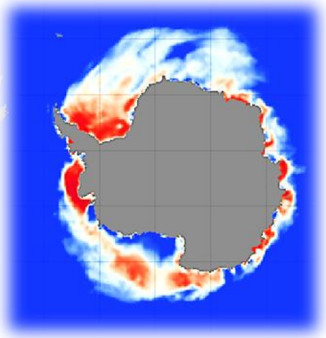

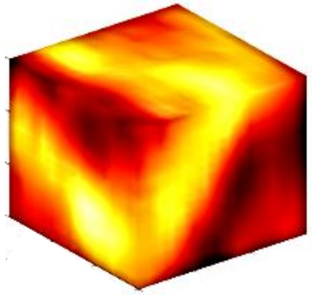

Spectral Learning for Dynamical Systems Via Infinite-Dimensional Numerical Linear Algebra

Matthew Colbrook

9th Sep 2025

UNIVERSITY OF CAMBRIDGE

"To classify is to bring order into chaos." - **George Pólya**

Cast of great collaborators!

Alex Townsend
(Cornell)

Igor Mezić
(UC Santa Barbara)

Alexei Stepanenko
(Cam. -> Industry)

Nicolas Boullé
(Imperial)

Gustav Conradie
(PhD student at Cambridge)

- C., Townsend. *"Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems."* **Communications on Pure and Applied Mathematics,** 2024.
- C., Mezić, Stepanenko, *"Adversarial Dynamical Systems Reveal Limits and Rules for Trustworthy Data-Driven Learning."* (under revision at **Nature Communications**).
- Boullé, C., Conradie, *"Convergent Methods for Koopman Operators on Reproducing Kernel Hilbert Spaces."* (***SpecRKHS*** **- hot off the press: https://arxiv.org/abs/2506.15782**)

What is a Koopman operator?

- $\mathcal{X}$ – *the state space*
- $\mathcal{X} \ni x$ – *the state*

cts $F: \mathcal{X} \to \mathcal{X}$ – *the dynamics:* $x_{n+1} = F(x_n)$

Henri Poincaré
(Sorbonne)

• Poincaré, *''Les méthodes nouvelles de la mécanique céleste,''* **Vol. 2. Gauthier-Villars et fils, imprimeurs-libraires**, 1893.

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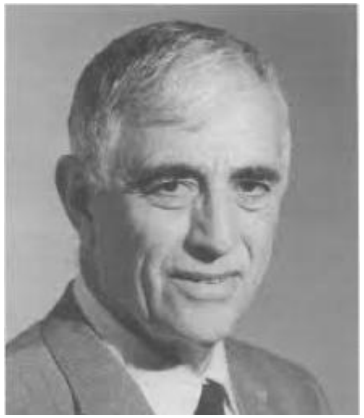
- Functions $g: \mathcal{X} \to \mathbb{C}$ a.k.a "observables"
- Koopman operator $\mathcal{K}_F$: $[\mathcal{K}_F g](x) = g(F(x))$

LINEAR!

Observe g one time step forward

Bernard Koopman
(Columbia)



John von Neumann
(IAS)

- Koopman, "*Hamiltonian systems and transformation in Hilbert space,*" **Proc. Natl. Acad. Sci. USA**, 1931.
- Koopman, v. Neumann, "*Dynamical systems of continuous spectra,*" **Proc. Natl. Acad. Sci. USA**, 1932.

What is a Koopman operator?

- $\mathcal{X}$ – *the state space*
- $\mathcal{X} \ni x$ – *the state*
- ***Unknown*** cts $F: \mathcal{X} \to \mathcal{X}$ – *the dynamics:* $x_{n+1} = F(x_n)$
- Functions $g: \mathcal{X} \to \mathbb{C}$ a.k.a "observables"
- Koopman operator $\mathcal{K}_F$: $[\mathcal{K}_F g](x) = g(F(x))$ LINEAR!
- ***Available*** snapshot data: $\left\{\left(x^{(m)}, y^{(m)} = F\left(x^{(m)}\right)\right) : m = 1, \ldots, M\right\}$

Can we compute spectral properties from trajectory data?

Why?

$g(x_n) = [\mathcal{K}^n g](x_0)$

If $\|\mathcal{K}g - \lambda g\| \leq \varepsilon$, then $g(x_n) = [\mathcal{K}^n g](x_0) = \lambda^n g(x_0) + \mathcal{O}(n\varepsilon)$

Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

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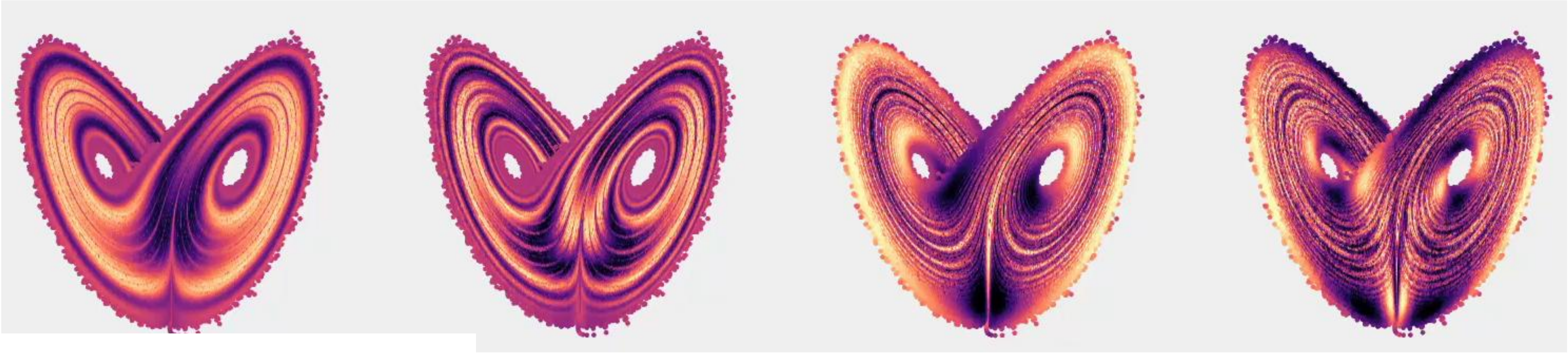
Why?

If $||\mathcal{K}g - \lambda g|| \leq \varepsilon$, then $g(x_n) = [\mathcal{K}^n g](x_0) = \lambda^n g(x_0) + \mathcal{O}(n\varepsilon)$



Coherent features!

