{-1} & a_0 & \ddots & \\ a_{-2} & a_{-1} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{pmatrix}.$$

For ease of understanding, take, say, $a_m = 2^{-|m|}$, $b_0 = 1$, and $b_m = 0$ for $m > 0$. If we know x_n then the LHS would yield values for negative as well as positive m. So we write

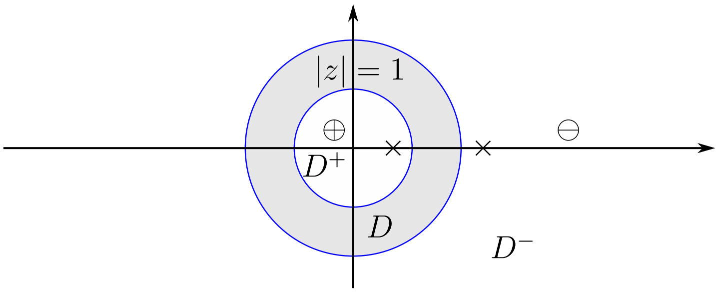
$$\sum_{n=0}^{\infty} 2^{-|m-n|} x_n = \begin{cases} \delta_{n0} & m \geq 0; \\ c_m & m < 0, \end{cases}$$

where c_m are as yet unknown constants. Now multiply by z^m and sum over all m. The left-hand side gives

$$\sum_{m=-\infty}^{\infty} z^m \sum_{n=0}^{\infty} 2^{|m-n|} x_n = \sum_{n=0}^{\infty} x_n \sum_{m=-\infty}^{\infty} z^m 2^{-|m-n|},$$

(note the similarity to the previous problem), and let $u = m - n$. Then the equation becomes

$$\begin{aligned} \sum_{n=0}^{\infty} x_n \sum_{u=-\infty}^{\infty} z^{u+n} 2^{-|u|} &= \sum_{n=0}^{\infty} x_n z^n \sum_{u=-\infty}^{\infty} 2^{-|u|} z^u, \\ &= \sum_{m=0}^{\infty} b_m z^m + \sum_{m=-\infty}^{-1} c_m z^m, \\ &= \boxed{1 + \sum_{m=1}^{\infty} c_{-m} z^{-m} = \sum_{m=0}^{\infty} x_n z^n \sum_{u=-\infty}^{\infty} 2^{-|u|} z^u.} \end{aligned}$$



If z is a complex variable then $\sum_{n=0}^{\infty} x_n z^n = X^+(z) =$ a function analytic in some disc in the complex z-plane. We assume the radius of convergence is outside $|z| = 1$.

Similarly $\sum_{m=1}^{\infty} c_m z^{-m}$ is the Laurent expansion of $C^-(z)$ which is assumed to be analytic outside some circle $|z| < 1$. Further,

$$\begin{aligned}\sum_{u=-\infty}^{\infty} 2^{-|u|} z^u &= \sum_{u=-\infty}^{-1} 2^u z^u + \sum_{z=0}^{\infty} 2^{-u} z^u, \\ &= \frac{2^{-1} z^{-1}}{1 - 2^{-1} z^{-1}} + \frac{1}{(1 - 2^{-1} z)}, \\ &= \frac{\frac{1}{2}}{z - \frac{1}{2}} - \frac{2}{z - 2} = -\frac{3z}{(z - \frac{1}{2})(z - 2)}.\end{aligned}$$

Note that $\frac{\frac{1}{2}}{z - \frac{1}{2}}$ is free of singularities outside the radius of convergence ($\ominus$ region is $|z| > 1/2$) and $\frac{2}{z-2}$ is free of singularities inside the radius of convergence ($\oplus$ region is $|z| < 2$).

The Wiener–Hopf functional equation is therefore

$$\boxed{-\frac{3z}{2(z - \frac{1}{2})(z - 2)} X^+(z) = 1 + C^-(z).}$$

Here the kernel is

$$K(z) = -\frac{3z}{2(z - \frac{1}{2})(z - 2)},$$

which naturally separates into a product of $\oplus \times \ominus$ functions, and so rearranging gives

$$\underbrace{-\frac{3X^+(z)}{2(z-2)} - 1}_{\oplus} = \underbrace{\left(\frac{z - \frac{1}{2}}{z}\right)(1 + C^-(z)) - 1}_{\ominus}.$$

By construction, the left-hand side is analytic in the $\oplus$ region, and the right-hand side is analytic in the $\ominus$ region, with common overlap D. Thus, analytic continuation from the left and right sides tells us that this is equivalent to an entire function E, say. We know that $C^-(z)$ behaves as $\mathcal{O}(z^{-1})$ as $|z| \to \infty$ in D^-, and hence the right-hand side tends to zero in this limit (which is why we subtract a constant on each side). Liouville's theorem then tells us that $E \equiv 0$. Hence

$$X^+(z) = -\frac{2}{3}(z-2).$$

Immediately we see that

$$x_0 = \frac{4}{3}, \qquad x_1 = -\frac{2}{3},$$

recalling that

$$\sum_{m=0}^{\infty} x_m z^m = X^+(z).$$

All other x_m are zero for $m \geq 2$. In more complicated problems, we could require

$$\boxed{x_p = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} X^+(z) z^{-p-1}\,\mathrm{d}z \;\forall\; p,}$$

which is the inverse z-transform.

Check:

$$\sum_{n=0}^{\infty} 2^{-|n-m|} x_n = 2^{-|m|}\frac{4}{3} - 2^{-|1-m|}\frac{2}{3} = \begin{cases} 1, & m = 0 \\ 0, & m > 0. \end{cases}$$

Critical Steps:

- Factorisation of the transform of the kernel or matrix: $K(z) = K^+(z)K^-(z)$.
- Rearrangement of the equation into $\oplus = \ominus$ which allows analytic continuation into the whole complex plane and hence gives that both sides are equivalent to an entire function.
- Application of Liouville's theorem.

There is an alternative factorisation procedure. Rewrite the equation as

$$\underbrace{X^+(z)}_{\oplus}\left\{\underbrace{\frac{\frac{1}{2}}{z-\frac{1}{2}}}_{\ominus} - \underbrace{\frac{2}{z-2}}_{\oplus}\right\} = \underbrace{1}_{\oplus} + \underbrace{C^-(z)}_{\ominus},$$

then remove the pole at $z = 1/2$ from the left-hand side, and rearrange as

$$\frac{1}{2}\cdot\frac{X^+(z) - X^+(1/2)}{z-\frac{1}{2}} - \frac{2X^+(z)}{z-2} - 1 = -\frac{X^+(1/2)}{2(z-\frac{1}{2})} + C^-(z) \equiv 0.$$

Here the left-hand side is $\oplus$, the right-hand side is $\ominus$ and we have used analytic continuation and Liouville's theorem to set them equal to zero. We can now write this as

$$-\frac{3z}{2(z-\frac{1}{2})(z-2)}X^{+}(z) = 1 + \frac{X^{+}(1/2)}{2\left(z-\frac{1}{2}\right)},$$

which implies that

$$X^{+}(z) = -\frac{(z-2)}{3z}\left(2\left(z-\frac{1}{2}\right) + X^{+}(1/2)\right).$$

This musn't have a pole at $z=0$, so we require $X^{+}(1/2)=1$, hence

$$\boxed{X^{+}(z) = -\frac{2}{3}(z-2).}$$

This agrees with the value obtained from our original approach.

Continuous Wiener–Hopf Problems

We now return to Equation (1), and use the left-hand side to extend to $x<0$.

$$\frac{3}{2}\int_0^{\infty} \mathrm{e}^{|x-y|} f(y)\,\mathrm{d}y = \begin{cases} 5\mathrm{e}^{-x/2} - f(x), & x>0; \\ g(x), & x<0. \end{cases}$$

Here $g(z)$ is a new unknown function. Introduct the Fourier transforms.

$$\underbrace{G^{-}(\alpha) = \int_{-\infty}^{0} g(z)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x,}_{\text{no singularities in the LHP: }\ominus} \qquad \underbrace{F^{+}(\alpha) = \int_{0}^{\infty} f(x)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x,}_{\text{no singularities in the UHP: }\oplus}$$

and so the Fourier transform of the integral equation gives

$$\underbrace{G^{-}(\alpha)}_{\ominus} + \underbrace{\frac{5\mathrm{i}}{(\alpha+\frac{\mathrm{i}}{2})}}_{\oplus} - \underbrace{F^{+}(\alpha)}_{\oplus} = \frac{3F^{+}(\alpha)}{\alpha^2+1},$$

or

$$G^{-}(\alpha) + \frac{5\mathrm{i}}{(\alpha+\frac{\mathrm{i}}{2})} = \left(\frac{\alpha^2+4}{\alpha^2+1}\right)F^{+}(\alpha).$$

Here the function $K(z)$ (often loosely called the kernel) can be factorised as

$$K(z) = \left(\frac{\alpha^2+4}{\alpha^2+1}\right) = \left(\frac{\alpha+2\mathrm{i}}{\alpha+\mathrm{i}}\right)\left(\frac{\alpha-2\mathrm{i}}{\alpha-\mathrm{i}}\right).$$

Hence we have

$$\underbrace{\left(\frac{\alpha-\mathrm{i}}{\alpha-2\mathrm{i}}\right)G^-(\alpha)}_{\ominus}+\underbrace{\frac{5\mathrm{i}}{(\alpha+\frac{\mathrm{i}}{2})}\left(\frac{\alpha-\mathrm{i}}{\alpha-2\mathrm{i}}\right)}_{\oplus/\ominus}=\underbrace{\left(\frac{\alpha+2\mathrm{i}}{\alpha+\mathrm{i}}\right)F^+(\alpha)}_{\oplus}.$$

