
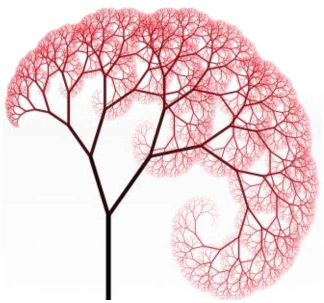
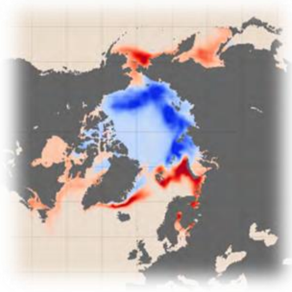
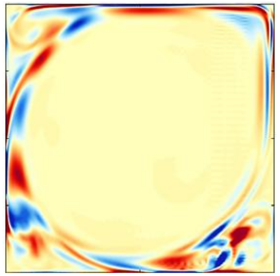

What can we learn about a dynamical system from data?

Matthew Colbrook

University of Cambridge

4/2/2025

"To classify is to bring order into chaos." - **George Pólya**

For papers and talk slides/videos, visit:
http://www.damtp.cam.ac.uk/user/mjc249/home.html

Data-driven dynamical systems

- Compact metric space $(\mathcal{X}, d)$ – the state space
- $x \in \mathcal{X}$ – the state

  cts $F: \mathcal{X} \to \mathcal{X}$ – the dynamics: $x_{n+1} = F(x_n)$

Dynamics (geometry)
19th century

- Borel measure ω on $\mathcal{X}$
- Function space $L^2 = L^2(\mathcal{X}, \omega)$ (elements g called "observables")
- Koopman operator $\mathcal{K}_F: L^2 \to L^2$; $[\mathcal{K}_F g](x) = g(F(x))$
- ***Available*** snapshot data: $\left\{\left(x^{(m)}, y^{(m)} = F\left(x^{(m)}\right)\right): m = 1, \ldots, M\right\}$

NB: Pointwise definition of $\mathcal{K}_F$ needs $F\#\omega \ll \omega$ – this will hold throughout.
NB: $\mathcal{K}_F$ bounded equivalent to $\mathrm{d}F\#\omega/\mathrm{d}\omega \in L^\infty$ – this will hold throughout (can be dropped).

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Analysis
20th century

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Analysis
20th century

- ***Available*** snapshot data: $\left\{\left(x^{(m)}, y^{(m)} = F\left(x^{(m)}\right)\right): m = 1, \dots, M\right\}$

Data
21st century

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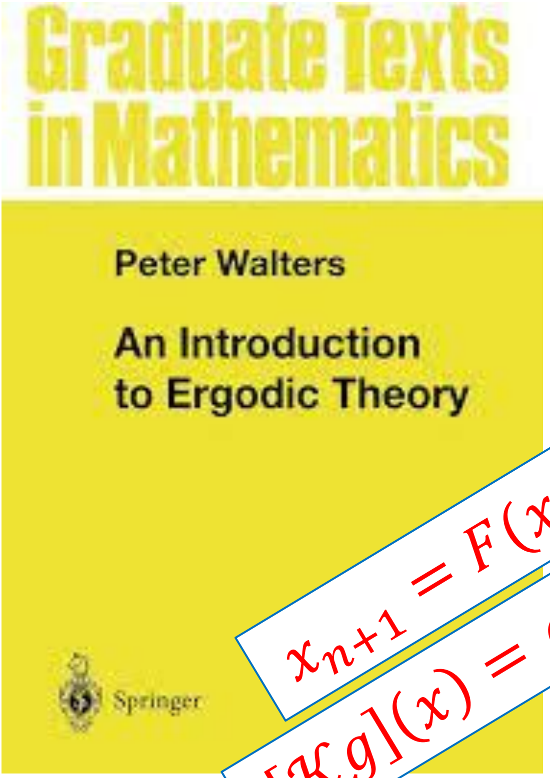
Why you should care about Koopman

Fundamental in ergodic theory



$x_{n+1} = F(x_n)$

$[\mathcal{K}g](x) = g(F(x))$

E.g., key to ergodic theorems of Birkhoff and von Neumann.

Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

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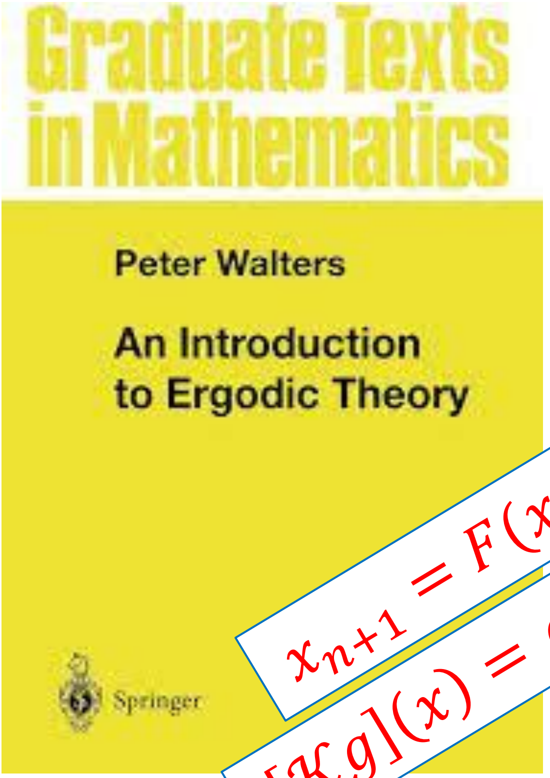

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Can provide a *diagonalization* of a nonlinear system.

eigenfunction of $\mathcal{K}$

continuous spectrum

$$g(x) = \sum_{\text{eigenvalues } \lambda_j} c_{\lambda_j} \varphi_{\lambda_j}(x) + \int_{-\pi}^{\pi} \phi_{\theta,g}(x)\, \mathrm{d}\theta$$

$$g(x_n) = [\mathcal{K}^n g](x_0)$$

$$= \sum_{\text{eigenvalues } \lambda_j} c_{\lambda_j} \boxed{\lambda_j{}^n} \varphi_{\lambda_j}(x_0) + \int_{-\pi}^{\pi} \boxed{e^{in\theta}} \phi_{\theta,g}(x_0)\, \mathrm{d}\theta$$

Spectral properties encode: geometric features, invariant measures, transient behavior, long-time behavior, coherent structures, quasiperiodicity, etc.

Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

Why you should care about Koopman

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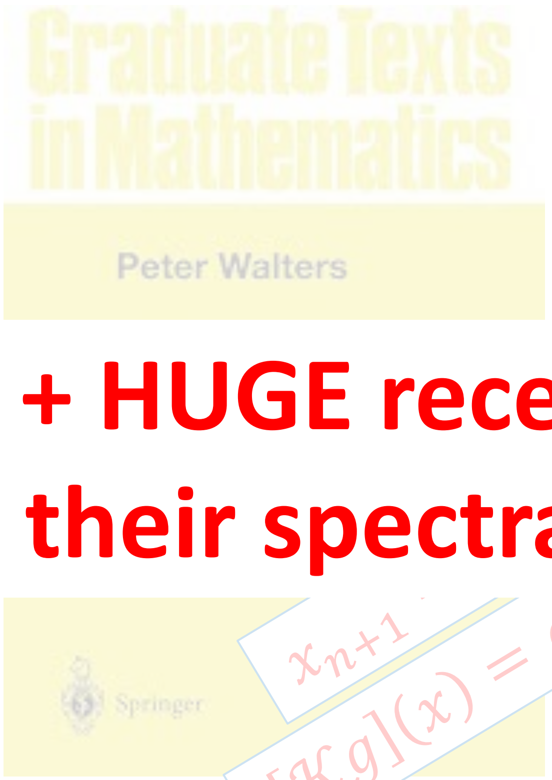


E.g., key to ergodic theorems of Birkhoff and von Neumann.

Can provide a *diagon*

$g(x) = \sum$ eigenvalues

$g](x$

c_{λ_j}

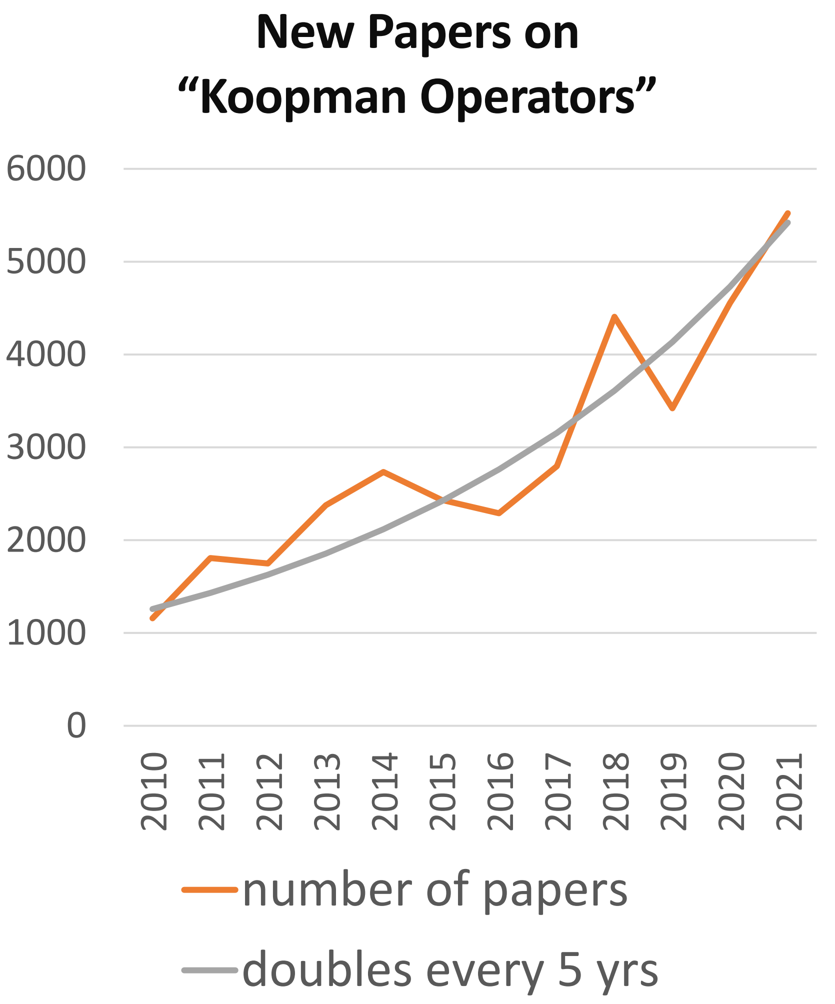
$\sum$ eigenvalues λ_j

x_{n+1}

$[\mathcal{K}g](x) = g($

$d\theta$

+ HUGE recent interest in their spectral properties!