Lorenz attractor

Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

$g(x_n) = [\mathcal{K}^n g](x_0)$

Why?

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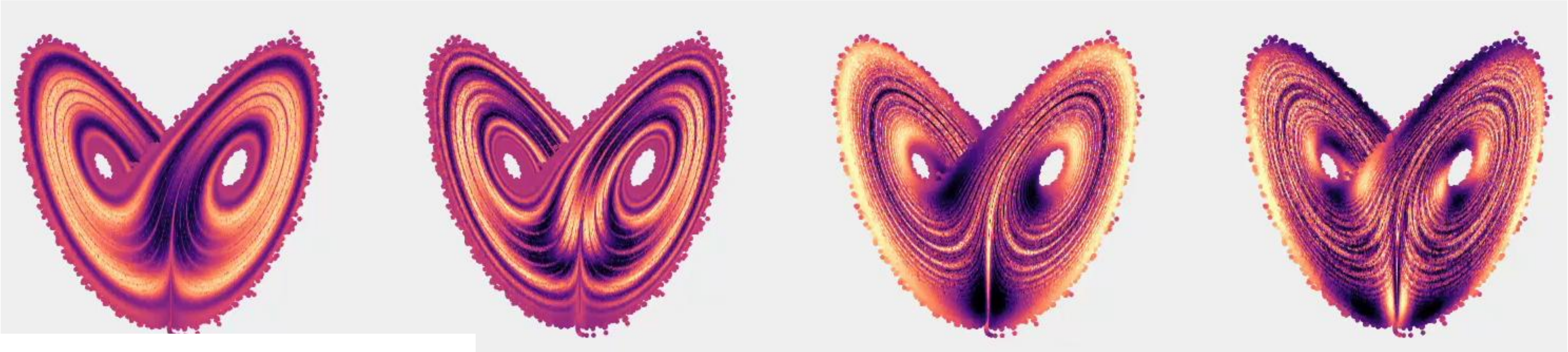


Coherent features!

$$\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}) = \{z \in \mathbb{C} : \exists g, \|g\| = 1, \|\mathcal{K}g - zg\| \leq \varepsilon\}$$

Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

Koopman Mode Decomposition

- Find (g_j, λ_j) with $\|\mathcal{K} g_j - \lambda_j g_j\| \leq \varepsilon$

Verified Eigenfunctions

- Expand state:

$$x \approx \sum_j c_j g_j(x)$$

coefficients, called "Koopman modes"

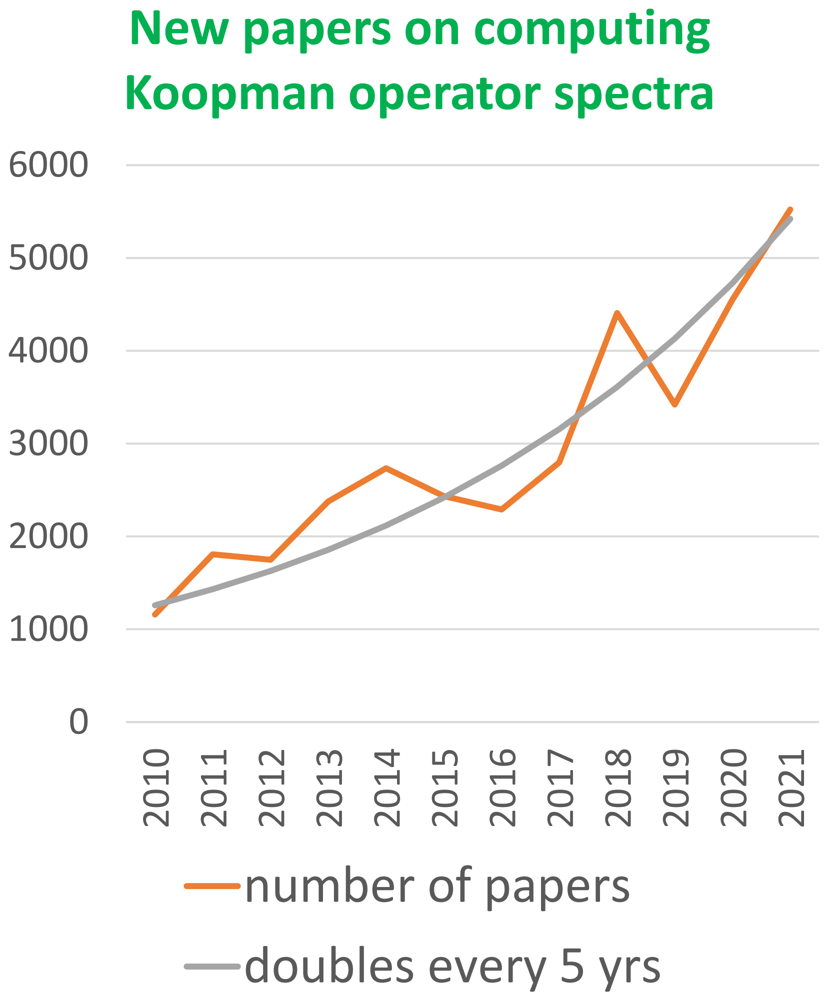
- Forecasts:

$$x_n = \sum_j \lambda_j^n c_j g_j(x) + \mathcal{O}(n\varepsilon)$$

$g(x_n) = [\mathcal{K}^n g](x_0)$

Intuition: A nonlinear separation of variables through a linear operator!

Koopmania*: A revolution in the big data era?



Very* little on convergence guarantees. *WHY?

1. Koopman operators have been largely used in applied domains + distinct from NLA.

2. Infinite-dimensional spec. comp. notoriously hard …

Only recently have the tools been developed

GOAL: Compute spectral properties and figure out how hard this is.