The final step of the rearrangement is to write the forcing term as

$$\frac{5\mathrm{i}}{(\alpha+\frac{\mathrm{i}}{2})}\left(\frac{\alpha-\mathrm{i}}{\alpha-2\mathrm{i}}\right)=\underbrace{\frac{2\mathrm{i}}{\alpha-2\mathrm{i}}}_{\ominus}+\underbrace{\frac{3\mathrm{i}}{\alpha+\mathrm{i}/2}}_{\oplus}.$$

Then we write

$$\underbrace{\left(\frac{\alpha-\mathrm{i}}{\alpha-2\mathrm{i}}\right)G^-(\alpha)+\frac{2\mathrm{i}}{\alpha-2\mathrm{i}}}_{\ominus}=\underbrace{\left(\frac{\alpha+2\mathrm{i}}{\alpha+\mathrm{i}}\right)F^+(\alpha)-\frac{3\mathrm{i}}{\alpha+\mathrm{i}/2}}_{\oplus}.$$

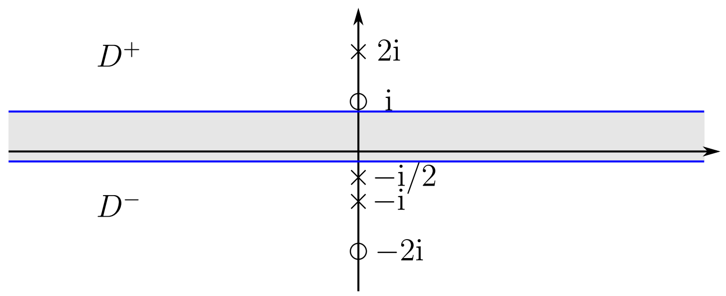


Figure 3: Location of the poles and zeros in the complex α-plane.

If $f(x)$ and $g(x)$ are bounded as $x\to\pm 0$, then we know

$$F^+(\alpha)=\mathcal{O}\left(\frac{1}{\alpha}\right),\quad |\alpha|\to\infty \text{ in } D^+,$$
$$G^-(\alpha)=\mathcal{O}\left(\frac{1}{\alpha}\right),\quad |\alpha|\to\infty \text{ in } D^-.$$

Both sides tend to zero as $|\alpha|\to\infty$ which implies $E(\alpha)\equiv 0$ by Liouville's theorem, and hence

$$F^+(\alpha)=\frac{3\mathrm{i}(\alpha+\mathrm{i})}{(\alpha+2\mathrm{i})(\alpha+\mathrm{i}/2)}.$$

Taking the inverse transform, and using residue calculus, we obtain the solution

$$f(x)\equiv 2\mathrm{e}^{-2x}+\mathrm{e}^{-x/2}.$$

Now return to Simplification 1 (in which we changed from Milne's kernel to $K(x) = \frac{1}{2}\mathrm{e}^{-|x|}$). We have

$$\frac{1}{2}\int_0^\infty \mathrm{e}^{-|x-y|} f(y)\,\mathrm{d}y = \begin{cases} f(x), & x > 0; \\ g(x), & x < 0. \end{cases}$$

This is an inhomogeneous equation (where the solution is $f(x) = 1 + x$). Take the Fourier transform as before.

$$G^-(\alpha) + F^+(\alpha) = \frac{F^+(\alpha)}{\alpha^2 + 1} \implies \boxed{G^-(\alpha) = -\frac{\alpha^2 F^+(\alpha)}{1 + \alpha^2}.}$$

This implies, by analytic continuation, that

$$\underbrace{(\alpha - \mathrm{i})G^-(\alpha)}_{\ominus} = \underbrace{-\frac{\alpha^2}{\alpha + \mathrm{i}} F^+(\alpha)}_{\oplus} = E(\alpha).$$

Both sides tell us that $E(\alpha) =$ constant which is not zero. Writing $E = -\mathrm{i}$ without loss of generality gives

$$\boxed{F^+(\alpha) = \frac{\mathrm{i}(\alpha + 1)}{\alpha^2}.}$$

The inverse transform contour must pass above the origin if it is a $\oplus$ function.

Hence, residue calculus again yields

$$f(x) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^\infty \underbrace{\left(\frac{\alpha + \mathrm{i}}{\alpha^2}\right)}_{1/\alpha + \mathrm{i}/\alpha^2} \mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}x = 1 + x.$$

note that $f(x) = 0$ for $x < 0$ as required. Now let us return to Milne's integral equation, which was the original equation studied by Wiener and Hopf. Recall that we have

$$f(x) = \int_0^\infty k(x - y) f(y)\,\mathrm{d}y, \qquad \boxed{k(x) = \frac{1}{2}\int_{|x|}^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t.}$$

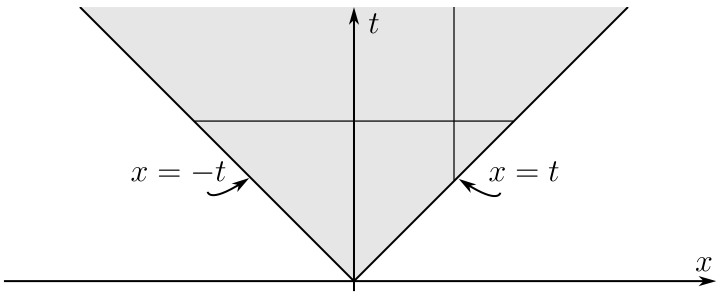
Take the Fourier transform of the kernel $k(x)$ to get $K(\alpha)$.

$$\frac{1}{2}\int_{-\infty}^\infty \int_{|x|}^\infty \frac{\mathrm{e}^{-t}}{t}\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}t\,\mathrm{d}x = \frac{1}{2}\int_0^\infty \frac{\mathrm{e}^{-t}}{t}\left(\int_{-t}^t \mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x\right)\mathrm{d}t.$$

This change of integration order, and limits, can be seen from the following diagram.

The integral can be evaluated as follows. The transformed kernel is

$$K(\alpha) = \frac{1}{2}\int_0^\infty \frac{\mathrm{e}^{-t}}{t}\left(\int_{-t}^t \mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x\right)\mathrm{d}t = \frac{1}{2\mathrm{i}\alpha}\int_0^\infty \frac{1}{t}\left[\mathrm{e}^{\mathrm{i}(\alpha+\mathrm{i})t} - \mathrm{e}^{-\mathrm{i}(\alpha-\mathrm{i})t}\right]\mathrm{d}t,$$



but we label the integral as $I(\alpha)$:

$$I(\alpha) = \int_0^\infty \frac{1}{t}\left[e^{i(\alpha+i)t} - e^{-i(\alpha-i)t}\right] dt = 2i\alpha K(\alpha).$$

Differentiating $I(\alpha)$ with respect to α gives

$$\frac{dI}{d\alpha} = i\int_0^\infty \left(e^{i(\alpha+1)t} + e^{-i(\alpha-1)t}\right) dt,$$

which is easily evaluated as

$$\frac{dI}{d\alpha} = \frac{2i}{\alpha^2+1}.$$

This can now be integrated, using the fact that $I(0) = 0$, to yield

$$\boxed{K(\alpha) = \frac{i}{2\alpha}\log\left(\frac{i+\alpha}{i-\alpha}\right).}$$

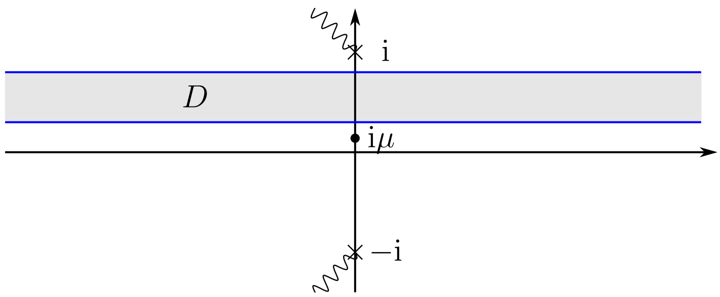


Figure 4: The strip D lies in the region $\mu < \mathrm{Im}(\alpha) < 1$.

Note that we choose branch cuts as shown in Figure 4, and the sheet of Riemann surface that has $K(0) = 1$. Hence

$$K(\alpha) \sim \begin{cases} \frac{\pi}{2\alpha}, & \alpha \to \infty, \\ \frac{\pi}{2|\alpha|}, & \alpha \to -\infty. \end{cases}$$

The transformed integral equation is thus

$$F^+(\alpha) + G^-(\alpha) = \frac{\mathrm{i}}{2\alpha} \log\left(\frac{\mathrm{i}+\alpha}{\mathrm{i}-\alpha}\right) F^+(\alpha),$$

where we make the initial assumption that $f(x)$ is bounded at the origin and grows like $f(x) \sim \mathrm{e}^{\mu x}$, where $0 < \mu < 1$. Hence $F^+(\alpha) = \int_0^\infty f(x)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x$ exists for $\mathrm{Im}(\alpha) > \mu$, and the singularities in $K(\alpha)$ imply that the strip D is $\mu < \mathrm{Im}(\alpha) < 1$. Now rearrange:

$$Q(\alpha)F^+(\alpha) = -G^-(\alpha),$$

where

$$Q(\alpha) = 1 - \frac{\mathrm{i}}{2\alpha} \log\left(\frac{\mathrm{i}+\alpha}{\mathrm{i}-\alpha}\right).$$

How do we factorise $Q(\alpha) = Q^+(\alpha)Q^-(\alpha)$? We use the Cauchy integral formula. The sum split of $a(\alpha)$ is

$$a(\alpha) = a^+(\alpha) + a^-(\alpha) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{a(\zeta)}{\zeta-\alpha}\,\mathrm{d}\zeta - \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{a(\zeta)}{\zeta-\alpha}\,\mathrm{d}\zeta,$$

where the symbol $\smile$ ($\frown$) denotes that α lies above (below) the integration contour.