Spectral properties
invariant measures,
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Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

Perils of discretization: Warmup on $\ell^2(\mathbb{Z})$

$$\begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{C}^{N\times N}$$

Two-way infinite

- Spectrum is unit circle.
- Spectrum is stable.
- Continuous spectra.
- Unitary evolution.

- Spectrum is $\{0\}$.
- Spectrum is unstable.
- Discrete spectra.
- Nilpotent evolution.

Lots of Koopman operators are built up from operators like these!

Perils of discretization: Warmup on $\ell^2(\mathbb{Z})$

Issue: *Many practitioners view Koopman as a magic bullet, but standard algorithms typically fail to converge! (Inf-dim spectral problems.)*

Question: *When can we reliably learn Koopman spectral properties from system data, and when is it impossible?*

- Spectrum is stable.
- Continuous spectra.
- Unitary evolution.

- Spectrum is unstable.
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Extended Dynamic Mode Decomposition (EDMD)

Functions $\psi_j : \mathcal{X} \to \mathbb{C},\ j = 1, \dots, N$

$$\{x^{(m)}, y^{(m)} = F(x^{(m)})\}_{m=1}^{M}$$

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quadrature points

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j\left(x^{(m)}\right)} \psi_k\left(x^{(m)}\right) = \Bigg[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}^*}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X} \Bigg]_{jk}$$

quadrature weights

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j\left(x^{(m)}\right)} \underbrace{\psi_k\left(y^{(m)}\right)}_{[\mathcal{K}\psi_k](x^{(m)})} = \Bigg[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}^*}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \psi_1(y^{(1)}) & \cdots & \psi_N(y^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(y^{(M)}) & \cdots & \psi_N(y^{(M)}) \end{pmatrix}}_{\Psi_Y} \Bigg]_{jk}$$

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Galerkin Approximation

$$\mathcal{K} \longrightarrow \mathbb{K} = (\Psi_X{}^* W \Psi_X)^{-1} \Psi_X{}^* W \Psi_Y = (\sqrt{W}\Psi_X)^{\dagger} \sqrt{W}\Psi_Y \in \mathbb{C}^{N \times N}$$

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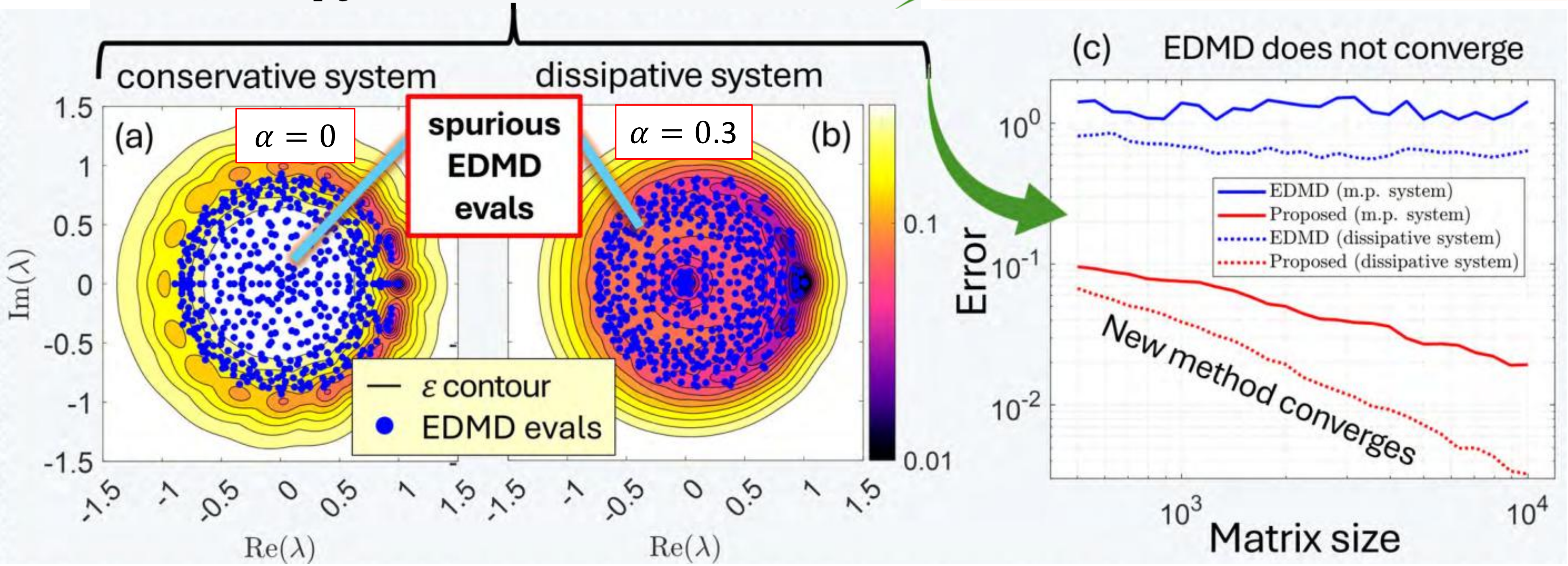
Example: EDMD does NOT converge

- Duffing oscillator: $\dot{x} = y, \dot{y} = -\alpha y + x(1 - x^2)$, sampled $\Delta t = 0.3$.
- Gaussian radial basis functions, Monte Carlo integration ($M = 50000$)

Compute Sp_ε, local adaptive control on $\varepsilon \downarrow 0$

$\mathrm{Sp}_\varepsilon(\mathcal{K}_F) = \{z \in \mathbb{C} : \|(\mathcal{K}_F - zI)^{-1}\|^{-1} \leq \varepsilon\}$



Residual DMD (ResDMD)

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \psi_k\left(x^{(m)}\right) = [\underbrace{\Psi_X{}^* W \Psi_X}_{G}]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \underbrace{\psi_k\left(y^{(m)}\right)}_{[\mathcal{K}\psi_k](x^{(m)})} = [\underbrace{\Psi_X{}^* W \Psi_Y}_{K_1}]_{jk}$$

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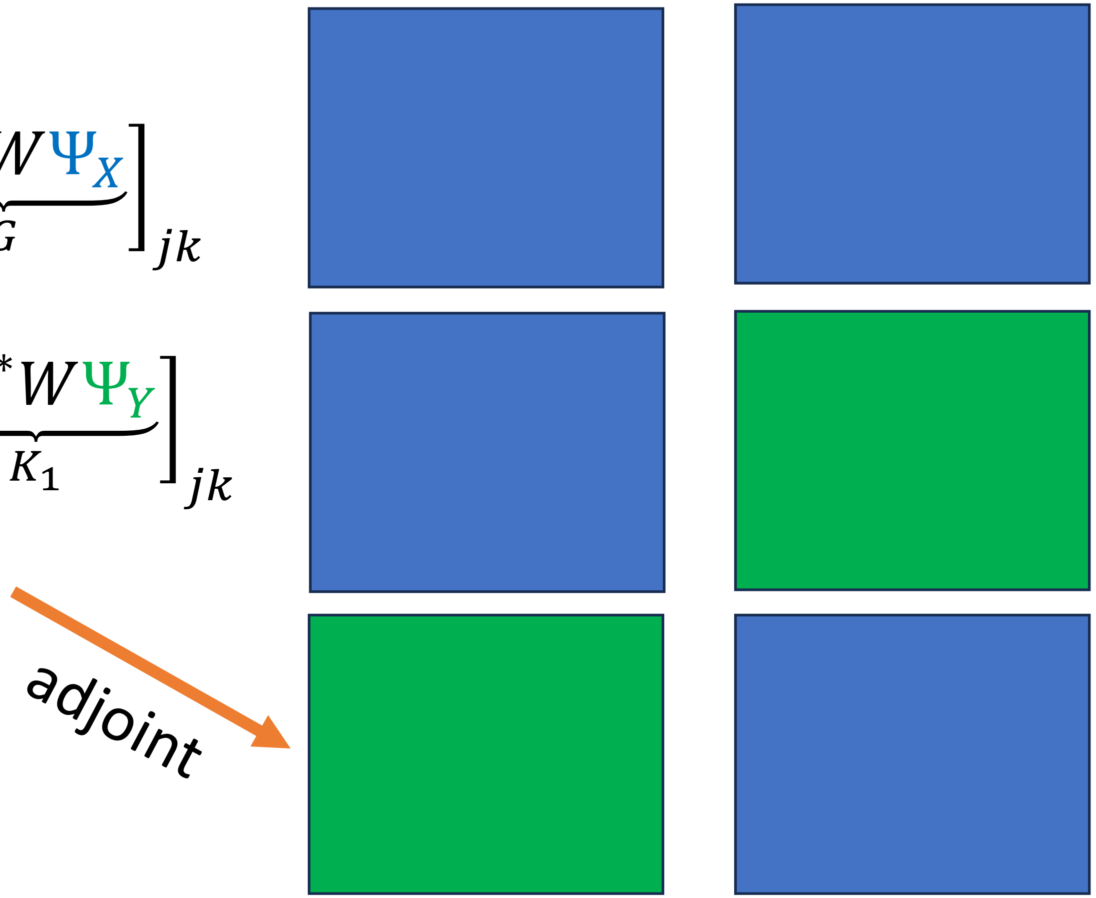