The Standard (naïve?) Pipeline

DATA + DISCRETIZE $\mathcal{K}$

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DATA + DISCRETIZE $\mathcal{K}$

FINITE-DIMENSIONAL NUMERICAL LINEAR ALGEBRA

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DATA + DISCRETIZE $\mathcal{K}$

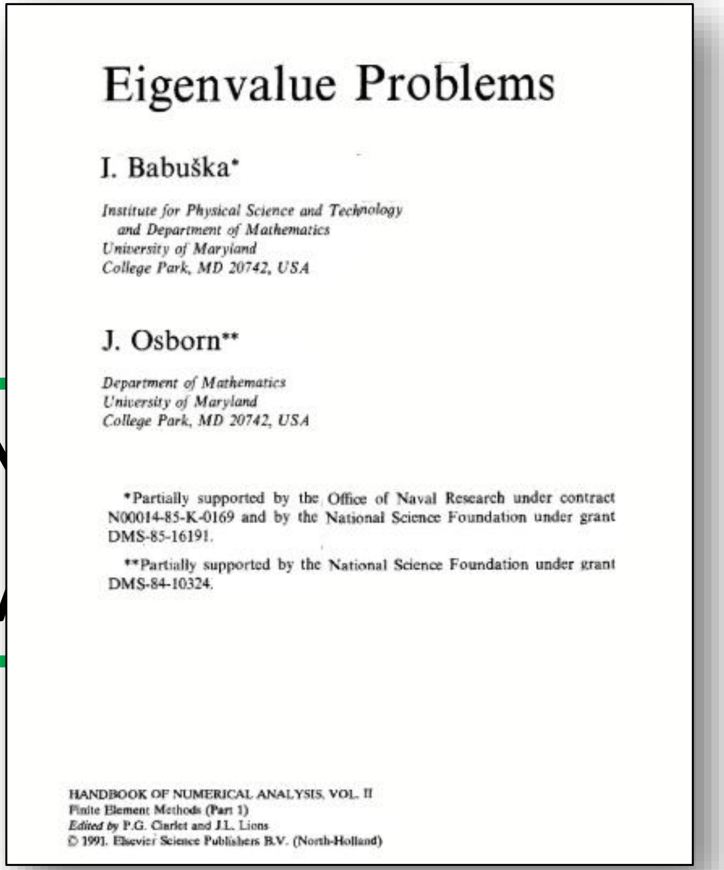
FINITE-DIMENSIONAL NUMERICAL LINEAR ALGEBRA

EIGENVALUES etc.

The Standard (naïve?) Pipeline

DATA + DISCRETIZE $\mathcal{K}$

FINITE-DIMEN
NUMERICAL LINE


Eigenvalue Problems

J. Babuška*

Institute for Physical Science and Technology
and Department of Mathematics
University of Maryland
College Park, MD 20742, USA

J. Osborn**

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University of Maryland
College Park, MD 20742, USA

*Partially supported by the Office of Naval Research under contract N00014-85-K-0169 and by the National Science Foundation under grant DMS-85-16191.

**Partially supported by the National Science Foundation under grant DMS-84-10324.

HANDBOOK OF NUMERICAL ANALYSIS, VOL. II
Finite Element Methods (Part 1)
Edited by P.G. Ciarlet and J.L. Lions
© 1991, Elsevier Science Publishers B.V. (North-Holland)

EIGENVALUES etc.

Works great if you have a self-adjoint operator that is compact or has compact resolvent!

The Standard (naïve?) Pipeline

DATA + DISCRETIZE $\mathcal{K}$

FINITE-DIMENSIONAL NUMERICAL LINEAR ALGEBRA

EIGENVALUES etc.

For Koopman?

Perils of discretization: Warmup on $\ell^2(\mathbb{Z})$

$$\begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{C}^{N\times N}$$

Two-way infinite

- Spectrum is unit circle.
- Spectrum is stable.
- Continuous spectra.
- Unitary evolution.

- Spectrum is $\{0\}$.
- Spectrum is unstable.
- Discrete spectra.
- Nilpotent evolution.

Lots of Koopman operators are built up from operators like these!

Explicit example: Matrix approximation of $\mathcal{K}$ (EDMD)

Observables $\psi_j : \mathcal{X} \to \mathbb{C},\ j = 1, \ldots, N$

$$\left\{x^{(m)}, y^{(m)} = F\left(x^{(m)}\right)\right\}_{m=1}^{M}$$

quadrature points

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j\left(x^{(m)}\right)} \psi_k\left(x^{(m)}\right) = \Bigg[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}^*}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X} \Bigg]_{jk}$$

quadrature weights

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j\left(x^{(m)}\right)} \underbrace{\psi_k\left(y^{(m)}\right)}_{[\mathcal{K}\psi_k](x^{(m)})} = \Bigg[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}^*}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \psi_1(y^{(1)}) & \cdots & \psi_N(y^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(y^{(M)}) & \cdots & \psi_N(y^{(M)}) \end{pmatrix}}_{\Psi_Y} \Bigg]_{jk}$$

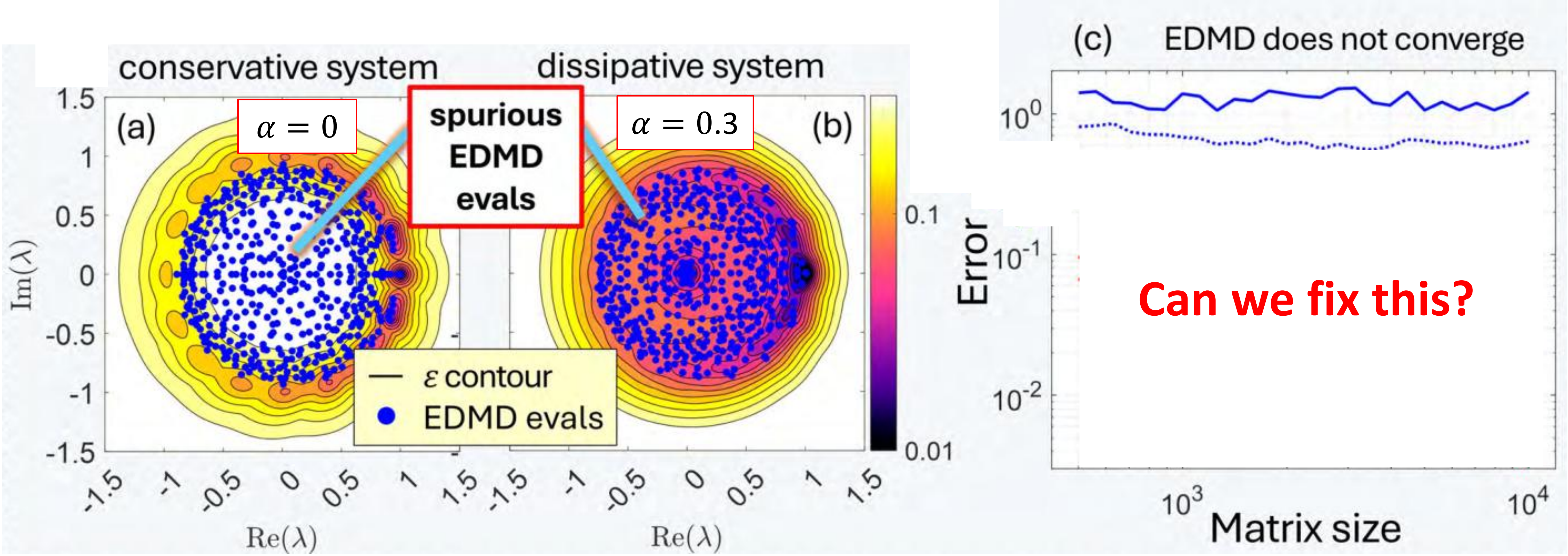
Galerkin Approximation

$$\mathcal{K} \longrightarrow \left({\Psi_X}^* W \Psi_X\right)^{-1} {\Psi_X}^* W \Psi_Y \in \mathbb{C}^{N\times N}$$