The product split of a function $b(\alpha)$ can be found by taking logarithms and exponentiating: $b(\alpha) = b^+(\alpha)b^-(\alpha)$, where

$$b^+(\alpha) = \exp\left\{\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\log b(\zeta)}{\zeta-\alpha}\,\mathrm{d}\zeta\right\},$$

$$b^-(\alpha) = \exp\left\{-\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\log b(\zeta)}{\zeta-\alpha}\,\mathrm{d}\zeta\right\}.$$

Note that b must not branch on the contour, and tend to unit as $|\zeta| \to \infty$ for convergence.

To use this formula, it is easiest to rearrange $Q(\alpha)$. Note that

$$\begin{aligned}
Q(\alpha) &= 1 - \frac{\mathrm{i}}{2\alpha}\log\left(\frac{1-\mathrm{i}\alpha}{1+\mathrm{i}\alpha}\right), \\
&\sim 1 - \frac{\mathrm{i}}{2\alpha}\left\{-\mathrm{i}\alpha + \frac{\alpha^2}{2} + \frac{\mathrm{i}\alpha^3}{3} + \cdots - \left(\mathrm{i}\alpha + \frac{\alpha^2}{2} - \frac{\mathrm{i}\alpha^3}{3} + \cdots\right)\right\}, \\
&= 1 - 1 + \frac{\alpha^2}{3}, \\
&\sim \frac{\alpha^2}{3} \text{ as } \alpha \to 0.
\end{aligned}$$

So it has a double zero at the origin. Hence we may write

$$Q(\alpha) = \frac{\alpha^2}{\alpha^2+1} \underbrace{\left\{ \left(\frac{\alpha^2+1}{\alpha^2}\right)\left(1 - \frac{\mathrm{i}}{2\alpha}\log\left(\frac{\mathrm{i}+\alpha}{\mathrm{i}-\alpha}\right)\right)\right\}}_{R(\alpha):\text{ well-behaved in the strip } |\operatorname{Im}(\alpha)|<1},$$

where R can be factorised into $R = R^+R_-$ by the integral formulae, and so

$$Q^+(\alpha) = \frac{\alpha^2}{\alpha+\mathrm{i}}R^+(\alpha), \qquad Q^-(\alpha) = \frac{1}{\alpha-\mathrm{i}}R^-(\alpha).$$

This yields

$$\underbrace{\frac{\alpha^2}{\alpha+\mathrm{i}}R^+(\alpha)F^+(\alpha)}_{\oplus} = -\underbrace{\frac{(\alpha-\mathrm{i})G^-(\alpha)}{R^-(\alpha)}}_{\ominus} = E(\alpha),$$

and we know

$$R^{\pm}(\alpha) \to 1, \quad F^+(\alpha) = \mathcal{O}\left(\frac{1}{\alpha}\right), \quad G^-(\alpha) = \mathcal{O}\left(\frac{1}{\alpha}\right), \quad |\alpha| \to \infty \text{ in } D^{\pm}.$$

So the LHS $= \mathcal{O}(1)$ and the RHS $= \mathcal{O}(1)$ as $|\alpha| \to \infty$ in $D^{\pm}$, and hence we may take $E = \mathrm{i}$ without loss of generality, which gives

$$F^+(\alpha) = \frac{\mathrm{i}(\alpha+\mathrm{i})}{\alpha^2 R^+(\alpha)}.$$

It will be useful to rewrite this as

$$\begin{aligned} F^+(\alpha) &= \frac{\mathrm{i}(\alpha+\mathrm{i})R^-(\alpha)}{\alpha^2 R(\alpha)}, \\ &= \frac{\mathrm{i}(\alpha+\mathrm{i})R^-(\alpha)}{\alpha^2\left(\frac{\alpha^2+1}{\alpha^2}\right)\left(1-\frac{\mathrm{i}}{2\alpha}\log\left(\frac{\mathrm{i}+\alpha}{\mathrm{i}-\alpha}\right)\right)}, \\ &= \frac{\mathrm{i}R^-(\alpha)}{(\alpha-\mathrm{i})\left(1-\frac{\mathrm{i}}{2\alpha}\log\left(\frac{\mathrm{i}+\alpha}{\mathrm{i}-\alpha}\right)\right)}. \end{aligned}$$

Hence we have

$$f(x) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \frac{R^-(\alpha)\mathrm{e}^{-\mathrm{i}\alpha x}}{(\alpha-\mathrm{i})\left(1-\frac{\mathrm{i}}{2\alpha}\log\left(\frac{\mathrm{i}+\alpha}{\mathrm{i}-\alpha}\right)\right)}\,\mathrm{d}\alpha.$$

where we deform the integration contour $\mathcal{C}$ into the LHP, picking up the residue from the double pole at the origin, and the branch cut contribution, as shown in Figure 5

Near the origin, the integrand is approximately

$$\frac{\mathrm{i}[R^-(0)+\alpha(R^-)'(0)](1-\mathrm{i}\alpha x)(1-\mathrm{i}\alpha)}{\alpha^2/3} = \frac{3\mathrm{i}}{\alpha^2}\left\{R^-(0) - \underbrace{\mathrm{i}\alpha(1+x)R^-(0)+\alpha(R^-)'(0)}_{\text{residue term}}\right\}.$$

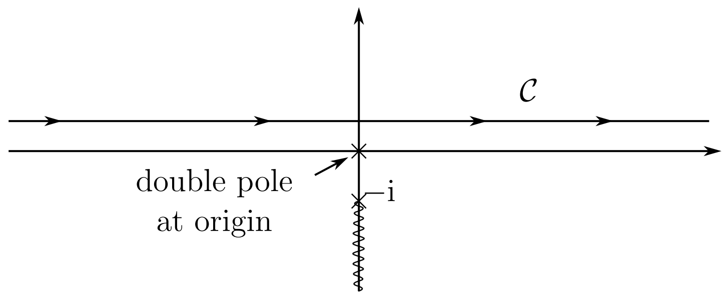


Figure 5: The branch cut of $F^+(\alpha)$ lies in the LHP and we have a double pole at the origin.

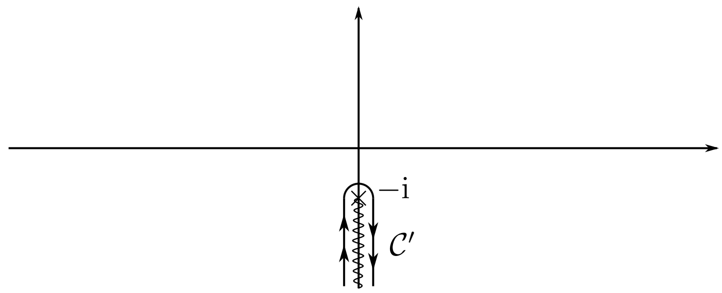


Figure 6: The integration contour $\mathcal{C}'$ has been deformed to follow the branch cut.

Hence

$$f(x) = -2\pi\mathrm{i} \cdot \frac{\mathrm{i}}{2\pi} \cdot 3\mathrm{i}\left[-\mathrm{i}(1+x)R^-(0) + (R^-)'(0)\right] + \frac{\mathrm{i}}{2\pi}\int_{\mathcal{C}'} \frac{R^-(\alpha)\mathrm{e}^{-\mathrm{i}\alpha x}}{(\alpha-\mathrm{i})Q(\alpha)}\,\mathrm{d}\alpha,$$

where the integration contour $\mathcal{C}'$ is as in Figure 6.

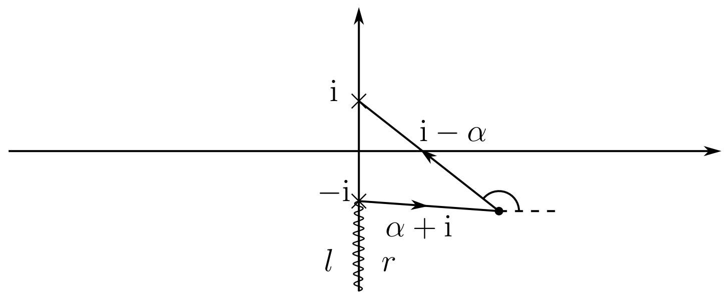
So

$$f(x) = 3\{(1+x)R^-(0) + \mathrm{i}(R^-)'(0)\} + \frac{\mathrm{i}}{2\pi}\int_{-\mathrm{i}}^{-\mathrm{i}\infty} \frac{R^-(\alpha)\mathrm{e}^{-\alpha x}}{\alpha-\mathrm{i}}\left\{\frac{1}{Q^r(\alpha)} - \frac{1}{Q^l(\alpha)}\right\}\mathrm{d}\alpha.$$

We know that

$$(\mathrm{i}-\alpha) = |\mathrm{i}-\alpha|\mathrm{e}^{\mathrm{i}\pi/2} \text{ on } r \text{ and } l,$$

$$(\mathrm{i}+\alpha) = \begin{cases} |\alpha+\mathrm{i}|\mathrm{e}^{-\mathrm{i}\pi/2} \text{ on } r, \\ |\alpha+\mathrm{i}|\mathrm{e}^{3\mathrm{i}\pi/2} \text{ on } l. \end{cases}$$