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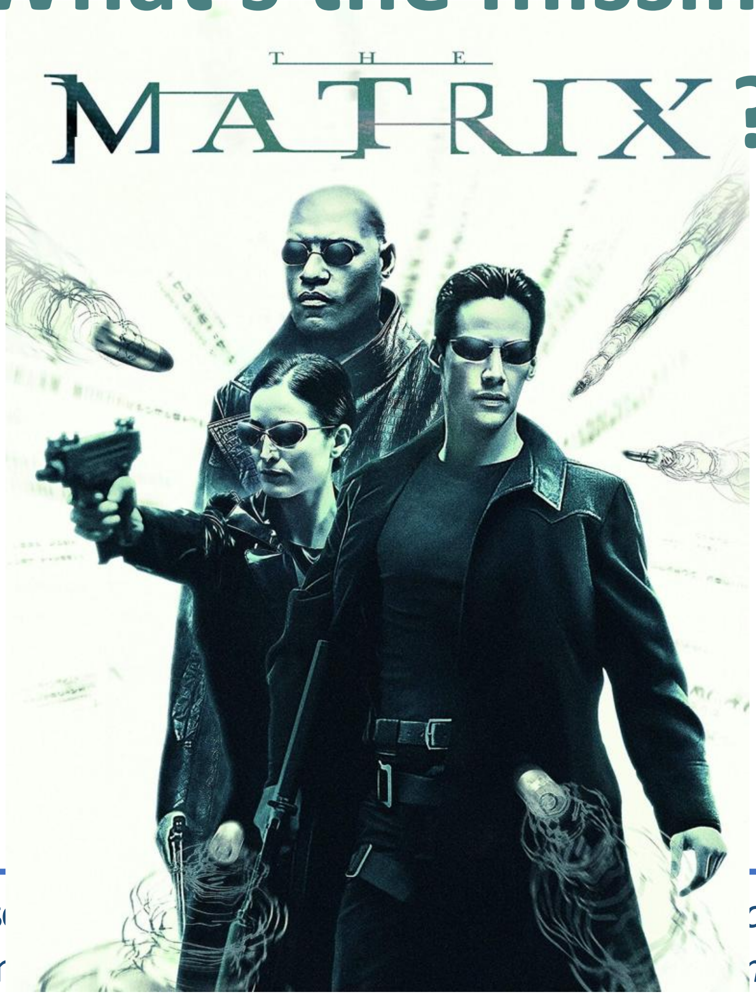
Residual DMD (ResDMD)

What's the missing



?

$$= \Big[\underbrace{\Psi_X{}^{*} W \Psi_X}_{G}\Big]_{jk}$$

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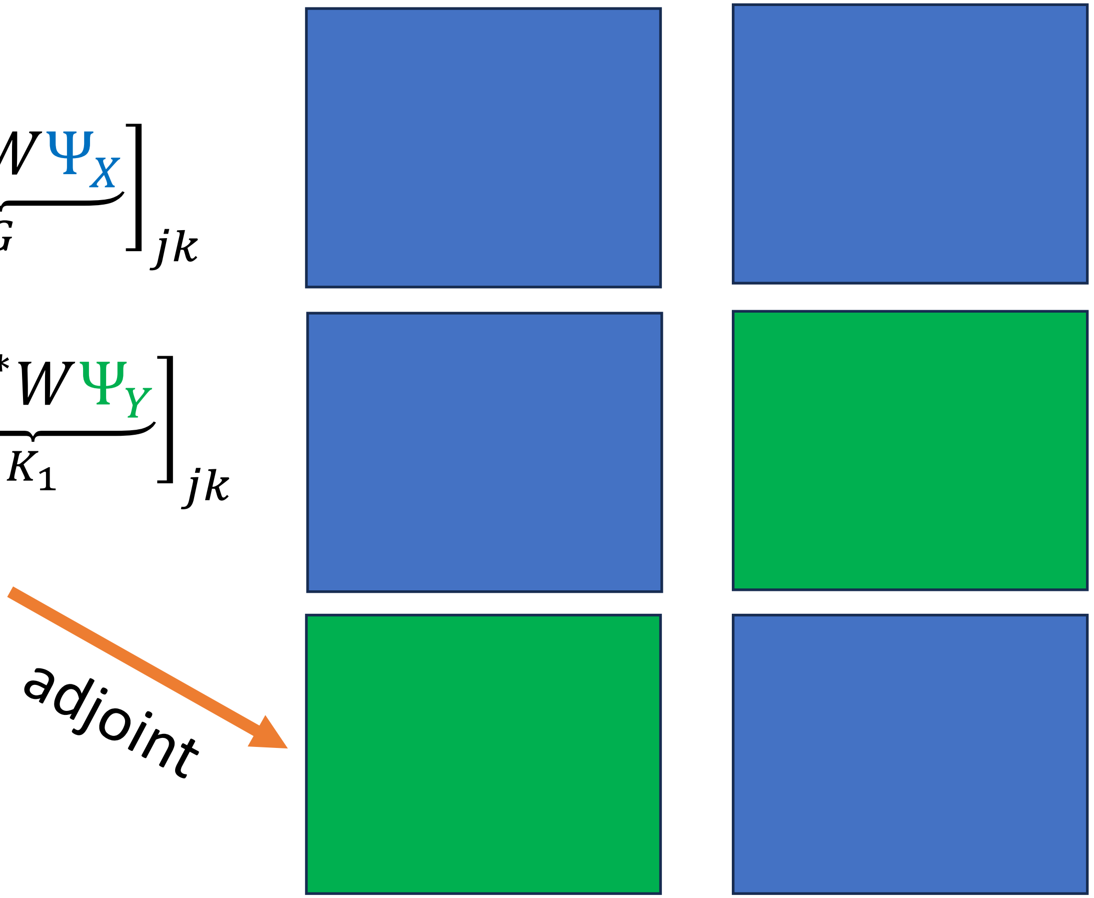
)



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Residuals: $g = \sum_{j=1}^{N} \mathbf{g}_j \psi_j$, $\|\mathcal{K}g - \lambda g\|^2 = \langle \mathcal{K}g - \lambda g, \mathcal{K}g - \lambda g \rangle$

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Bound projection errors!

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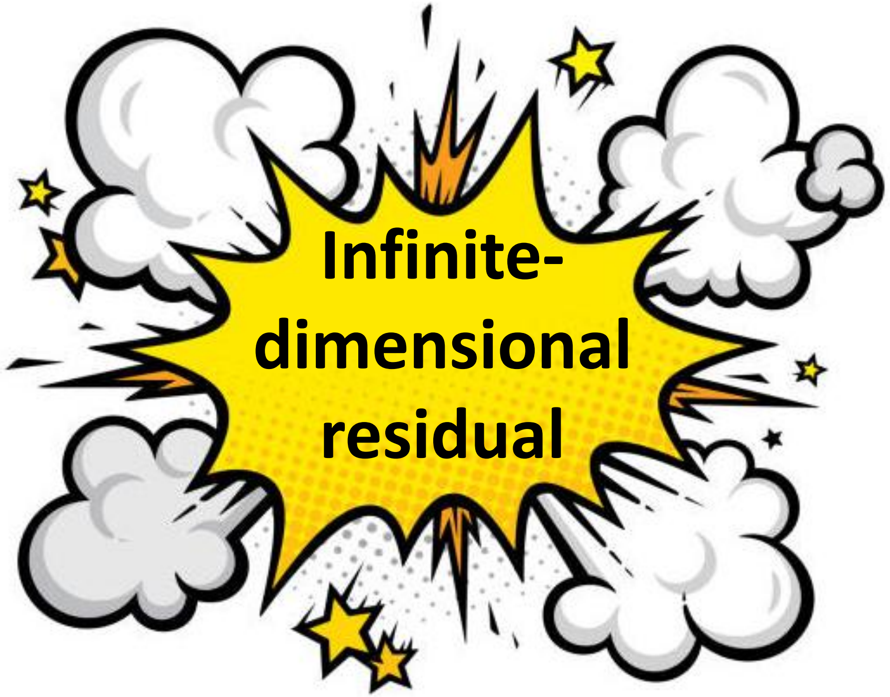

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- Code*:* https://github.com/MColbrook/Residual-Dynamic-Mode-Decomposition

Proof sketch

Theorem A: There **exists** *deterministic* algorithms $\{\Gamma_{n_2,n_1}\}$ using snapshots such that $\lim_{n_2\to\infty}\lim_{n_1\to\infty}\Gamma_{n_2,n_1}(F)=\mathrm{Sp}(\mathcal{K}_F)$ for all measure-preserving systems.