- Schmid, "*Dynamic mode decomposition of numerical and experimental data,*" **J. Fluid Mech.**, 2010.
- Rowley, Mezić, Bagheri, Schlatter, Henningson, "*Spectral analysis of nonlinear flows,*" **J. Fluid Mech.**, 2009.
- Williams, Kevrekidis, Rowley "*A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition,*" **J. Nonlinear Sci.**, 2015.

EDMD doesn't converge!

- Duffing oscillator: $\dot{x} = y, \dot{y} = -\alpha y + x(1 - x^2)$, sampled $\Delta t = 0.3$.
- Gaussian radial basis functions, Monte Carlo integration ($M = 50000$)



The fix: Residual DMD (ResDMD)

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \psi_k\left(x^{(m)}\right) = \Big[\underbrace{\Psi_X{}^* W \Psi_X}_{G}\Big]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \underbrace{\psi_k\left(y^{(m)}\right)}_{[\mathcal{K}\psi_k](x^{(m)})} = \Big[\underbrace{\Psi_X{}^* W \Psi_Y}_{K_1}\Big]_{jk}$$

- C., Townsend, "*Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems*," **Commun. Pure Appl. Math.**, 2023.
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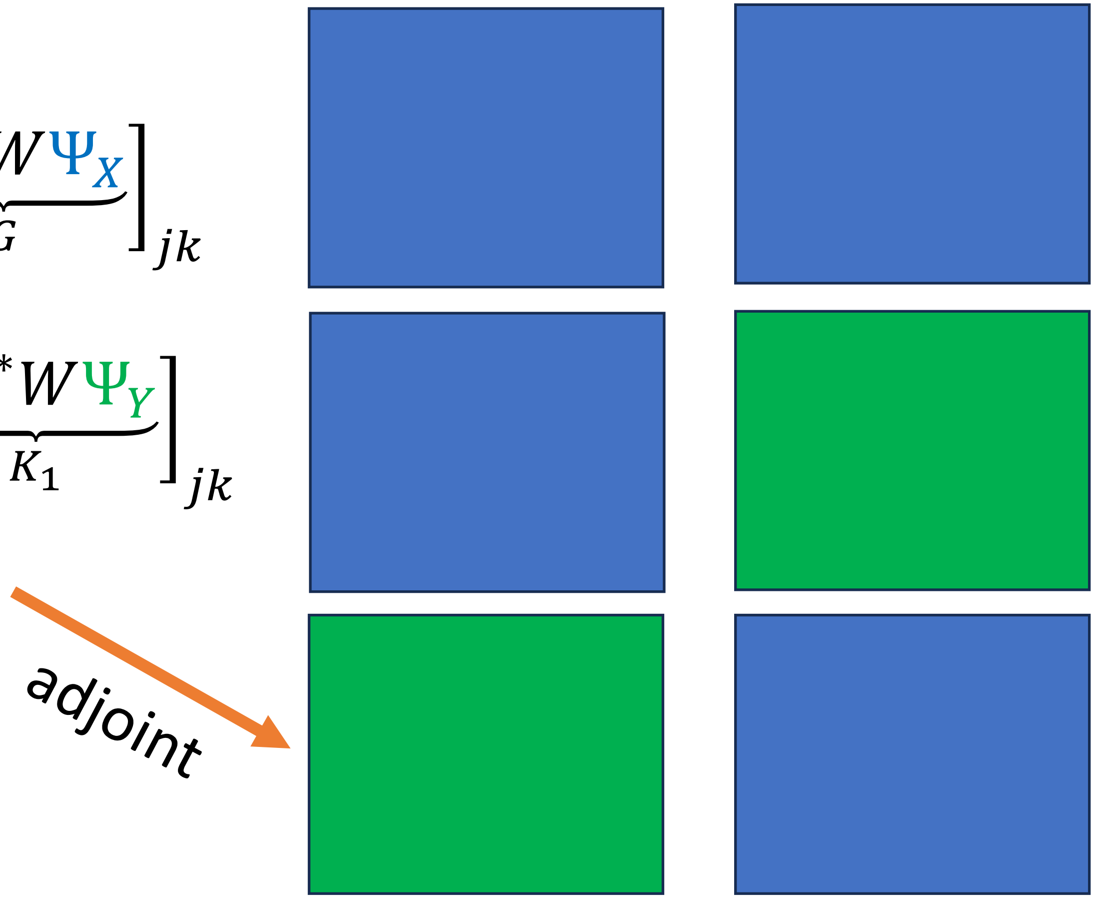

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Residuals: $g = \sum_{j=1}^{N} \mathbf{g}_j \psi_j$, $\|\mathcal{K}g - \lambda g\|^2 = \langle \mathcal{K}g - \lambda g, \mathcal{K}g - \lambda g \rangle$