Now let $\alpha = -\mathrm{i}(1+u)$. So

$$\begin{aligned}\frac{1}{Q^r(\alpha)} - \frac{1}{Q^l(\alpha)} &= \frac{1}{\left\{1 - \frac{\mathrm{i}}{(-2\mathrm{i})(1+u)}\left(\log\left(\frac{u\mathrm{e}^{-\mathrm{i}\pi/2}}{(2+u)\mathrm{e}^{\mathrm{i}\pi/2}}\right)\right)\right\}} - \frac{1}{\left\{1 + \frac{1}{2(1+u)}\log\left(\frac{u\mathrm{e}^{\mathrm{i}3\pi/2}}{(2+u)\mathrm{e}^{\mathrm{i}\pi/2}}\right)\right\}} \\ &= \frac{2(1+u)}{\left\{2(1+u) + \log\left(\frac{u}{2+u}\right) - \mathrm{i}\pi\right\}} - \frac{2(1+u)}{\left\{2(1+u) + \log\left(\frac{u}{2+u}\right) + \mathrm{i}\pi\right\}} \\ &= \frac{4\mathrm{i}\pi(1+u)}{\left(\left[2(1+u) + \log\left(\frac{u}{2+u}\right)\right]^2 + \pi^2\right)},\end{aligned}$$

and hence

$$\begin{aligned}f(x) = 3&\left\{(1+x)R^-(0) + \mathrm{i}(R^-)'(0)\right\} \\ &+ \frac{\mathrm{i}}{2\pi}\int_0^\infty \frac{(-\mathrm{i})R^-(-\mathrm{i}(1+u))\mathrm{e}^{-(1+u)x}}{(-\mathrm{i})(2+u)}\left\{\frac{1}{Q^r(-\mathrm{i}(1+u))} - \frac{1}{Q^l(-\mathrm{i}(1+u))}\right\}\mathrm{d}u.\end{aligned}$$

This can equivalently be written as

$$\boxed{f(x) = 3\left\{(1+x)R^-(0) + \mathrm{i}(R^-)'(0)\right\} - 2\int_0^\infty \frac{R^-(-\mathrm{i}(1+u))\mathrm{e}^{-(1+u)x}(1+u)}{(2+u)\left(\left[2(1+u) + \log\left(\frac{u}{2+u}\right)\right]^2 + \pi^2\right)}\,\mathrm{d}u.}$$

This is close to a linear profile as the integral term decays exponentially. So

$$f(x) \sim 3\left(R^-(0) + \mathrm{i}(R^-)'(0)\right) + 3R^-(0)x, \qquad x \to \infty.$$

This can be numerically evaluated:

$$f(x) \sim 1.23053 + 1.73205x,$$

but note that at the origin

$$\begin{aligned}f(0) &= 3\left(R^-(0) + \mathrm{i}(R^-)'(0)\right) - 2\int_0^\infty \frac{R^-(-\mathrm{i}(1+u))(1+u)}{(2+u)\left(\left[2(1+u) + \log\left(\frac{u}{2+u}\right)\right]^2 + \pi^2\right)}\,\mathrm{d}u, \\ &= 1.\end{aligned}$$

There is one final point. We know

$$R^-(\alpha) = \exp\left\{ -\frac{1}{2\pi \mathrm{i}} \fint_{-\infty}^{\infty} \frac{\log R(\zeta)}{\zeta - \alpha}\,\mathrm{d}\zeta \right\},$$

so

$$(R^-)'(\alpha) = -\frac{1}{2\pi \mathrm{i}} \fint_{-\infty}^{\infty} \frac{\log R(\zeta)}{(\zeta - \alpha)^2}\,\mathrm{d}\zeta \cdot R^-(\alpha).$$

Furthermore,

$$R^-(0) = \exp\left\{ -\frac{1}{2\pi \mathrm{i}} \fint_{-\infty}^{\infty} \frac{\log R(\zeta)}{\zeta}\,\mathrm{d}\zeta \right\},$$

and

$$(R^-)'(0) = -\frac{1}{2\pi \mathrm{i}} \fint_{-\infty}^{\infty} \frac{\log R(\zeta)}{\zeta^2}\,\mathrm{d}\zeta \cdot R^-(0).$$

These are all straightforward to evaluate numerically.

Lecture 3

Matrix Systems

For scalar Wiener–Hopf equations, we have a systematic process for obtaining a solution:

$$\text{Integral equation} \xrightarrow{\text{Fourier transform}} \underbrace{K(\alpha)}_{\text{transform of } k(x)} F^+(\alpha) = G^-(\alpha) + \underbrace{A^+(\alpha)}_{\text{forcing}}.$$

The strip is defined by the assumptions made on the behaviour of $F^+(\alpha)$ **and** the singularities of the kernel and forcing. Then we perform the following.

1. **Product split:**

$$K = K^+K^- \implies K^+(\alpha)F^+(\alpha) = \frac{G^-(\alpha)}{K^-(\alpha)} + \frac{A^+(\alpha)}{K^-(\alpha)}.$$

2. **Sum split:**

$$\frac{A^+(\alpha)}{K^-(\alpha)} = B^+(\alpha) + B^-(\alpha) \implies K^+(\alpha)F^+(\alpha) - B^+(\alpha) = \frac{G^-(\alpha)}{K^-(\alpha)} + B^-(\alpha).$$

3. **Analytic continuation:**

$$K^+F^+ - B^+ = E \implies F^+ = \frac{E + B^+}{K^+} \xrightarrow{\text{Fourier inverse}} f(x), \quad x > 0.$$

The most critical step in the Wiener–Hopf procedure is the product factorisation of the Fourier transform of the kernel. But what happens in the matrix case?

Also, in general we don't go from the physical problem to the integral equation to the functional Wiener–Hopf equation. We now usually directly from the physical/boundary value problem to the Wiener–Hopf equation in the transformed plane (as devised by Jones/Reuter) [4].

As an example, let us derive a scalar Wiener–Hopf equation directly from a very simple problem in acoustics. This is shown in Figure 7, where ϕ^t is the velocity potential.

We have a two-dimensional steady (time-harmonic) problem for a compressible fluid, with rigid boundaries on $y = -h$, $-\infty < x < \infty$, and on $y = 0$, $x < 0$. The independent variables are scaled so that the velocity potential ϕ^t satisfies the Helmholtz equation $(\nabla^2 + 1)\phi^t = 0$.

Defining ϕ^t = the total potential = $\phi + \mathrm{e}^{\mathrm{i}x}$ in $0 > y > -h$, where ϕ is the scattered potential, and $\phi^t = \phi$ in $y > 0$. Then we have the following boundary value problem as in Figure 8.

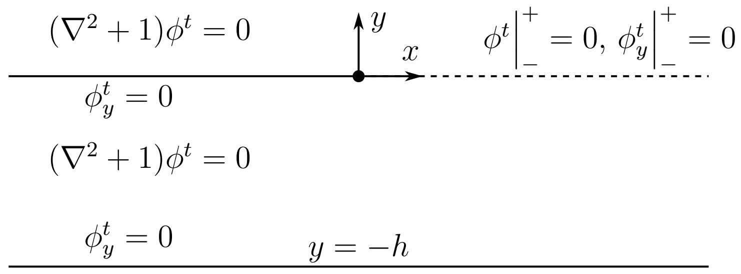


Figure 7: An acoustic scattering problem to be solved via the Wiener–Hopf technique.

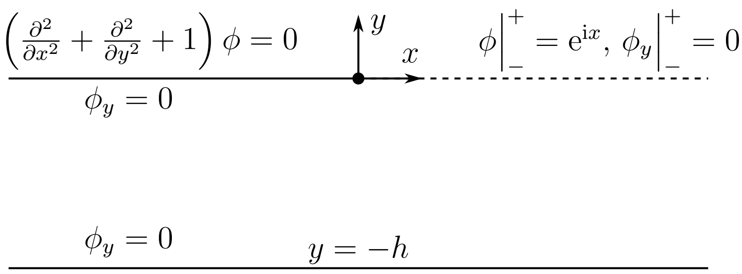


Figure 8: The boundary value problem after subtracting out the incoming potential $\mathrm{e}^{\mathrm{i}x}$.

Carry out a Fourier transform in the x-direction such that

$$\Phi(\alpha,y)=\int_{-\infty}^{\infty}\phi(x,y)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x,$$

and introduce the following unknowns:

$$\phi(x,0^+)-\phi(x,0^-)=b(x),\quad x<0;\quad \phi_y(x,0)=a(x),\quad x>0.$$

In $y>0$, we have

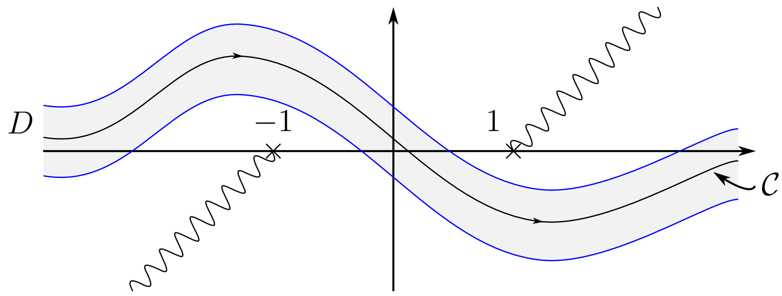
$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}+(1-\alpha^2)\right)\Phi=0\implies \Phi=C(\alpha)\mathrm{e}^{-(\alpha^2-1)^{1/2}y}+H(\alpha)\mathrm{e}^{(\alpha^2-1)^{1/2}y}.$$

Define $\gamma(\alpha)=(\alpha^2-1)^{1/2}$ such that the first term is outgoing/decaying as $y\to\infty$. We have $\gamma(0)=-\mathrm{i}$ and $\gamma\sim|\alpha|$ as $|\alpha|\to\infty$ in D.