- *Residuals* $\longrightarrow \lim_{N\to\infty}\lim_{M\to\infty}\gamma_{N,M}(z,F)=\|(\mathcal{K}_F-zI)^{-1}\|^{-1}$.

  N = size of basis, M = amount of data (quadrature).
- Measure-preserving $\Rightarrow \|(\mathcal{K}_F-zI)^{-1}\|^{-1}=\mathrm{dist}(z,\mathrm{Sp}(\mathcal{K}_F))$.
- Local N-adaptive minimisation of $\gamma_{N,M}(z,F)$ to approximate $\mathrm{Sp}(\mathcal{K}_F)$.

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Double limit $\lim_{N\to\infty}\lim_{M\to\infty}$ **Can we do better?**

Theorem B (impossibility)

Implies $\mathcal{K}$ is unitary

Class of systems: $\Omega_{\mathbb{D}} = \{F: \overline{\mathbb{D}} \to \overline{\mathbb{D}} \mid F \text{ cts, measure preserving, invertible}\}.$

Data an algorithm can use: $\mathcal{T}_F = \{(x, y_m) \mid x \in \overline{\mathbb{D}}, \|F(x) - y_m\| \leq 2^{-m}\}.$

Theorem B: There **does not exist** any sequence of deterministic algorithms $\{\Gamma_n\}$ using $\mathcal{T}_F$ such that $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F) \ \forall F \in \Omega_{\mathbb{D}}$.

NB: Similarly, no random algorithms converging with probability $> 1/2$.

Double limit is necessary!

Proof idea: Constructing an adversary

F_0: rotation by π, $\mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

Phase transition lemma: Let $X = \{x_1, \dots, x_N\}$,$Y = \{y_1, \dots, y_N\}$ be distinct points in annulus $\mathcal{A} = \{x \in \mathbb{D} | 0 < R < \|x\| < r < 1\}$ with $X \cap Y = \emptyset$. There exists a measure-preserving homeomorphism H such that H acts as the identity on $\mathbb{D} \backslash \mathcal{A}$ and $H(y_j) = F_0(H(x_j)), j = 1, \dots, N$.

Conjugacy of data ($x_j \rightarrow y_j$) with F_0

Idea: Use lemma to trick any algorithm into oscillating between spectra.

- Brown and Halperin. "*On certain area-preserving maps.*" **Annals of Mathematics**, 1935.

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty}\Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

Build an adversarial F…

$\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

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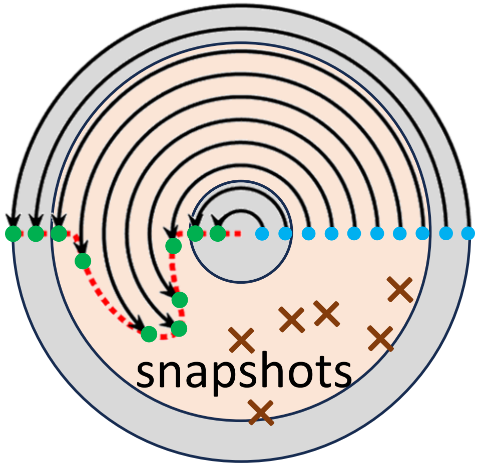


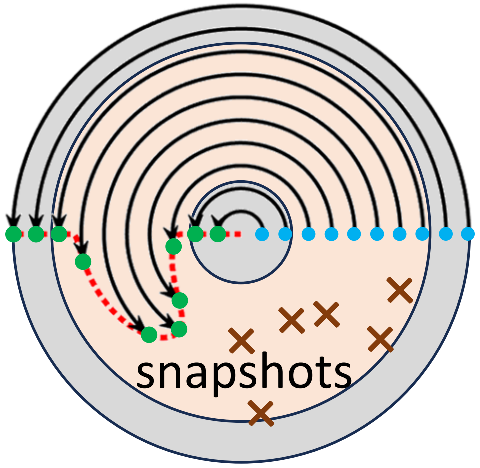
$\widetilde{F_1}(r,\theta) = \big(r, \theta + \pi + \phi(r)\big)$, $\mathrm{supp}(\phi) \subset [1/4, 3/4]$
$\mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) = \mathbb{T}$ (unit circle).

$\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{J}_F$, $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

Build an adversarial F...



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$\mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) = \mathbb{T}$ (unit circle).

$\lim_{n\to\infty} \Gamma_n\big(\widetilde{F_1}\big) = \mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) \Rightarrow \exists n_1$ s.t. $\mathrm{dist}(i, \Gamma_{n_1}\big(\widetilde{F_1}\big)) \le 1$.

BUT Γ_{n_1} uses finite amount of info to output $\Gamma_{n_1}\big(\widetilde{F_1}\big)$.

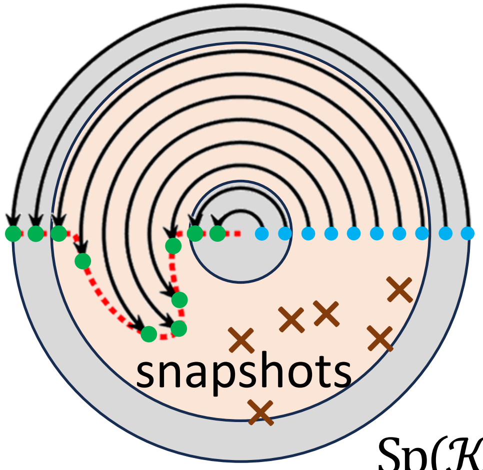
Let *X*, *Y* correspond to these snapshots.

$\mathcal{J}_F = \{(x, y_m) \mid \|F(x) - y_m\| \le 2^{-m}\}$

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty}\Gamma_n(F)=\mathrm{Sp}(\mathcal{K}_F)\ \forall F\in\Omega_{\mathbb{D}}$.

Build an adversarial F…



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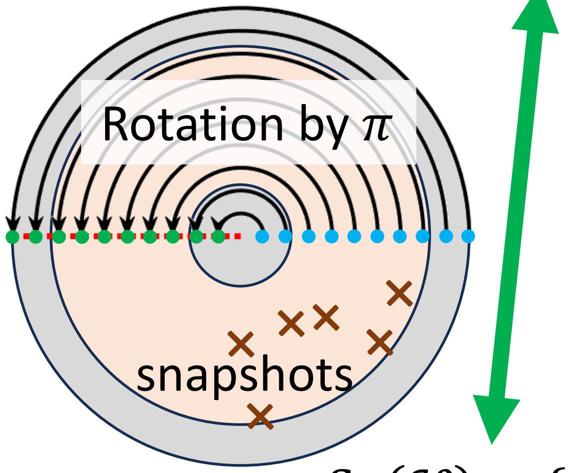


Lemma: $F_1=H_1^{-1}\circ F_0\circ H_1$ on annulus $\mathcal{A}_1$.
Consistent data $\Rightarrow\Gamma_{n_1}(F_1)=\Gamma_{n_1}\big(\widetilde{F_1}\big)$, $\mathrm{dist}(i,\Gamma_{n_1}(F_1))\le 1$
BUT $\mathrm{Sp}(\mathcal{K}_{F_1})=\mathrm{Sp}(\mathcal{K}_{F_0})=\{\pm1\}$

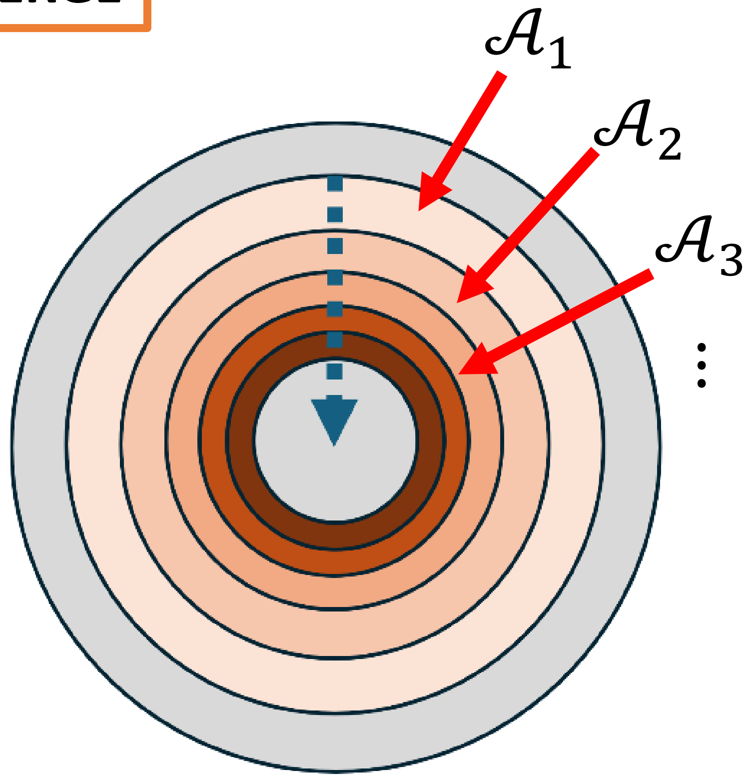
Proof idea: Constructing an adversary

Inductive step: Repeat on annuli, $F_k = H_k^{-1} \circ F_0 \circ H_k$ on $\mathcal{A}_k$. $F = \lim_{k\to\infty} F_k$

Consistent data $\Rightarrow \Gamma_{n_k}(F) = \Gamma_{n_k}(\widetilde{F_k})$, $\mathrm{dist}(i, \Gamma_{n_k}(F)) \leq 1$, $n_k \to \infty$