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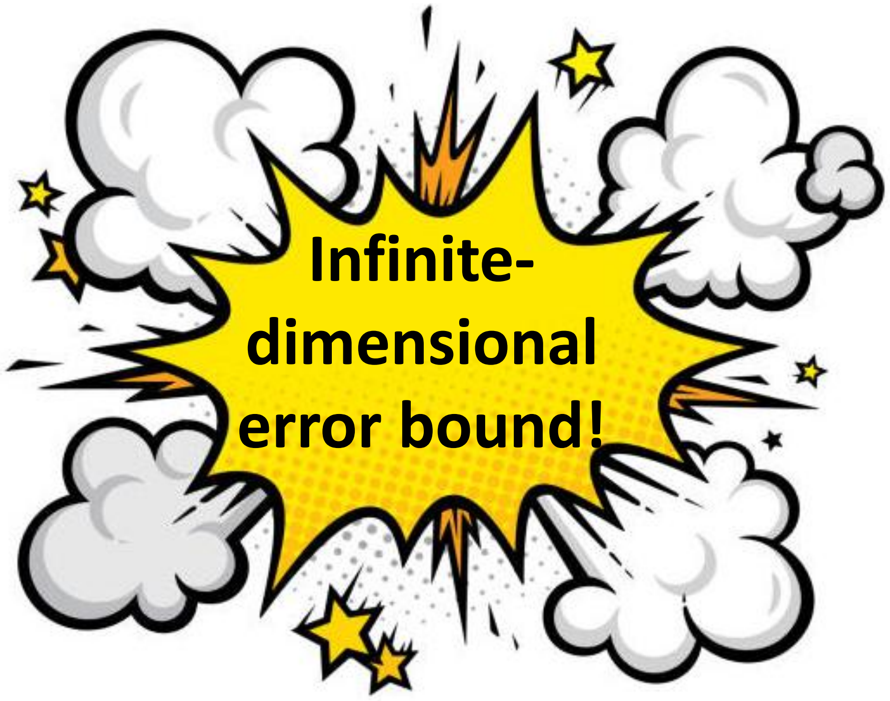

Residuals: $g = \sum_{j=1}^{N} \mathbf{g}_j \psi_j, \ \|\mathcal{K}g - \lambda g\|^2 = \lim_{M\to\infty} \mathbf{g}^* [K_2 - \lambda K_1{}^* - \bar{\lambda} K_1 + |\lambda|^2 G]\mathbf{g}$

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ResDMD does converge!

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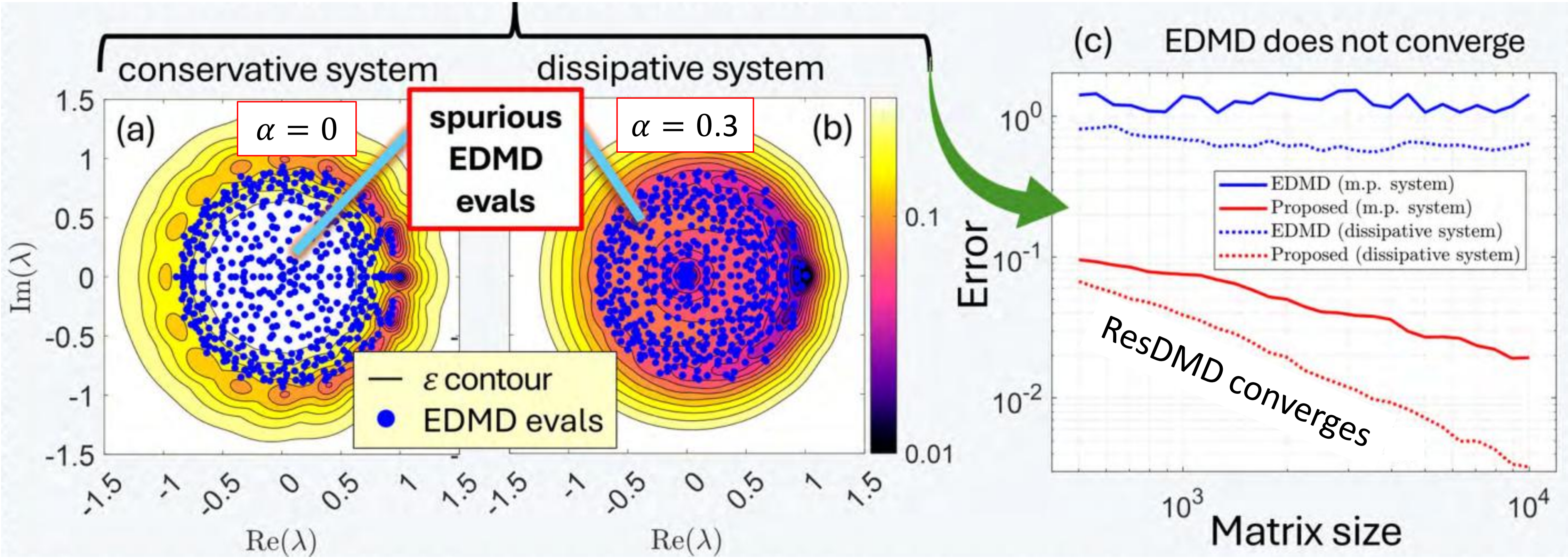
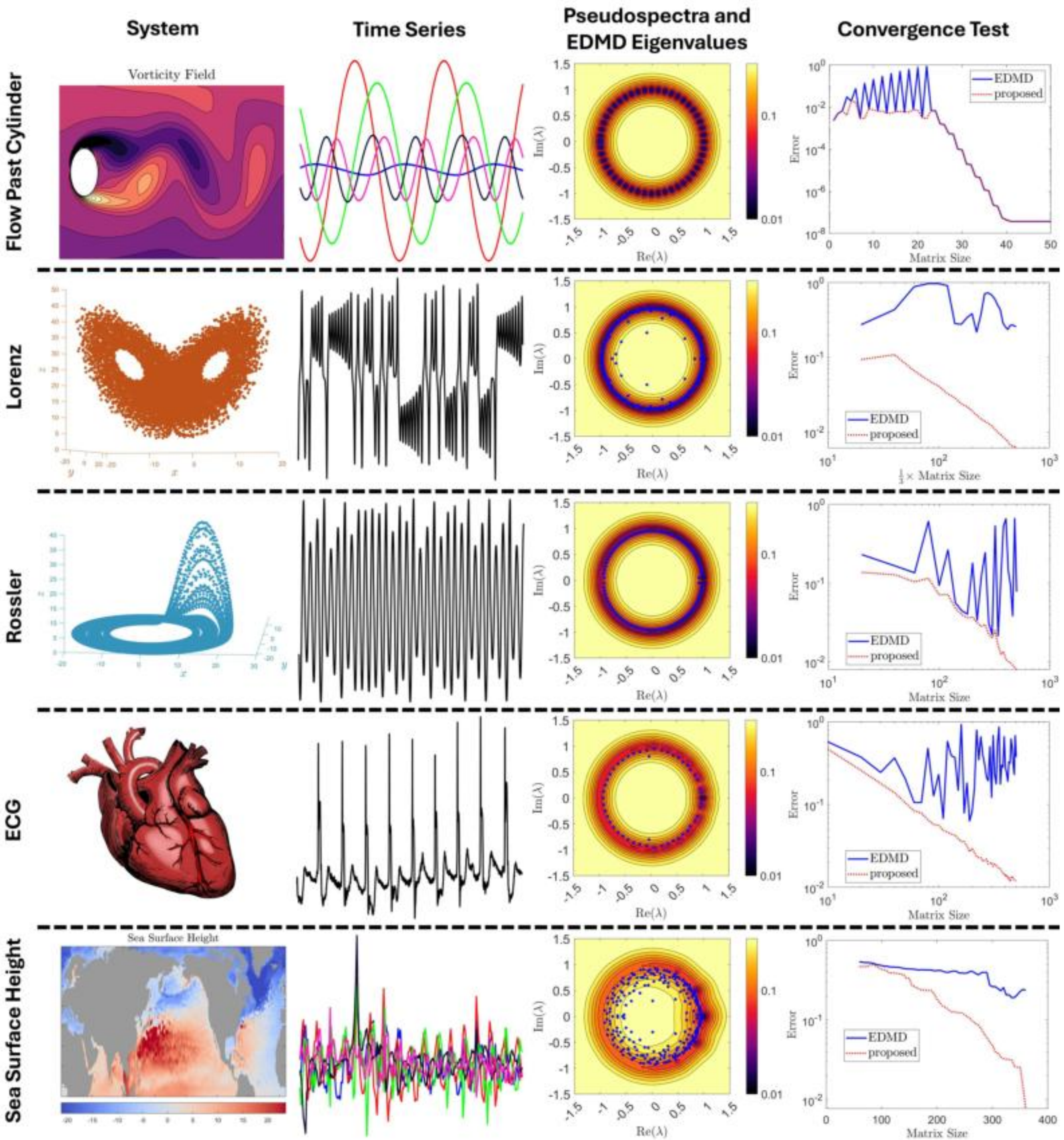
- Gaussian radial basis functions, Monte Carlo integration ($M = 50000$)