So $H(\alpha)\equiv 0$ to satisfy the radiation condition.

For $y<0$, we have

$$\Phi=G(\alpha)\cosh\gamma(y+h).$$



This satisfies the boundary condition on $y=-h$. And on $y=0$, we have

$$\Phi\Big|_-^+ = \int_{-\infty}^{0} \underbrace{[\phi(x,0^+)-\phi(x,0^-)]}_{b(x)} \mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x + \int_0^\infty \mathrm{e}^{\mathrm{i}x}\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x,$$

$$= \boxed{B^-(\alpha) + \frac{\mathrm{i}}{(\alpha+1)_+} = C(\alpha) - G(\alpha)\cosh(\gamma h).}$$

Using $\Phi_y\Big|_-^+ = 0$ gives

$$\boxed{G(\alpha)\sinh\gamma h = -C(\alpha).}$$

Finally,

$$\Phi_y(\alpha,0) = \int_0^\infty a(x)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x,$$

$$= \boxed{A^+(\alpha) = \gamma G(\alpha)\sinh\gamma h = -\gamma C(\alpha).}$$

Eliminate the unknowns $C(\alpha)$ and $G(\alpha)$:

$$B^-(\alpha) + \frac{\mathrm{i}}{(\alpha+1)_+} = -\frac{A^+(\alpha)}{\gamma} - \frac{\cosh\gamma h}{\gamma\sinh\gamma h}A^+(\alpha) = -\frac{A^+(\alpha)}{\gamma \mathrm{e}^{-\gamma h}\sinh\gamma h} = K(\alpha)A^+(\alpha),$$

and so

$$(K(\alpha))^{-1} = -\frac{\gamma(1-\mathrm{e}^{-2\gamma h})}{2}.$$

As we were interested only on the direct derivation of this equation, we do not continue to the solution; however, it follows in a standard fashion. Now we consider the following acoustic scattering problem, shown in Figure 9, which is essentially a vector Wiener–Hopf system. We wish to find $R\mathrm{e}^{-\mathrm{i}x}$ and $T\mathrm{e}^{\mathrm{i}x}$ for a barrier of height $y=a<h$, $x=0$. We send in a plane wave from $x=-\infty$ and work with the scattered potential.

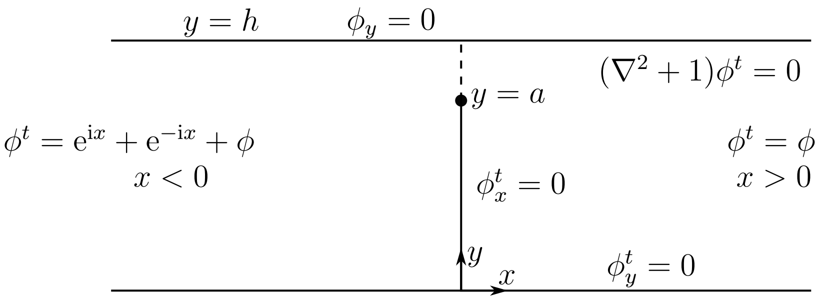


Figure 9: The physical acoustic problem.

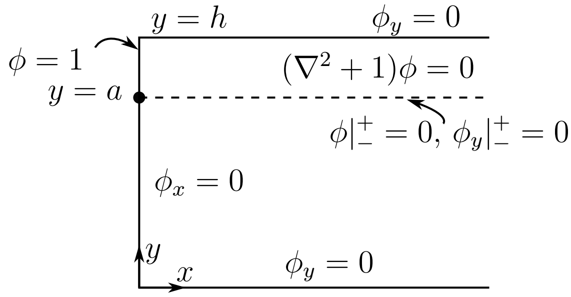


Figure 10: After using symmetry to simplify the physical problem.

We find that $\phi\Big|^+_- = 2$ on $x = 0$, $y > a$, and $\phi_x\Big|^+_- = 0$ on $x = 0$, $y > a$. The forcing and geometry indicate that $\phi(x) = -\phi(-x)$ (odd in x), so we just need to look at $x \geq 0$, as in Figure 10.

We introduce the unknowns β and δ as the potential and normal derivative on $y = a$, and take the cosine transform Φ_c in $y < a$ and the sine transform Φ_s in $y > a$, where the transforms are defined as

$$\Phi_c(\alpha, y) = \int_0^\infty \phi(x, y) \cos \alpha x \, \mathrm{d}x,$$
$$\Phi_s(\alpha, y) = \int_0^\infty \phi(x, y) \sin \alpha x \, \mathrm{d}x.$$

This further reduces into the following problem as in Figure 11.

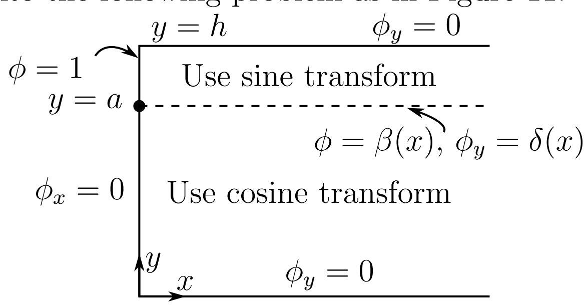


Figure 11: After a further simplification.

In $y > a$, we have

$$\int_0^\infty \frac{\partial^2 \phi}{\partial x^2} \sin \alpha x \, \mathrm{d}x = \left[\frac{\partial \phi}{\partial x} \sin \alpha x\right]_0^\infty - \alpha \left[\phi \cos \alpha x\right]_0^\infty - \alpha^2 \int_0^\infty \phi \sin \alpha x \, \mathrm{d}x,$$
$$= \alpha - \alpha^2 \Phi_s(\alpha, y).$$

So we have

$$\frac{\mathrm{d}^2 \Phi_s}{\mathrm{d}y^2} + (1 - \alpha^2)\Phi_s = -\alpha$$

and hence the general solution is

$$\boxed{\Phi_s(\alpha, y) = -\frac{\alpha}{(1-\alpha^2)} + C(\alpha) \cosh \gamma(h - y),}$$

where we have written $\gamma = (\alpha^2 - 1)^{1/2}$ (hence there are branch points at $\alpha = \pm 1$). This also satisfies the boundary condition on $y = h$. For $y < a$:

$$\frac{\mathrm{d}^2 \Phi_c}{\mathrm{d}y^2} + (1 - \alpha^2)\Phi_c = 0,$$

and so

$$\boxed{\Phi_c = A(\alpha) \cosh \gamma y.}$$

This satisfies the boundary condition on $y = 0$.

Now consider the cosine transform on the line $y = a$:

$$\Phi_c(\alpha, a) = \int_0^\infty \beta(x) \cos \alpha x \, \mathrm{d}x = \frac{1}{2} \int_0^\infty \beta(x)(\mathrm{e}^{\mathrm{i}\alpha x} + \mathrm{e}^{-\mathrm{i}\alpha x}) \, \mathrm{d}x.$$

This gives

$$\boxed{\Phi_c(\alpha, a) = \frac{1}{2}\left(B^+(\alpha) + B^+(-\alpha)\right) = A(\alpha) \cosh \gamma a,}$$

and additionally

$$\Phi_{c,y}(\alpha, a) = \int_0^\infty \delta(x) \cos \alpha x \, \mathrm{d}x = \boxed{\frac{1}{2}(D^+(\alpha) + D^+(-\alpha)) = \gamma A(\alpha) \sinh \gamma a.}$$

Similarly,

$$\Phi_s(\alpha, a) = \int_0^\infty \beta(x) \sin \alpha x \, \mathrm{d}x = \boxed{\frac{1}{2\mathrm{i}}\left(B^+(\alpha) - B^+(-\alpha)\right) = -\frac{\alpha}{1-\alpha^2} + C(\alpha) \cosh \gamma(h - a),}$$

and furthermore

$$\Phi_{s,y}(\alpha, a) = \int_0^\infty \delta(x) \sin \alpha x \, \mathrm{d}x = \boxed{\frac{1}{2\mathrm{i}}\left(D^+(\alpha) - D^+(-\alpha)\right) = -\gamma C(\alpha) \sinh \gamma(h-a).}$$

Eliminate the unknowns $A(\alpha)$ and $C(\alpha)$:

$$\underbrace{D^+(\alpha)}_{\oplus} + \underbrace{D^+(-\alpha)}_{\ominus} = \frac{\gamma \sinh \gamma a}{\cosh \gamma a}(B^+(\alpha) + B^+(-\alpha)),$$

$$\underbrace{B^+(\alpha)}_{\oplus} - \underbrace{B^+(-\alpha)}_{\ominus} = \frac{2\alpha \mathrm{i}}{\alpha^2 - 1} - \frac{\cosh \gamma(h-a)}{\gamma \sinh \gamma(h-a)}(D^+(\alpha) - D^+(-\alpha)).$$