BUT $\mathrm{Sp}(\mathcal{K}_F) = \mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

CANNOT CONVERGE



Cascade of disks

Proof idea: Constructing an adversary

Inductive step: Repeat on annuli, $F_k = H_k^{-1} \circ F_0 \circ H_k$ on $\mathcal{A}_k$. $F = \lim_{k\to\infty} F_k$

Consistent data $\Rightarrow \Gamma_{n_k}(F) = \Gamma_{n_k}(\widetilde{F_k})$, $\mathrm{dist}(i, \Gamma_{n_k}(F)) \leq 1$, $n_k \to \infty$

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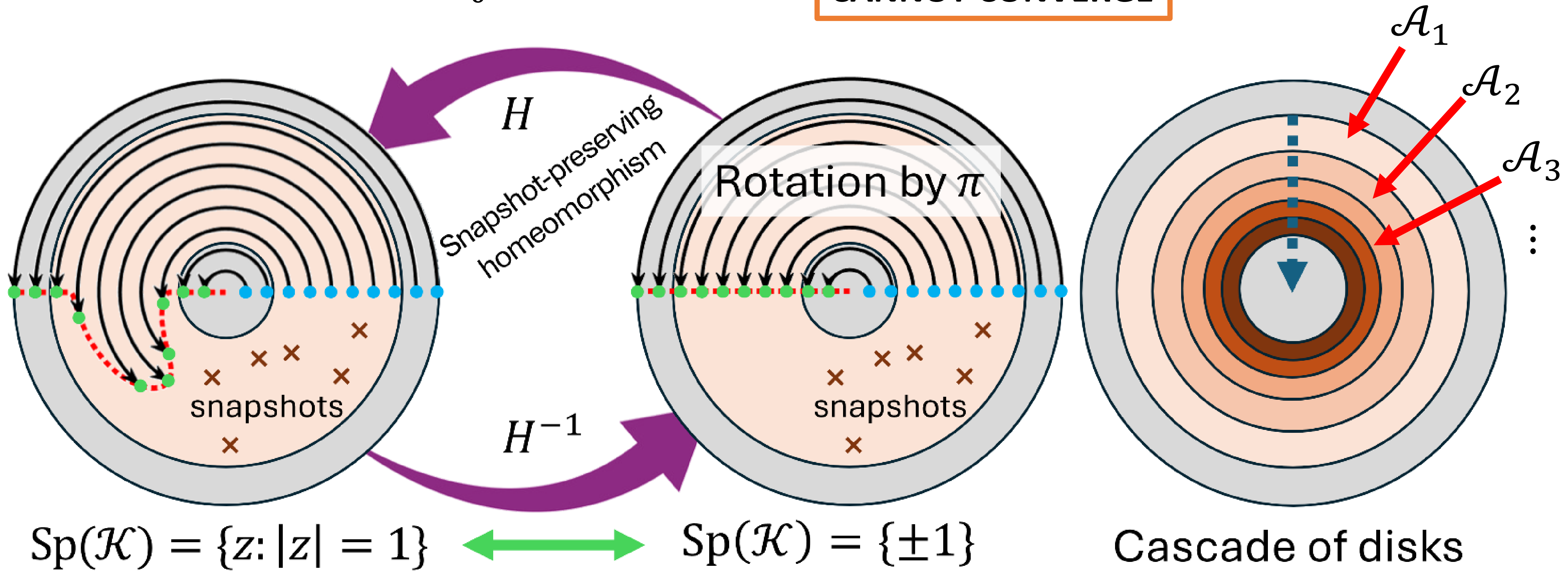

Successive limits seems unavoidable!?!?

Def: $\{\Gamma_{n_k,\ldots,n_1}\}$ with $\lim_{n_k\to\infty} \ldots \lim_{n_1\to\infty} \Gamma_{n_k,\ldots,n_1}$ convergent a ***tower of algorithms.***

algorithms

First appeared in dynamical systems theory:


Steve Smale

"Is there any purely iterative convergent rational map for polynomial zero finding?"


Curtis McMullen

"Yes for cubic, no for higher degree. Quartic and quintic can be solved using towers of algorithms. Sextic cannot be solved in any number of limits."

- Smale, *"On the efficiency of algorithms of analysis."* **Bull. Am. Math. Soc.**, 1985.
- McMullen, *"Families of rational maps and iterative root-finding algorithms."* **Annals Math.**, 1987.
- McMullen, *"Braiding of the attractor and the failure of iterative algorithms."* **Invent. Math.** 1988.
- Doyle, McMullen, *"Solving the quintic by iteration."* **Acta Math.**, 1989.

Classifications: *Solvability Complexity Index (SCI)*

SCI: Fewest number of limits needed to solve a computational problem.

- Hansen, "On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators." **J. Am. Math. Soc.**, 2011.
- C., "*The foundations of infinite-dimensional spectral computations,*" **PhD diss.**, University of Cambridge, 2020.
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Theorem A: $\mathrm{SCI} \leq 2$

Theorem B: $\mathrm{SCI} > 1$

combine

$\mathrm{SCI} = 2$

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Theorem A: SCI ≤ 2

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combine

SCI $= 2$

So far literature has only proven upper bounds, that need not be sharp…

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Koopman literature has tonnes of upper bounds!

SCI: Fewest number of limits needed to solve a computational problem.

Algorithm	Comments/Assumptions	Spectral Problem's Corresponding SCI Upper Bound			
		KMD	*Spectrum*	*Spectral Measure (if m.p.)*	*Spectral Type (if m.p.)*
Extended DMD [47]	general L^2 spaces	$\mathrm{SCI} \leq 2^*$	N/C	N/C	n/a
Residual DMD [44]	general L^2 spaces	$\mathrm{SCI} \leq 2^*$	$\mathrm{SCI} \leq 3^*$	$\mathrm{SCI} \leq 2^*$	varies, see [84] e.g., a.c. density: $\mathrm{SCI} \leq 2^*$
Measure-preserving EDMD [45]	m.p. systems	$\mathrm{SCI} \leq 1$	N/C	$\mathrm{SCI} \leq 2^*$ (general) $\mathrm{SCI} \leq 1$ (delay-embedding)	n/a
Hankel DMD [85]	m.p. ergodic systems	$\mathrm{SCI} \leq 2^*$	N/C	N/C	n/a
Periodic approximations [86]	m.p. + ω a.c.	$\mathrm{SCI} \leq 2$	N/C	$\mathrm{SCI} \leq 2$ (see [87])	a.c. density: $\mathrm{SCI} \leq 3$
Christoffel–Darboux kernel [40]	m.p. ergodic systems	$\mathrm{SCI} \leq 3$	n/a	$\mathrm{SCI} \leq 2$	e.g., a.c. density: $\mathrm{SCI} \leq 2$
Generator EDMD [88]	cts.-time, samples ∇F (otherwise additional limit)	$\mathrm{SCI} \leq 2$	N/C	$\mathrm{SCI} \leq 2$ (see [89])	n/a
Compactification [42]	cts.-time, m.p. ergodic systems	$\mathrm{SCI} \leq 4$	N/C	$\mathrm{SCI} \leq 4$	n/a
Resolvent compactification [43]	cts.-time, m.p. ergodic systems	$\mathrm{SCI} \leq 5$	N/C	$\mathrm{SCI} \leq 5$	n/a
Diffusion maps [90] (see also [10])	cts.-time, m.p. ergodic systems	$\mathrm{SCI} \leq 3$	n/a	n/a	n/a

Are these sharp?

Previous techniques prove upper bounds on SCI.
"N/C": method need not converge. "n/a": algorithm not applicable to problem.

Also in Ulam's method for Markov processes, SRB measure computation, control,…

Classifications: *Solvability Complexity Index (SCI)*

SCI: Fewest number of limits needed to solve a computational problem.

- Δ_1: One limit, full error control. E.g., $d(\Gamma_n(F), \mathrm{Sp}(\mathcal{K}_F)) \leq 2^{-n}$.
- Δ_{m+1}: SCI $\leq m$.
- Σ_m: SCI $\leq m$, final limit from below.

  E.g., Σ_1: $\sup_{z \in \Gamma_n(F)} \mathrm{dist}(z, \mathrm{Sp}(\mathcal{K}_F)) \leq 2^{-n}$.
- Π_m: SCI $\leq m$, final limit from above.