Compute $\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K})$, local adaptive control on $\varepsilon \downarrow 0$




System
Time Series
Pseudospectra and EDMD Eigenvalues
Convergence Test
Flow Past Cylinder
Vorticity Field
Lorenz
Rossler
ECG
Sea Surface Height
Im(λ)
Re(λ)
Error
Matrix Size
⅓ × Matrix Size
EDMD
proposed

Can maths help guide the way?

Consider space of observables with finite energy: $L^2(\mathcal{X}, \omega)$

Theorem: There **exists** algorithms $\Gamma_{N,M}$ using snapshots such that

$$\lim_{N\to\infty}\lim_{M\to\infty}\Gamma_{N,M}(F) = \mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}_F)$$

for all systems.

N = size of basis, **M = amount of data (quadrature)**

$$\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}) = \{z \in \mathbb{C}: \exists g, \|g\| = 1, \|\mathcal{K}g - zg\| \le \varepsilon\}$$

- C., Mezić, Stepanenko, "Adversarial Dynamical Systems Reveal Limits and Rules for Trustworthy Data-Driven Learning," **preprint**, 2025.

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Double limit $\lim_{N\to\infty}\lim_{M\to\infty}$ **Can we do better?**

- C., Mezić, Stepanenko, "Adversarial Dynamical Systems Reveal Limits and Rules for Trustworthy Data-Driven Learning," **preprint**, 2025.

Adversaries: Double limit is necessary!

Implies $\mathcal{K}$ is unitary

Class of systems: $\Omega_{\mathbb{D}} = \{F: \overline{\mathbb{D}} \to \overline{\mathbb{D}} \mid F \text{ cts, measure preserving, invertible}\}.$

Data an algorithm can use: $\mathcal{T}_F = \{(x, y_m) \mid x \in \overline{\mathbb{D}}, \|F(x) - y_m\| \leq 2^{-m}\}.$

Theorem: There **does not exist** any sequence of deterministic algorithms $\{\Gamma_n\}$ using $\mathcal{T}_F$ such that $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}_F)\ \forall F \in \Omega_{\mathbb{D}}$.

NB:

- n can index anything.
- Universal - any type of algorithm or computational model.
- Similarly, no random algorithms converging with probability $> 1/2$.

- C., Mezić, Stepanenko, "Adversarial Dynamical Systems Reveal Limits and Rules for Trustworthy Data-Driven Learning," **preprint**, 2025.

Proof idea: Constructing an adversary

F_0: rotation by π, $\mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

Phase transition lemma: Let $X = \{x_1, \dots, x_N\}$, $Y = \{y_1, \dots, y_N\}$ be distinct points in annulus $\mathcal{A} = \{x \in \mathbb{D} | 0 < R < \|x\| < r < 1\}$ with $X \cap Y = \emptyset$. There exists a measure-preserving homeomorphism H such that H acts as the identity on $\mathbb{D} \backslash \mathcal{A}$ and $H(y_j) = F_0(H(x_j)), j = 1, \dots, N$.

Conjugacy of data ($x_j \to y_j$) with F_0

Idea: Use lemma to trick any algorithm into oscillating between spectra.

- Brown and Halperin. "*On certain area-preserving maps.*" **Annals of Mathematics**, 1935.

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty}\Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

Build an adversarial F…

$\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \le 2^{-m}\}$

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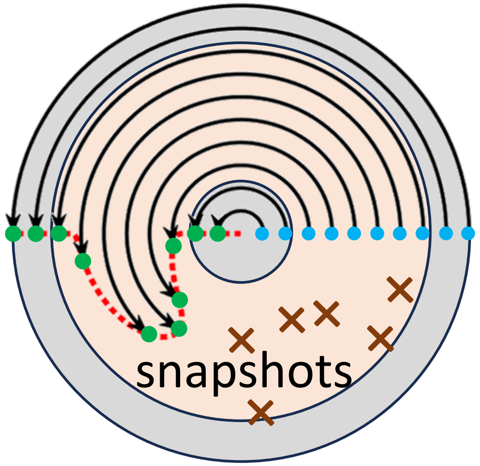

$\widetilde{F_1}(r,\theta) = \left(r, \theta + \pi + \phi(r)\right)$, $\mathrm{supp}(\phi) \subset$ [1/4, 3/4]

$\mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) = \mathbb{T}$ (unit circle).