This can be written in terms of matrices:

$$\underbrace{\begin{pmatrix} 1 & -\frac{\cosh \gamma a}{\gamma \sinh \gamma a} \\ 1 & \frac{\cosh \gamma(h-a)}{\sinh \gamma(h-a)} \end{pmatrix}}_{\mathsf{M}} \begin{pmatrix} B^+(\alpha) \\ D^+(\alpha) \end{pmatrix} = \mathsf{JM}(\alpha) \begin{pmatrix} B^+(-\alpha) \\ D^+(-\alpha) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{2\mathrm{i}\alpha}{\alpha^2-1} \end{pmatrix},$$

where

$$\mathsf{J} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence we have the matrix equation

$$\underbrace{\mathsf{M}^{-1}\mathsf{JM}}_{\mathsf{K}}\, \mathsf{D}^+ = \mathsf{D}^- + \text{forcing},$$

where K is

$$\mathsf{K} = \frac{1}{\Delta} \begin{pmatrix} \gamma \sinh \gamma(h-2a) & 2\cosh \gamma(h-a) \cosh \gamma a \\ 2\gamma^2 \sinh \gamma(h-a) \sinh \gamma a & -\gamma \sinh \gamma(h-2a) \end{pmatrix}, \qquad \Delta = \gamma \sinh \gamma h.$$

How do we write $\mathsf{K}(\alpha) = \mathsf{K}^-(\alpha)\mathsf{K}^+(\alpha)$? If $a = h/2$, then this simplifies down to

$$\mathsf{K} = \frac{1}{\Delta} \begin{pmatrix} 0 & 2\cosh^2(\gamma h/2) \\ 2\gamma^2 \sinh^2(\gamma a/2) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\gamma}\coth(\gamma h/2) \\ \gamma \tanh(\gamma h/2) & 0 \end{pmatrix}.$$

Hence it decomposes into two scalar uncoupled equations. Note that

$$\mathsf{K}^2 = \mathsf{M}^{-1}\mathsf{JMM}^{-1}\mathsf{J} = \mathsf{I},$$

and so $K(\alpha)$ is its own inverse! The exact factorisation of this matrix is a long-standing problem!

Commutative Factorisation

Why are matrix Wiener–Hopf systems tricky? Well, we know how to perform a sum split for scalar terms:

$$a(\alpha) = a^+(\alpha) + a^-(\alpha)$$

and this goes through element-wise for a matrix $\mathsf{A}(\alpha)$. For product factorisation, we have

$$a(\alpha) = a^+(\alpha)a^-(\alpha).$$

Take logarithms:

$$\log(a) = \log(a^+) + \log(a^-)$$

and then perform a sum split to get

$$\log(a) = (\log a)_+ + (\log a)_-,$$

and then we obtain

$$a^\pm = \exp((\log a)_\pm).$$

Note: We need $a(\alpha)$ to be free of zeros in the strip as well as singularities. Then $a^+(\alpha)$ and $1/a^+(\alpha)$ are analytic in D^+.

If A is a matrix, then how do we take the log of A? We may define it by the following operation:

$$\log(\mathsf{I} + (\mathsf{A} - \mathsf{I})) = (\mathsf{A} - \mathsf{I}) - \frac{1}{2}(\mathsf{A} - \mathsf{I})^2 + \frac{1}{3}(\mathsf{A} - \mathsf{I})^3 + \cdots$$

then do the sum split on the RHS in the usual way to get

$$\log \mathsf{A} = (\log \mathsf{A})_+ + (\log \mathsf{A})_-.$$

But the final step fails because in general,

$$\exp\left[(\log \mathsf{A})_+ + (\log \mathsf{A})_-\right] \neq \exp\left((\log \mathsf{A})_+\right) \exp\left((\log \mathsf{A})_-\right).$$

There has been considerable effort spent by many researchers looking at special/general cases trying to make progress.

However, the procedure goes through for commutative factorisation. Consider the following example of diffraction by a semi-infinite strip with different conditions on each face (*e.g.* Rawlins, Hurd, Daniele) [6] as in Figure 12:

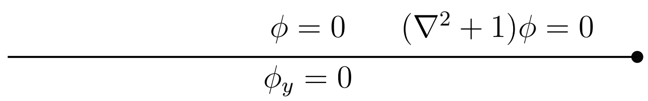


Figure 12: Acoustic diffraction by a semi-infinite strip with different conditions on each face.

As before $\gamma = (\alpha^2 - 1)^{1/2}$. Then the matrix kernel K is found to be

$$\mathsf{K}(\alpha) = \begin{pmatrix} 1 & \gamma(\alpha) \\ -\frac{1}{\gamma(\alpha)} & 1 \end{pmatrix} = \mathsf{I} + \frac{1}{\gamma} \underbrace{\begin{pmatrix} 0 & \gamma^2 \\ -1 & 0 \end{pmatrix}}_{\mathsf{J}(\alpha)}.$$

J is an entire matrix with the property

$$\mathsf{J}^2 = -\gamma^2 \mathsf{I} = -\Delta^2 \mathsf{I}.$$

In general we can take kernels of the form

$$\mathsf{K}(\alpha) = a(\alpha)\mathsf{I} + b(\alpha)\mathsf{J}(\alpha),$$

where $a(\alpha)$ and $b(\alpha)$ are arbitrary, J is an entire matrix and $\mathsf{J}^2 = -\Delta^2\mathsf{I}$.

Then pose

$$\begin{aligned} \mathsf{K} &= \mathsf{K}^+\mathsf{K}^- = \mathsf{K}^-\mathsf{K}^+, \\ &= a\mathsf{I} + b\mathsf{J} = (a_+\mathsf{I} + b_+\mathsf{J})(a_-\mathsf{I} + b_-\mathsf{J}), \end{aligned}$$

where

$$\mathsf{K}^\pm = a^\pm \left(\cos(\Delta b^\pm)\mathsf{I} + \frac{1}{\Delta}\sin(\Delta b^\pm)\mathsf{J}(\alpha) \right),$$

which is referred to as the Khrapkov–Daniele form. Multiplying terms gives

$$a^+a^- \cos[\Delta(b^+ + b^-)] = a, \qquad \frac{a^+a^-}{\Delta}\sin[\Delta(b^+ + b^-)] = b.$$

The split functions may now be uncoupled to satisfy

$$(a^+a^-)^2 = a^2 + \Delta^2 b^2, \qquad b^+ + b^- = \frac{1}{\Delta}\arctan\left(\frac{\Delta b}{a}\right).$$

Here for the Rawlins kernel, we have $a = 1$, $b = 1/\gamma$, $\Delta = \gamma$, and $\Delta^2 = \alpha^2 - 1$. Hence

$$a^\pm = 2^{1/4}, \qquad b^\pm = \frac{1}{2\gamma(\alpha)}\arctan\left(\frac{1-\alpha}{1+\alpha}\right)^{\pm 1/2}.$$

So we have

$$\mathsf{K} = \begin{pmatrix} 1 & \gamma \\ -1/\gamma & 1 \end{pmatrix} = \mathsf{K}^+\mathsf{K}^-,$$

with

$$\mathsf{K}^\pm(\alpha) = 2^{1/4}\left\{ \cos\left[\frac{1}{2}\arctan\left(\frac{1-\alpha}{1+\alpha}\right)^{\pm 1/2}\right]\mathsf{I} + \frac{1}{\gamma}\sin\left[\frac{1}{2}\arctan\left(\frac{1-\alpha}{1+\alpha}\right)^{\pm 1/2}\right]\begin{pmatrix} 0 & \gamma^2 \\ -1 & 0 \end{pmatrix} \right\}.$$

These factors have algebraic behaviour as $|\alpha| \to \infty$ in $D^\pm$ — this is essential! The Khrapkov method fails if Δ^2 grows too quickly. We now take another diversion to look at an alternative method for factorisation when the singularities are simple.

Pole Removal Method for Noncommutative Factorisation

Take the boundary value problem indicated in Figure 13.

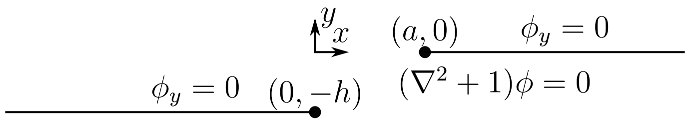


Figure 13: Acoustic diffraction by two staggered semi-infinite strips.

Here the kernel K is

$$\mathsf{K}(\alpha) = \begin{pmatrix} 2/\gamma & \mathrm{e}^{\mathrm{i}\alpha a - h\gamma} \\ \mathrm{e}^{-\mathrm{i}\alpha a - h\gamma} & -\frac{\gamma}{2}(1 - \mathrm{e}^{-2h\gamma}) \end{pmatrix},$$

where $-\frac{\gamma}{2}(1 - \mathrm{e}^{-2h\gamma})$ is a scalar $L = L^+L^-$, say. We want to find K^+ and K^- such that

$$\mathsf{K}\mathsf{K}^+(\alpha) = \mathsf{K}^-,$$

so we do this one column at a time:

$$\mathsf{K}\begin{pmatrix} p^+ \\ q^+ \end{pmatrix} = \begin{pmatrix} p^- \\ q^- \end{pmatrix}.$$

The bottom row gives

$$\mathrm{e}^{-\mathrm{i}\alpha a - h\gamma} p^+ + L^+L^- q^+ = q^-,$$

or

$$-\frac{2L^+\mathrm{e}^{-\mathrm{i}\alpha a - h\gamma}}{\gamma(1 - \mathrm{e}^{-2h\gamma})} p^+ + L^+ q^+ = \frac{q^-}{L^-}.$$

This gives

$$-\frac{L^+\mathrm{e}^{-\mathrm{i}\alpha a}}{\gamma \sinh(\gamma h)} p^+ + L^+ q^+ = \frac{q^-}{L^-}.$$