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verification

trust output

covers spectrum

- Hansen, "On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators." **J. Am. Math. Soc.**, 2011.
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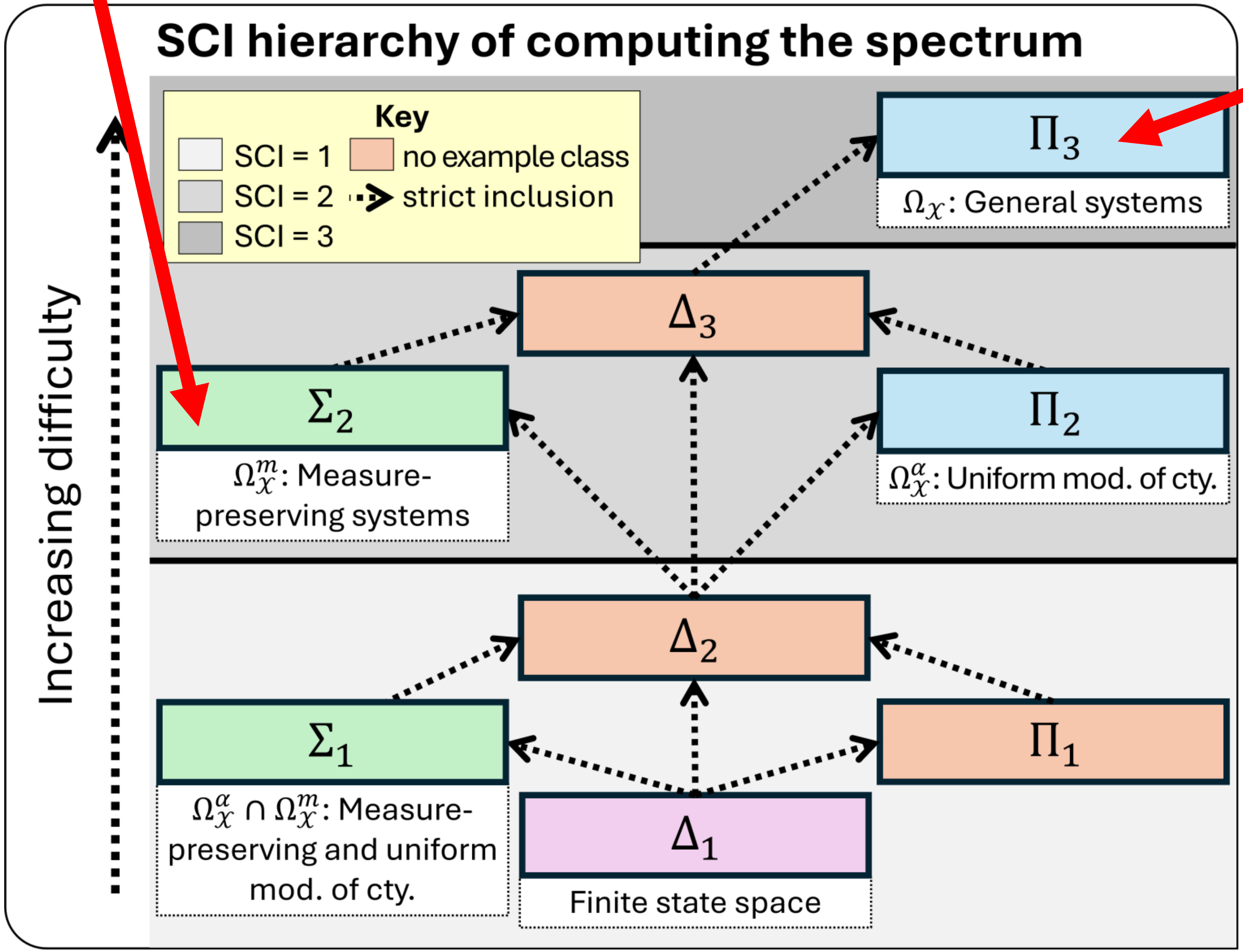
Classification for Koopman I

Theorems A + B



3 limits needed in general!

Different classes:

$$\Omega_{\mathcal{X}} = \{F:\mathcal{X}\to\mathcal{X} \mid F \text{ cts}\}$$

$$\Omega_{\mathcal{X}}^m = \{F:\mathcal{X}\to\mathcal{X} \mid F \text{ cts, m. p.}\}$$

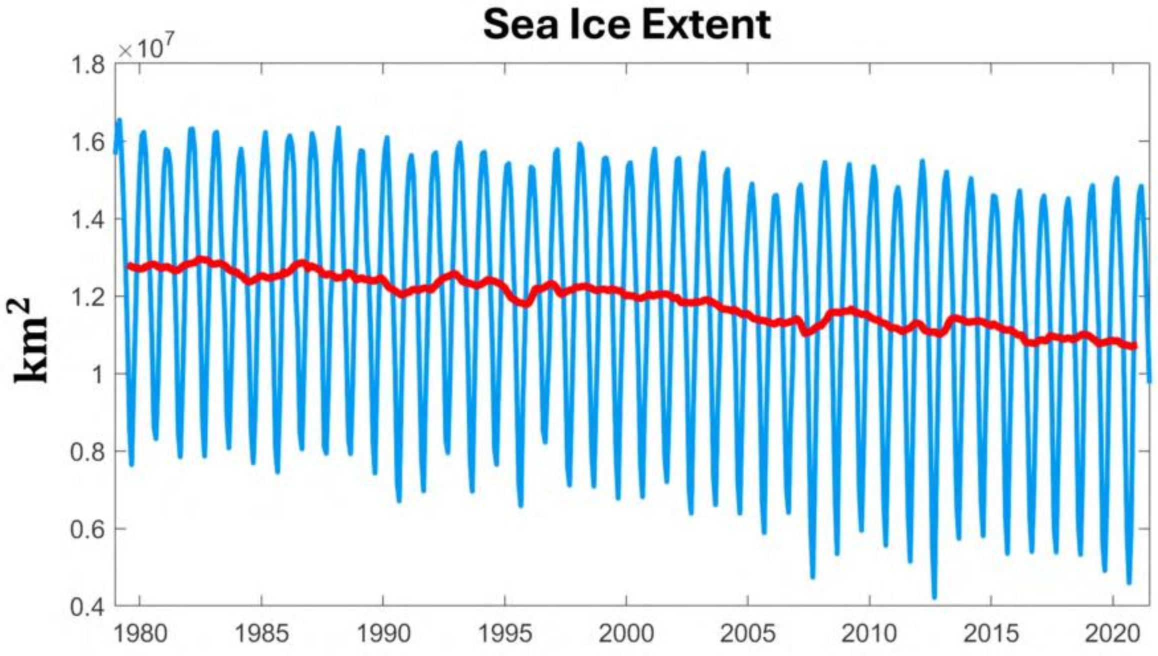
$$\Omega_{\mathcal{X}}^\alpha = \{F:\mathcal{X}\to\mathcal{X} \mid F \text{ mod. cty. } \alpha\}$$

$$[d_{\mathcal{X}}(F(x),F(y)) \leq \alpha(d_{\mathcal{X}}(x,y))]$$

Optimal algorithms and classifications of dynamical systems.

Practical Gains: Arctic Sea Ice Forecasting

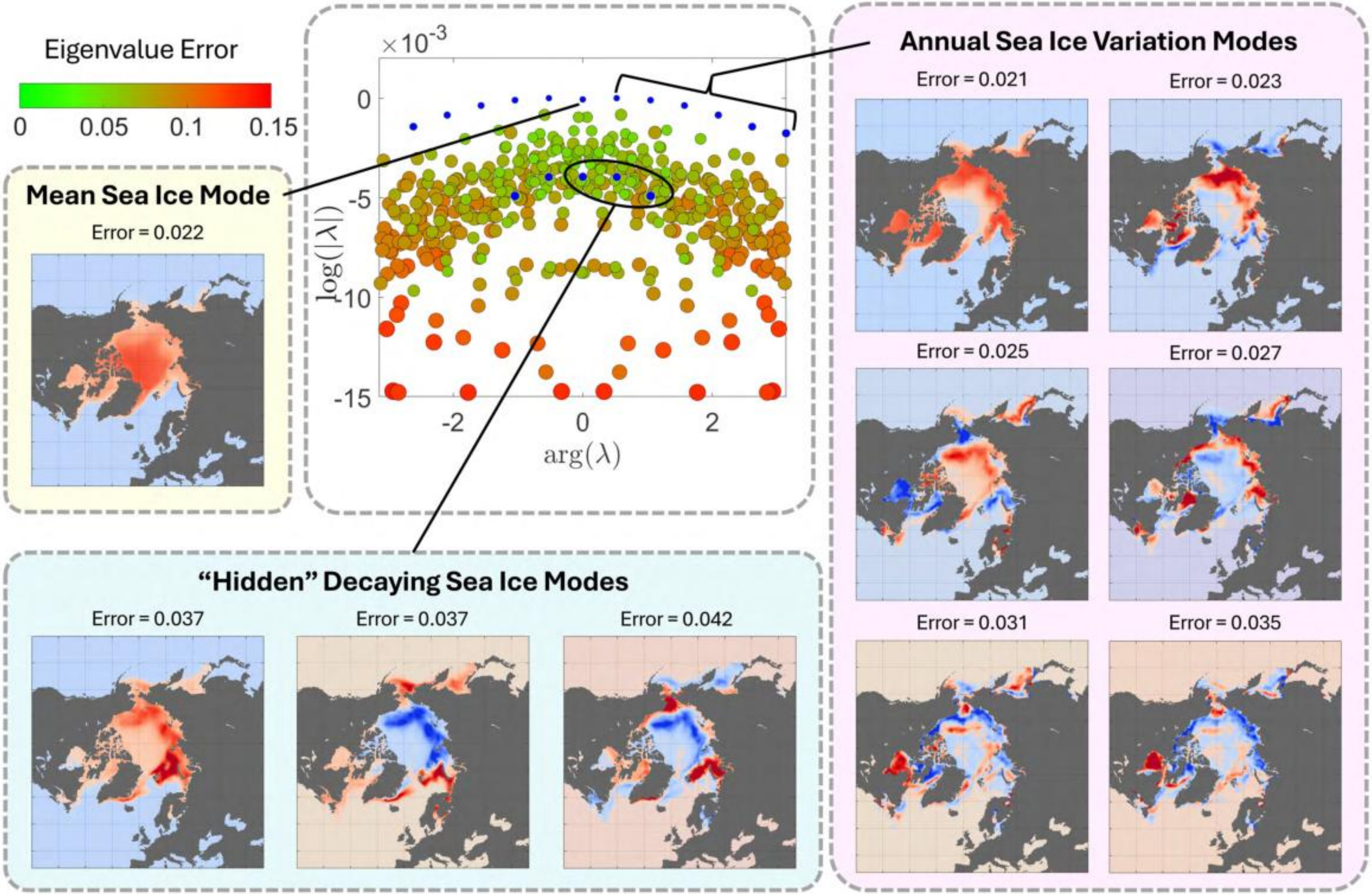
Monthly average from satellite passive microwave sensors. Data from the European Organisation for the Exploitation of Meteorological Satellites' (EUMETSAT) Ocean and Sea Ice Satellite Application Facilities (OSI-SAF) data record, comprising retrieval algorithms OSI-450 (1979–2015) and OSI-430-b (2016 onwards).