$\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$

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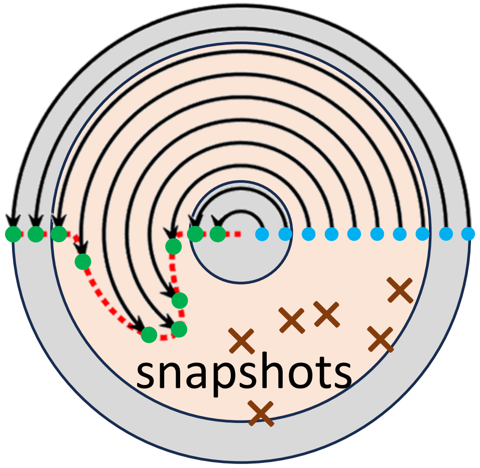
Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.
Build an adversarial F...



$\widetilde{F_1}(r,\theta) = \big(r, \theta + \pi + \phi(r)\big)$, $\mathrm{supp}(\phi) \subset [1/4, 3/4]$
$\mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) = \mathbb{T}$ (unit circle).

$\lim_{n\to\infty} \Gamma_n\big(\widetilde{F_1}\big) = \mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) \Rightarrow \exists n_1$ s.t. $\mathrm{dist}(i, \Gamma_{n_1}\big(\widetilde{F_1}\big)) \leq 1$.
BUT Γ_{n_1} uses finite amount of info to output $\Gamma_{n_1}\big(\widetilde{F_1}\big)$.
Let *X*, *Y* correspond to these snapshots.

$\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$

Proof idea: Constructing an adversary

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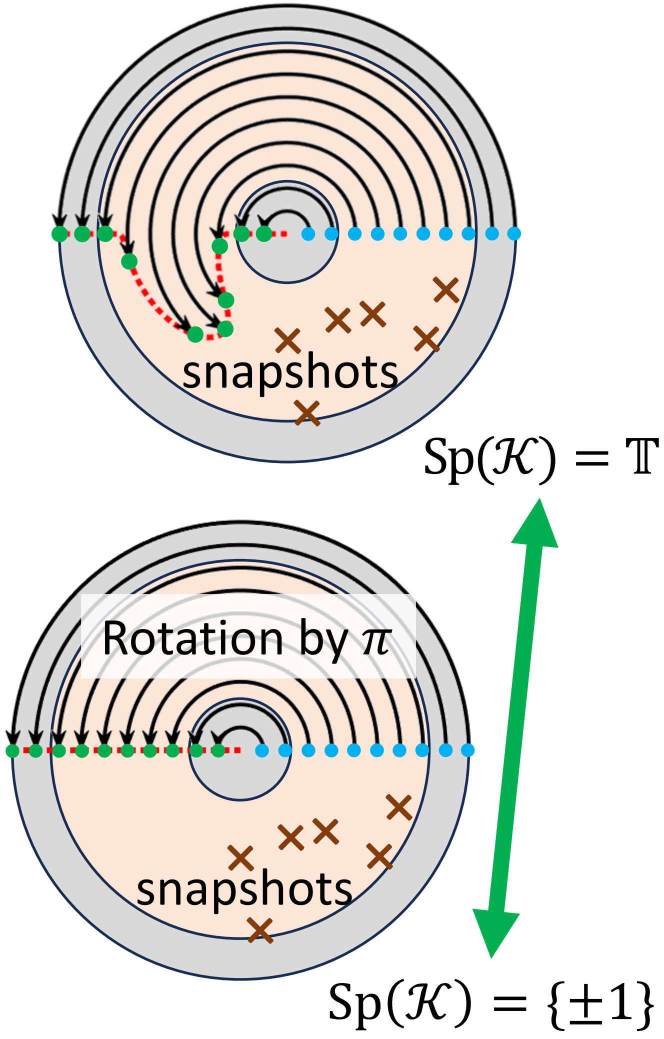


Lemma: $F_1 = H_1^{-1} \circ F_0 \circ H_1$ on annulus $\mathcal{A}_1$.
<u>Consistent data</u> $\Rightarrow \Gamma_{n_1}(F_1) = \Gamma_{n_1}\big(\widetilde{F_1}\big)$, $\mathrm{dist}(i, \Gamma_{n_1}(F_1)) \leq 1$
BUT $\mathrm{Sp}(\mathcal{K}_{F_1}) = \mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

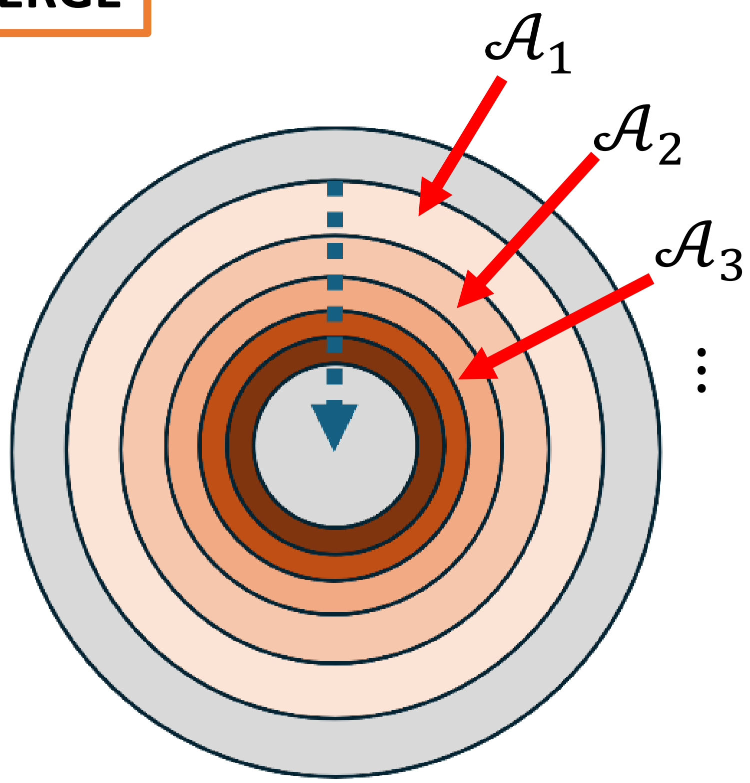
Proof idea: Constructing an adversary

Inductive step: Repeat on annuli, $F_k = H_k^{-1} \circ F_0 \circ H_k$ on $\mathcal{A}_k$. $F = \lim_{k\to\infty} F_k$

Consistent data $\Rightarrow \Gamma_{n_k}(F) = \Gamma_{n_k}(\widetilde{F_k})$, $\mathrm{dist}(i, \Gamma_{n_k}(F)) \leq 1$, $n_k \to \infty$