If $a < 0$ then the LHS is analytic in the UHP except for the poles at $\gamma \sinh \gamma h = 0$ lying in the upper region, or

$$\alpha = \lambda_n = \left(1 - \left(\frac{n\pi}{h}\right)^2\right)^{1/2} = \mathrm{i}\left(\left(\frac{n\pi}{h}\right)^2 - 1\right)^{1/2}, \quad n = 0, 1, 2, 3, \ldots$$

We can remove these offending poles as follows:

$$\underbrace{-\frac{L^+(\alpha)\mathrm{e}^{-\mathrm{i}\alpha a} p^+(\alpha)}{\gamma(\alpha)\sinh(\gamma(\alpha)h)} + \sum_{n=0}^{\infty} \frac{b_n}{\alpha - \lambda_n} + L^+(\alpha) q^+(\alpha)}_{\oplus} = \underbrace{\sum_{n=0}^{\infty} \frac{b_n}{\alpha - \lambda_n} + \frac{q^-(\alpha)}{L^-(\alpha)}}_{\ominus} \equiv E(\alpha),$$

where

$$\boxed{b_n = \frac{\varepsilon_n \mathrm{e}^{-\mathrm{i}a\lambda_n} L^+(\lambda_n) p^+(\lambda_n)}{2h\lambda_n(-1)^n}, \qquad \varepsilon_n = \begin{cases} 2, & n > 0, \\ 1, & n = 0. \end{cases}}$$

For construction of the kernel factors, we can set $E(\alpha) = B$, a constant say. So

$$\boxed{q^-(\alpha) = -L^-(\alpha) \sum_{n=0}^{\infty} \frac{b_n}{\alpha - \lambda_n} + L^-(\alpha) B,}$$

and similarly

$$q^+(\alpha) - \frac{\mathrm{e}^{-\mathrm{i}\alpha a} p^+(\alpha)}{\gamma(\alpha)\sinh(\gamma(\alpha)h)} + \frac{1}{L^+(\alpha)} \sum_{n=0}^{\infty} \frac{b_n}{\alpha - \lambda_n} = \frac{B}{L^+(\alpha)}.$$

Now repeat for the top row, but we invert $\mathsf{K}(\alpha)$ first and divide by $L^+(\alpha)$:

$$\frac{p^+(\alpha)}{L^+(\alpha)} = -\frac{L^-(\alpha)\mathrm{e}^{\mathrm{i}\alpha a} q^-(\alpha)}{\gamma(\alpha)\sinh(\gamma h)} - L^-(\alpha) p^-(\alpha),$$

and

$$\frac{p^+(\alpha)}{L^+(\alpha)} - \sum_{n=0}^{\infty} \frac{d_n}{\alpha + \lambda_n} = -\frac{L^-(\alpha)\mathrm{e}^{\mathrm{i}\alpha a} q^-(\alpha)}{\gamma \sinh(\gamma h)} - \sum_{n=0}^{\infty} \frac{d_n}{\alpha + \lambda_n} - L^-(\alpha) p^-(\alpha) = D,$$

where D is a constant and we have

$$\boxed{d_n = \frac{\varepsilon_n \mathrm{e}^{-\mathrm{i}a\lambda_n} L^+(\lambda_n) q^-(-\lambda_n)}{2h\lambda_n(-1)^n},}$$

where $L^-(-\lambda_n) = L^+(\lambda_n)$ by symmetry. So

$$\boxed{p^+(\alpha) = L^+(\alpha) \sum_{n=0}^{\infty} \frac{d_n}{\alpha + \lambda_n} + L^+(\alpha) D,}$$

and

$$p^-(\alpha) + \frac{\mathrm{e}^{\mathrm{i}\alpha a} q^-(\alpha)}{\gamma(\alpha)\sinh(\gamma(\alpha)h)} + \frac{1}{L^+(\alpha)} \sum_{n=0}^{\infty} \frac{d_n}{\alpha + \lambda_n} = -\frac{D}{L^-(\alpha)}.$$

We need to find b_n and d_n, which we get from the expressions from $q^-(\alpha)$ and $p^+(\alpha)$, which yields a pair of coupled linear systems of equations:

$$\begin{aligned} b_n &= \frac{\varepsilon_n \mathrm{e}^{-\mathrm{i}a\lambda_n} [L^+(\lambda_n)]^2}{2h\lambda_n(-1)^n} \left(D + \sum_{m=0}^{\infty} \frac{d_m}{\lambda_m + \lambda_n} \right), \\ d_n &= \frac{\varepsilon_n \mathrm{e}^{-\mathrm{i}a\lambda_n} [L^+(\lambda_n)]^2}{2h\lambda_n(-1)^n} \left(B + \sum_{m=0}^{\infty} \frac{b_m}{\lambda_m + \lambda_n} \right), \end{aligned}$$

where $n = 0, 1, 2, \ldots$

This is a L^2-system which may be solved by truncation for $a < 0$, which can be shown to be very rapid. For the first column, we take $B = D = 1$, say and for the second column we take $B = -1$, $D = 1$. This yields a noncommutative matrix product decomposition. But what happens if the matrix does not just contain simple poles?

Example of Construction of an Approximate Noncommutative Matrix Kernel Decomposition

We revisit the previous problems of diffraction by a semi-infinite screen with different boundary conditions on each face. This is now positioned at the interface between two similar media as shown in Figure 14.

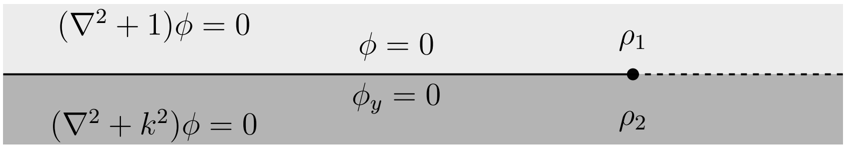


Figure 14: Acoustic scattering with two media of differing densities.

This is a simple example of a Wiener–Hopf-type BVP that leads to a matrix system of equations. We focus on the matrix decomposition, not the actual physical problem itself.

In this case, we find

$$\mathsf{K}(\alpha) = \mathsf{I} + \underbrace{\begin{pmatrix} 0 & \mu\gamma(\alpha) \\ -\frac{\mu}{\gamma(\alpha)} & 0 \end{pmatrix}}_{\mathsf{J}},$$

with the following definitions

$$\mu = \sqrt{\frac{\rho_2}{\rho_1}}, \quad \gamma(\alpha) = (\alpha^2 - 1)^{1/2}, \quad \delta(\alpha) = (\alpha^2 - k^2)^{1/2}, \quad k > 1,$$

where μ^2 is the ratio of the densities in the two regions, and k is the scaled wavenumber in region 2. Note that

$$\mathsf{J}^2 = \begin{pmatrix} -\mu^2\gamma/\delta & 0 \\ 0 & -\mu^2\gamma/\delta \end{pmatrix}.$$

We may rearrange K into the following form so that the factorisation reduces to Khrapkov form as $k \to 1$.

$$\mathsf{K} = \mathsf{I} + \frac{\mu}{\sqrt{\gamma\delta}} \begin{pmatrix} 0 & \gamma^2\sqrt{\delta/\gamma} \\ -\sqrt{\gamma/\delta} & 0 \end{pmatrix} = \mathsf{I} + \frac{\mu f(\alpha)}{\gamma(\alpha)} \underbrace{\begin{pmatrix} 0 & \gamma^2/f(\alpha) \\ -f(\alpha) & 0 \end{pmatrix}}_{\mathsf{J}(\alpha)}.$$

Here we have

$$f(\alpha) = \sqrt{\frac{\gamma}{\delta}} = \left(\frac{\alpha^2 - 1}{\alpha^2 - k^2}\right)^{1/4}.$$

Note that $f(0) = 1/\sqrt{k}$ and $f(\infty) = 1$, and $f(\alpha)$ posses finite branch cuts between $1 < |\alpha| < k$ on the real line (see Figure 15). Further, $\mathsf{J}^2 = -\gamma^2 \mathsf{I}$.

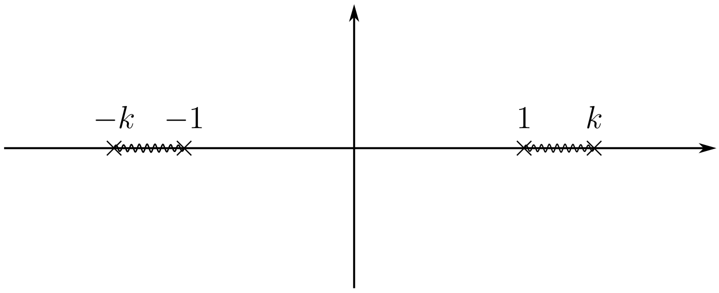


Figure 15: Poles and branch cuts in the complex α-plane.