Motivation: Arctic amplification, polar bears, local communities, effect on extreme weather in Northern hemisphere,…

Problems:

- Very hard to predict more than two months in advance.
- Which geographical regions are significant?



Eigenvalue Error
0
0.05
0.1
0.15
$\times 10^{-3}$
0
-5
-10
-15
$\log(|\lambda|)$
-2
0
2
$\arg(\lambda)$
Mean Sea Ice Mode
Error = 0.022
Annual Sea Ice Variation Modes
Error = 0.021
Error = 0.023
Error = 0.025
Error = 0.027
Error = 0.031
Error = 0.035
"Hidden" Decaying Sea Ice Modes
Error = 0.037
Error = 0.037
Error = 0.042

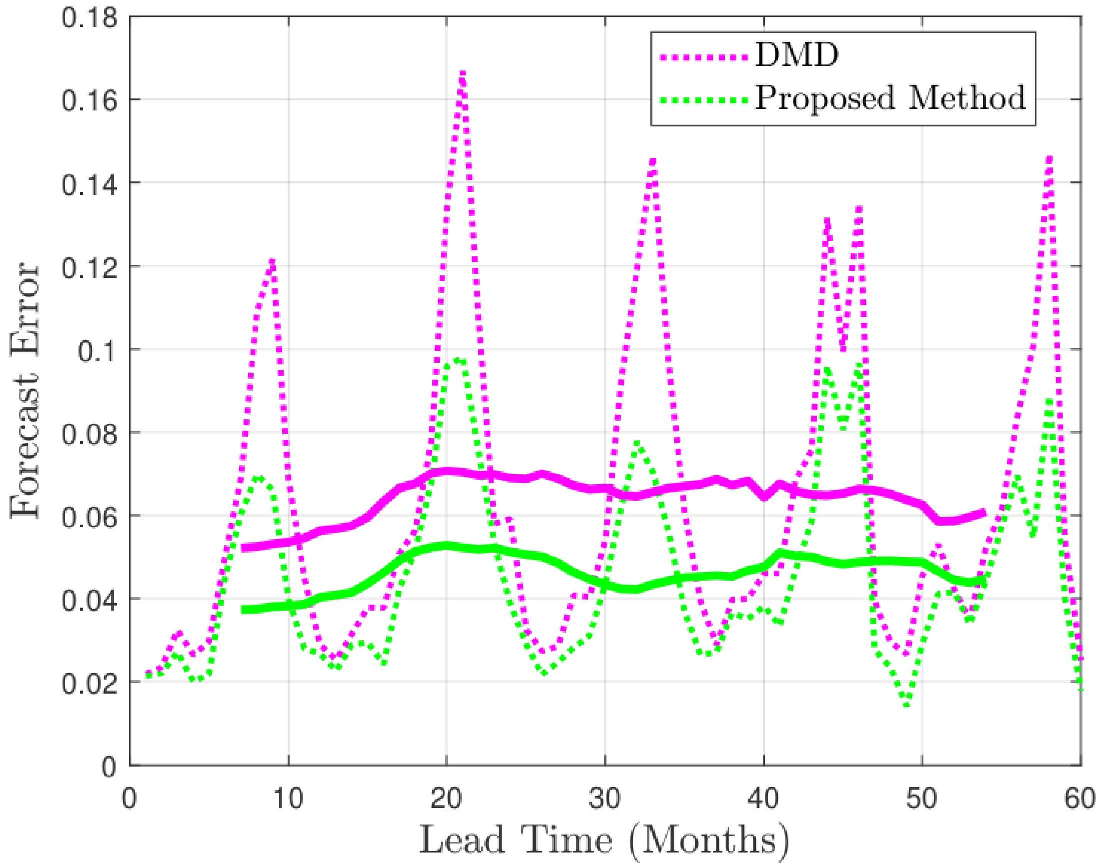


Figure 4: Forecast error for entire sea ice concentration. The relative mean squared error of forecasts over five years. The solid lines show the moving 12-month mean. In each case, the model is built using the data from the years 2005–2015, and then tested on 2016-2020. The proposed method consistently outperforms DMD.

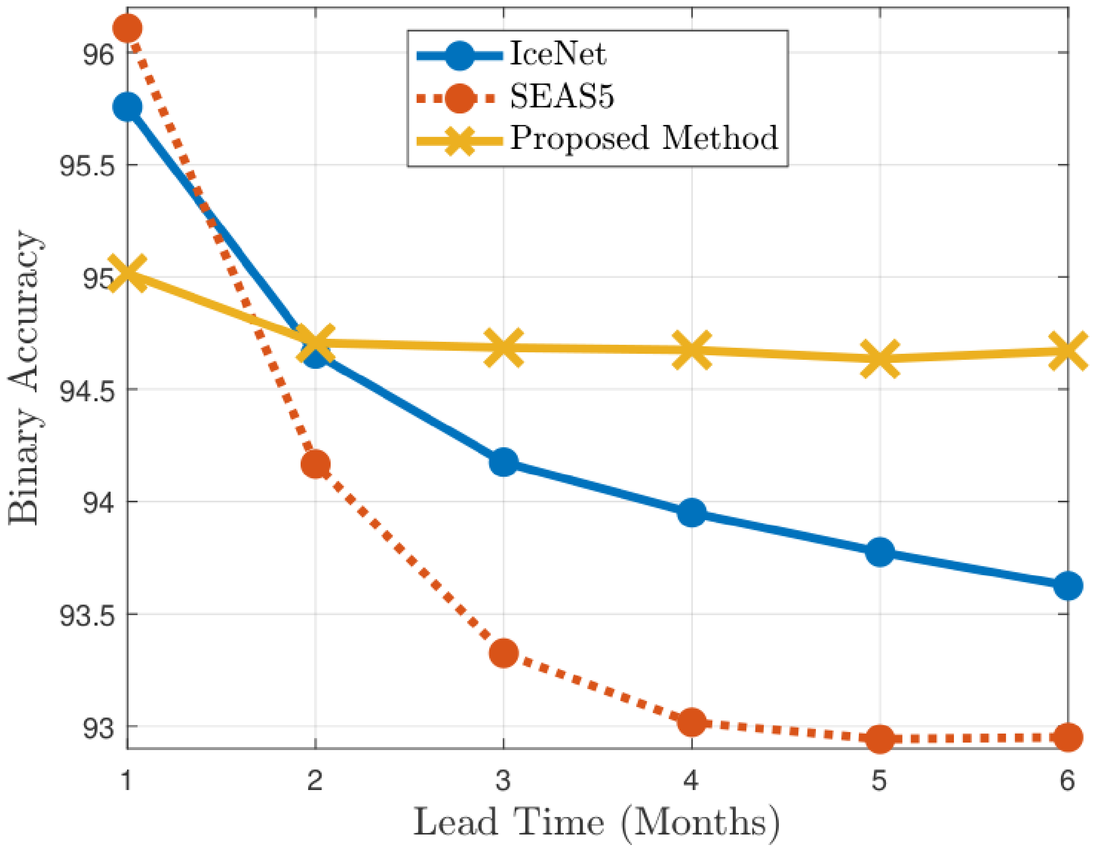
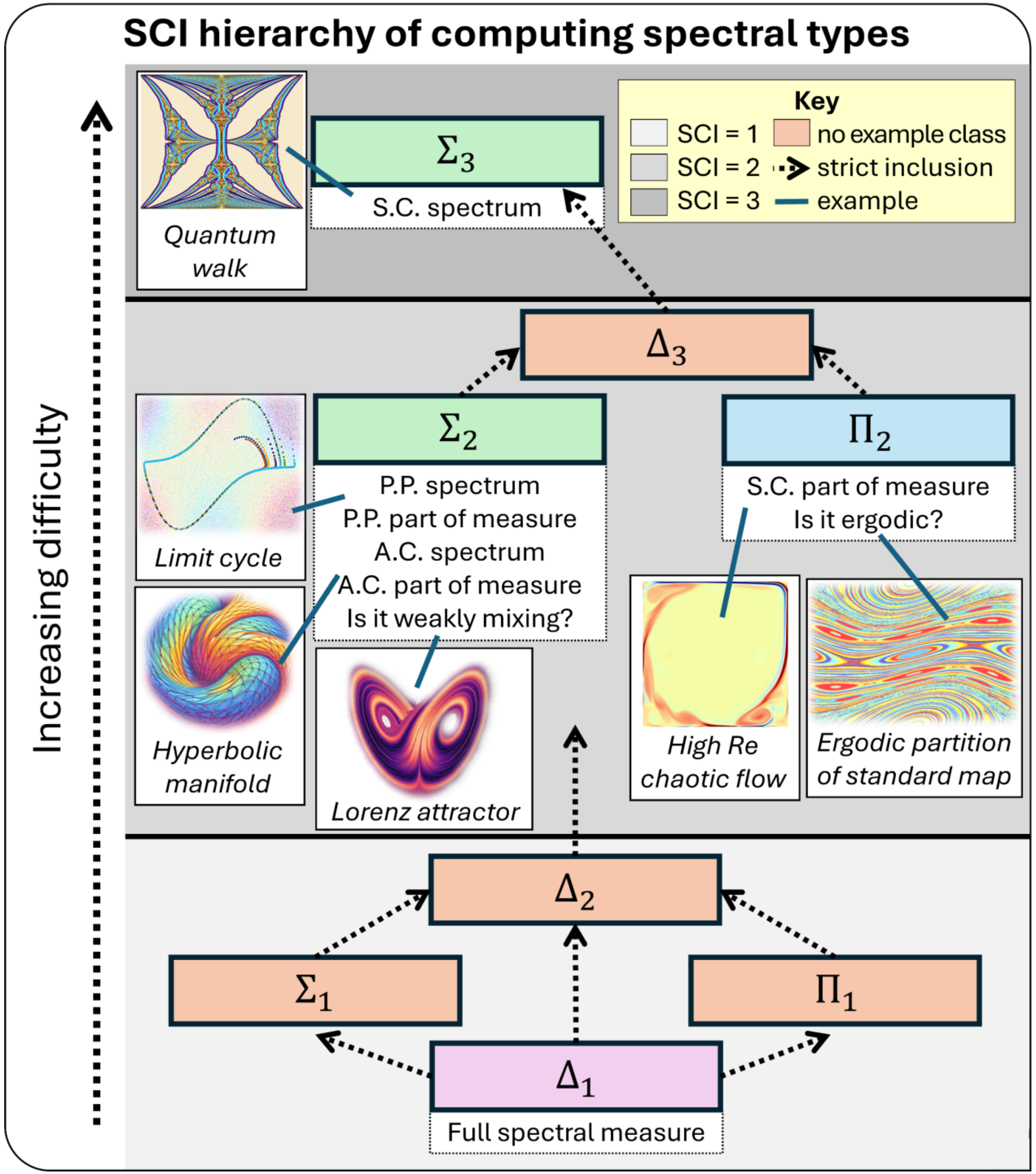


Figure 5: Comparison with machine learning and statistical prediction benchmarks. Mean binary accuracy over the test years 2012–2020, shown for IceNet, SEAS5, and our proposed method that avoids spurious Koopman eigenvalues. Our proposed method achieves better accuracy for lead times greater than one month, with very little increase of errors at larger lead times.

- Andersson et al, "Seasonal Arctic sea ice forecasting with probabilistic deep learning." **Nature Communications**, 2021.

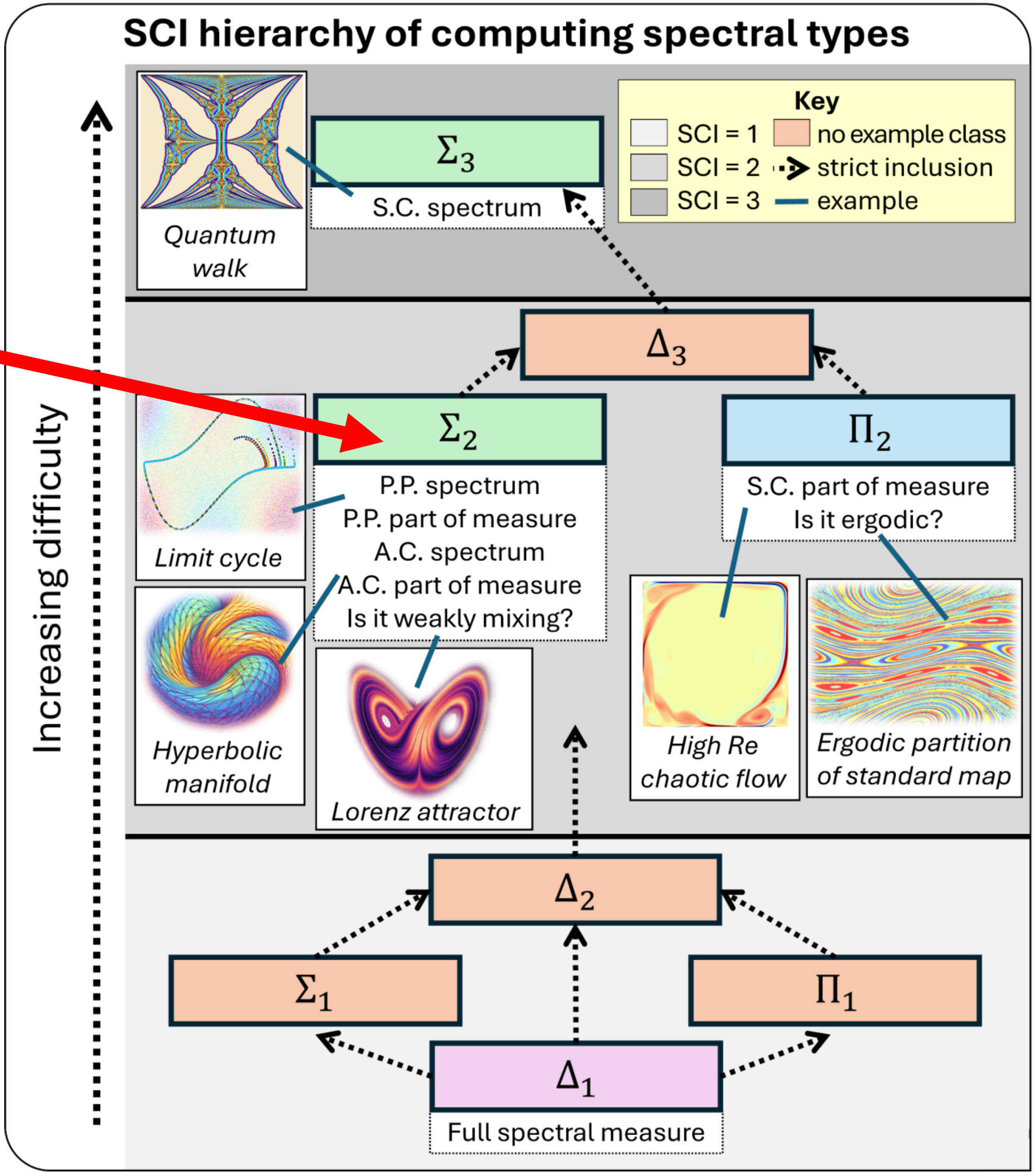
Classification for Koopman II



Classification for Koopman II

Example: Theorem C
For smooth, measure-preserving systems on a torus, learning eigenfunctions or even determining if there are any has SCI $= 2$ (even if we can sample derivatives).

Finding finite-dimensional embeddings in which the dynamics are linear (e.g., autoencoders, latent space representation) is very hard!



General tool in data-driven dynamical systems/PDEs

Adversarial arguments generalize to:

- "Learning the F". E.g., SINDy $(x_{n+1} = F(x_n))$
- Solving PDEs with neural networks (PINNs)
- Learning PDEs from forcing-solution pairs (e.g., hyperbolic)

- Brunton, Proctor, Kutz, "*Discovering governing equations from data by sparse identification of nonlinear dynamical systems,*" **Proc. Natl. Acad. Sci. USA**, 2016.
- Karniadakis, Kevrekidis, Lu, Perdikaris, Wang, Yang, "*Physics-informed machine learning,*" **Nature Reviews Physics**, 2021.
- Boulle, Halikias, Townsend, "*Elliptic PDE learning is provably data-efficient,*" **Proc. Natl. Acad. Sci. USA**, 2023.

Where does this leave us?

- Many problems **NECESSARILY** require multiple limits.
- New tools for **lower bounds** (impossibility results) for Koopman learning.
- Combine with **upper bounds** (algorithms)
  ⇒ **classify difficulty** of problems + **prove optimality** of algorithms.
- Ergodic theory + approximation theory + computational analysis
  ⇒ started to map out this terrain.

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- Future work:
  - Other function spaces.
  - Partial observations, continuous-time.
  - Control and uses of Koopman.
  - Other data-driven dynamical system methods.

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  - Partial observations, continuous-time.
  - Control and uses of Koopman.
  - Other data-driven dynamical system methods.

Where does your problem/method fit into the SCI hierarchy? Is it optimal?

References

[1] Colbrook, Matthew J., and Alex Townsend. "Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems." Communications on Pure and Applied Mathematics 77.1 (2024): 221-283.

[2] Colbrook, Matthew J., Lorna J. Ayton, and Máté Szőke. "Residual dynamic mode decomposition: robust and verified Koopmanism." Journal of Fluid Mechanics 955 (2023): A21.

[3] Colbrook, M. J., Li, Q., Raut, R. V., & Townsend, A. "Beyond expectations: residual dynamic mode decomposition and variance for stochastic dynamical systems." Nonlinear Dynamics 112.3 (2024): 2037-2061.

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[6] Colbrook, Matthew J., Catherine Drysdale, and Andrew Horning. "Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators." SIAM Journal on Applied Dynamical Systems 24, no. 2 (2025): 1150-1190.

[7] Boullé, Nicolas, and Matthew J. Colbrook. "Multiplicative Dynamic Mode Decomposition." SIAM Journal on Applied Dynamical Systems 24, no. 2 (2025): 1945-1968.

[8] Boullé, Nicolas and Matthew J. Colbrook, "On the Convergence of Hermitian Dynamic Mode Decomposition" Physica D: Nonlinear Phenomena, 472 (2025).

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