We may now perform a Khrapkov commutative factorisation for K, which yields nice factors except that J has the finite branch cuts in both D^+ and D^-. So $\mathsf{K}(\alpha) = \mathsf{Q}^-(\alpha)\mathsf{Q}^+(\alpha)$, with

$$\mathsf{Q}^\pm = r^\pm \left(\cos(\gamma s^\pm)\mathsf{I} + \frac{1}{\gamma}\sin(\gamma s^\pm)\mathsf{J}(\alpha)\right),$$

and

$$(r^+)^2(r^-)^2 = 1 + \mu^2 f^2(\alpha) = \frac{\mu^2\gamma + \delta}{\delta}.$$

In addition,

$$s^+ + s^- = \frac{1}{\gamma}\arctan(\mu f(\alpha))$$

Hence,

$$r^\pm(\alpha) = (1+\mu^2)^{1/4} \exp\left\{\pm\frac{1}{4\pi \mathrm{i}} \int \frac{\log\left[\frac{1+\mu^2 f^2(\zeta)}{1+\mu^2}\right]}{\zeta - a}\,\mathrm{d}\zeta\right\}$$

and

$$s^\pm(\alpha) = \pm\frac{1}{2\pi\mathrm{i}} \int \frac{\arctan(\mu f(\zeta))}{\gamma(\zeta)(\zeta - \alpha)}\,\mathrm{d}\zeta,$$

where as before, α lies above (below) the contour for r^+, s^+ (r^-, s^-). We now, somehow, need to remove the finite cut between 1 and k in $\mathsf{Q}^+(\alpha)$, and between -1 and $-k$ in $\mathsf{Q}^-(\alpha)$. This is difficult, but we do know how to remove offending poles as shown in an earlier example. So we approximate J by J_N, where the finite cuts in J are replaced by simple poles and zeros in J_N. We now replace K by K_N: $\mathsf{K} \approx \mathsf{K}_N = \mathsf{Q}_N^-(\alpha)\mathsf{Q}_N^+(\alpha)$. Thus,

$$\mathsf{Q}_N^\pm = r^\pm \left(\cos(\gamma s^\pm)\mathsf{I} + \frac{1}{\gamma}\sin(\gamma s^\pm)\mathsf{J}_N(\alpha)\right), \qquad \mathsf{J}_N = \begin{pmatrix} 0 & \gamma^2/f_N(\alpha) \\ -f_N(\alpha) & 0 \end{pmatrix}.$$

Note: Here $r^{\pm}$ and $s^{\pm}$ are not being approximated.

One way to introduce f_N is by use of Padé approximants:

$$f_N(\alpha) = \frac{P_N(\alpha)}{Q_N(\alpha)} = [N/N].$$

This is the Padé approximant of $f(\alpha) = (\gamma(\alpha)/\delta(\alpha))^{1/2}$. Here

$$\begin{aligned} P_N &= a_0 + a_1\alpha^2 + a_2\alpha^4 + \cdots + a_n\alpha^{2N}, \\ Q_N &= 1 + b_1\alpha^2 + \cdots \end{aligned}$$

and for given N, a_n and b_n are determined uniquely from the Taylor series expansion of $f(\alpha)$ at the origin. In fact, $f_N(\alpha)$ has N simple poles and N simple zeros interlaced between $-k < \alpha < -1$ and $1 < \alpha < k$, replacing the finite cuts. So $\mathsf{Q}_N^{\pm}(\alpha)$ are analytic in $D^{\pm}$ except for the simple zeros of $P_N(\alpha)$ and $Q_N(\alpha)$ in these regions. we can remove these by inserting a suitable matrix, and its inverse, in the factorisation, *i.e.*

$$\mathsf{K}_N = \underbrace{\left[\mathsf{Q}_N^{-}\mathsf{M}(\alpha)\right]}_{\mathsf{K}_N^{-}}\underbrace{\left[(\mathsf{M}^{-1})\mathsf{Q}_N^{+}\right]}_{\mathsf{K}_N^{+}}.$$

Finding $\mathsf{M}(\alpha)$ will lead to the exact noncommutative factorisation elements of $\mathsf{K}_N(\alpha)$, namely $\mathsf{K}_N^{\pm}(\alpha)$. Obviously $\mathsf{M}(\alpha)$ must have meromorphic elements, poles corresponding to zeros of P_N and Q_N. We pose an ansatz for M as

$$\mathsf{M} = \begin{pmatrix} 1 + \sum_{n=1}^{N} \frac{B_n}{\alpha^2 - p_n^2} & \alpha\sum_{n=1}^{N} \frac{\hat{B}_n}{\alpha^2 - p_n^2} \\ -\alpha\sum_{n=1}^{N} \frac{A_n}{\alpha^2 - q_n^2} & 1 - \sum_{n=1}^{N} \frac{\hat{A}_n}{\alpha^2 - q_n^2} \end{pmatrix},$$

Here p_n and q_n are the locations of the positive real zeros of $P_N(\alpha)$ and $Q_N(\alpha)$, ordered as $1 < p_1 < p_2 < \cdots < p_N < k$, and $1 < q_1 < q_2 < \cdots < q_N < k$. A_n, B_n, $\hat{A}_n$ and $\hat{B}_n$ are all as yet unknown. It is necessary to write f_N and $1/f_N$ in partial fraction form as

$$f_N = C + \sum_{n=1}^{N} \frac{\alpha_n}{\alpha^2 - q_n^2}, \quad \frac{1}{f_N} = \frac{1}{C} + \sum_{n=1}^{N} \frac{\beta_n}{\alpha^2 - p_n^2}.$$

We find

$$C = \frac{a_N}{b_N} \approx 1, \qquad \alpha_n = \frac{2q_n P_N(q_n)}{Q_N'(q_n)}, \qquad \beta_n = \frac{2p_n Q_N(p_n)}{P_N'(p_n)}.$$

Now, $\mathsf{Q}_N^{-}\mathsf{M}(\alpha)$ can be examined and pole removal enforced. Take the $(1,1)$ element:

$$r^{-}(\alpha)\cos[\gamma(\alpha)s^{-}(\alpha)]\left(1 + \sum_{n=1}^{N} \frac{B}{\alpha^2 - p_n^2}\right) + r^{-}(\alpha)\sin[\gamma(s)s^{-}(\alpha)]\frac{\gamma(\alpha)}{f_N(\alpha)}\left(-\alpha\sum_{n=1}^{N} \frac{A_n}{\alpha^2 - q_n^2}\right).$$

This does not have poles at $\alpha = \pm q_n$, as they are cancelled by the zeros of $1/f_N(\alpha)$. But it will have poles at $-p_m$, with $p = 1, \ldots, N$ in D^{-} unless the residues are zero. So set

$$r^{-}(-p_m)\left(\cos[\gamma(-p_m)s^{-}(-p_m)]\frac{B_m}{(-2p_m)} + \frac{\beta_m}{(-2p_m)}\gamma(-p_m)\sin[\gamma(-p_m)s^{-}(-p_m)]p_m\sum_{n=1}^{N} \frac{A_n}{p_m^2 - q_n^2}\right) = 0.$$

But $\gamma(-p_m) = \gamma(p_m)$ and $s^-(-p_m) = s^+(p_m)$, so

$$\boxed{B_m = p_m \beta_m \gamma(p_m) \tan[\gamma(p_m) s^+(p_m)] \sum_{n=1}^{N} \frac{A_n}{q_n^2 - p_m^2}, \qquad m = 1, \ldots, N.}$$

The $(2,1)$ element of $\mathsf{K}_N^-(\alpha)$ gives, for regularity in D^-:

$$\boxed{A_m = \frac{\alpha_m}{q_m \gamma(q_m)} \tan[\gamma(q_m) s^+(q_m)] \left(1 + \sum_{n=1}^{N} \frac{B_n}{q_m^2 - p_n^2}\right), \qquad m = 1, \ldots, N.}$$

These two equations are easily solved to obtain A_m and B_m. We can repeat for $\hat{A}_m$ and $\hat{B}_m$ which then gives $\mathsf{M}(\alpha)$ explicitly. Finally

$$\det(\mathsf{M}(\alpha)) = \left(1 + \sum_{n=1}^{N} \frac{B_n}{\alpha^2 - p_n^2}\right) \left(1 - \sum_{n=1}^{N} \frac{\hat{A}_n}{\alpha^2 - q_n^2}\right) + \alpha^2 \left(\sum_{n=1}^{N} \frac{\hat{B}_n}{\alpha^2 - p_n^2}\right) \left(\sum_{n=1}^{N} \frac{A_n}{\alpha^2 - q_n^2}\right),$$

and from the constraints on A_n, B_n, $\hat{A}_n$ and $\hat{B}_n$, it has no poles at either $\pm p_n$ or $\pm q_n$ for $n = 1, \ldots, N$. Thus by Liouville's theorem, $\det \mathsf{M} = 1$. In conclusion, we have exactly constructed $\mathsf{K}_N^\pm(\alpha)$ and as $N \to \infty$ we expect $\mathsf{K}_N^\pm \to \mathsf{K}^\pm$. It appears to converge well, *e.g.* $N \approx 5$ seems very accurate.

References

[1] T. Carleman. Sur la résolution de certaines équations intégrales. *Arkiv för Matematik, Astronomi och Fysik*, 16(26), 1922.

[2] T. Carleman. Über die abelsche integralgleichung mit konstanten integrationsgrenzen. *Mathematische Zeitschrift*, 15(1):111–120, Dec 1922.

[3] E. Hopf. *Mathematical problems of radiative equilibrium*. Cambridge tracts in mathematics ; 31. Cambridge University Press, Cambridge, 1934.

[4] D. S. Jones. A simplifying technique in the solution of a class of diffraction problems. *The Quarterly Journal of Mathematics*, 3(1):189–196, 01 1952.

[5] E. A. Milne. The reflection effect in eclipsing binaries. *Monthly Notices of the Royal Astronomical Society*, 87(1):43–55, 1926.

[6] A. D. Rawlins. The solution of a mixed boundary value problem in the theory of diffraction by a semi-infinite plane. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 346(1647):469–484, 1975.

[7] O. Toeplitz. Zur theorie der quadratischen und bilinearen formen von unendlichvielen veränderlichen. *Mathematische Annalen*, 70(3):351–376, Sep 1911.

[8] N. Wiener and E. Hopf. Über eine klasse singulärer integralgleichungen. *Sem–Ber Preuss Akad Wiss*, 31:696–706, 1931.