BUT $\mathrm{Sp}(\mathcal{K}_F) = \mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

CANNOT CONVERGE



Cascade of disks

Proof idea: Constructing an adversary

Inductive step: Repeat on annuli, $F_k = H_k^{-1} \circ F_0 \circ H_k$ on $\mathcal{A}_k$. $F = \lim_{k\to\infty} F_k$

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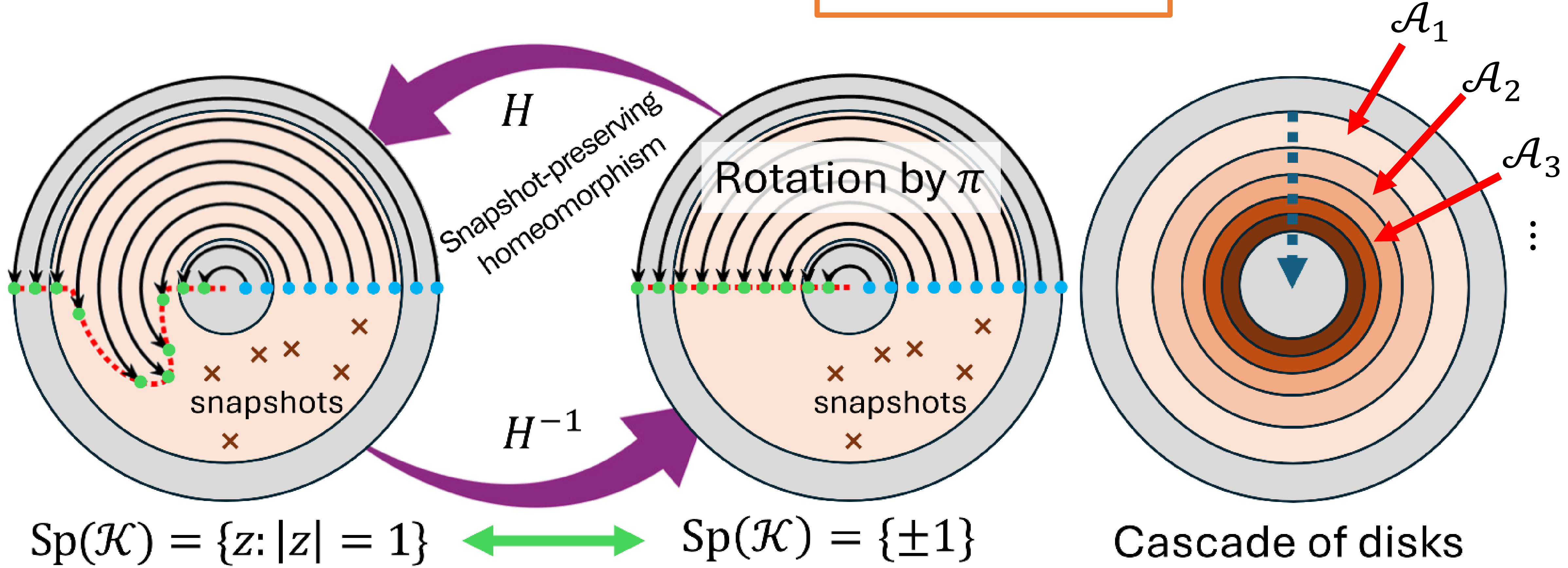

Classifications: *Solvability Complexity Index (SCI)*

SCI: Fewest number of limits needed to solve a computational problem.

- Δ_1: One limit, full error control. E.g., $d(\Gamma_n(F), \mathrm{Sp}(\mathcal{K}_F)) \leq 2^{-n}$.
- Δ_{m+1}: SCI $\leq m$.
- Σ_m: SCI $\leq m$, final limit from below.

$$\text{E.g., } \Sigma_1\text{: } \sup_{z\in\Gamma_n(F)} \mathrm{dist}(z, \mathrm{Sp}(\mathcal{K}_F)) \leq 2^{-n}.$$

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verification

trust output

covers spectrum

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Lots of SCI upper bounds lurking in Koopman literature!

SCI: Fewest number of limits needed to solve a computational problem.

Algorithm	Comments/Assumptions	Spectral Problem's Corresponding SCI Upper Bound			
		KMD	*Spectrum*	*Spectral Measure (if m.p.)*	*Spectral Type (if m.p.)*
Extended DMD [47]	general L^2 spaces	$\mathrm{SCI} \leq 2^*$	N/C	N/C	n/a
Residual DMD [44]	general L^2 spaces	$\mathrm{SCI} \leq 2^*$	$\mathrm{SCI} \leq 3^*$	$\mathrm{SCI} \leq 2^*$	varies, see [84] e.g., a.c. density: $\mathrm{SCI} \leq 2^*$
Measure-preserving EDMD [45]	m.p. systems	$\mathrm{SCI} \leq 1$	N/C	$\mathrm{SCI} \leq 2^*$ (general) $\mathrm{SCI} \leq 1$ (delay-embedding)	n/a
Hankel DMD [85]	m.p. ergodic systems	$\mathrm{SCI} \leq 2^*$	N/C	N/C	n/a
Periodic approximations [86]	m.p. + ω a.c.	$\mathrm{SCI} \leq 2$	N/C	$\mathrm{SCI} \leq 2$ (see [87])	a.c. density: $\mathrm{SCI} \leq 3$
Christoffel–Darboux kernel [40]	m.p. ergodic systems	$\mathrm{SCI} \leq 3$	n/a	$\mathrm{SCI} \leq 2$	e.g., a.c. density: $\mathrm{SCI} \leq 2$
Generator EDMD [88]	cts.-time, samples ∇F (otherwise additional limit)	$\mathrm{SCI} \leq 2$	N/C	$\mathrm{SCI} \leq 2$ (see [89])	n/a
Compactification [42]	cts.-time, m.p. ergodic systems	$\mathrm{SCI} \leq 4$	N/C	$\mathrm{SCI} \leq 4$	n/a
Resolvent compactification [43]	cts.-time, m.p. ergodic systems	$\mathrm{SCI} \leq 5$	N/C	$\mathrm{SCI} \leq 5$	n/a
Diffusion maps [90] (see also [10])	cts.-time, m.p. ergodic systems	$\mathrm{SCI} \leq 3$	n/a	n/a	n/a

Are these sharp?

Previous techniques prove upper bounds on SCI.

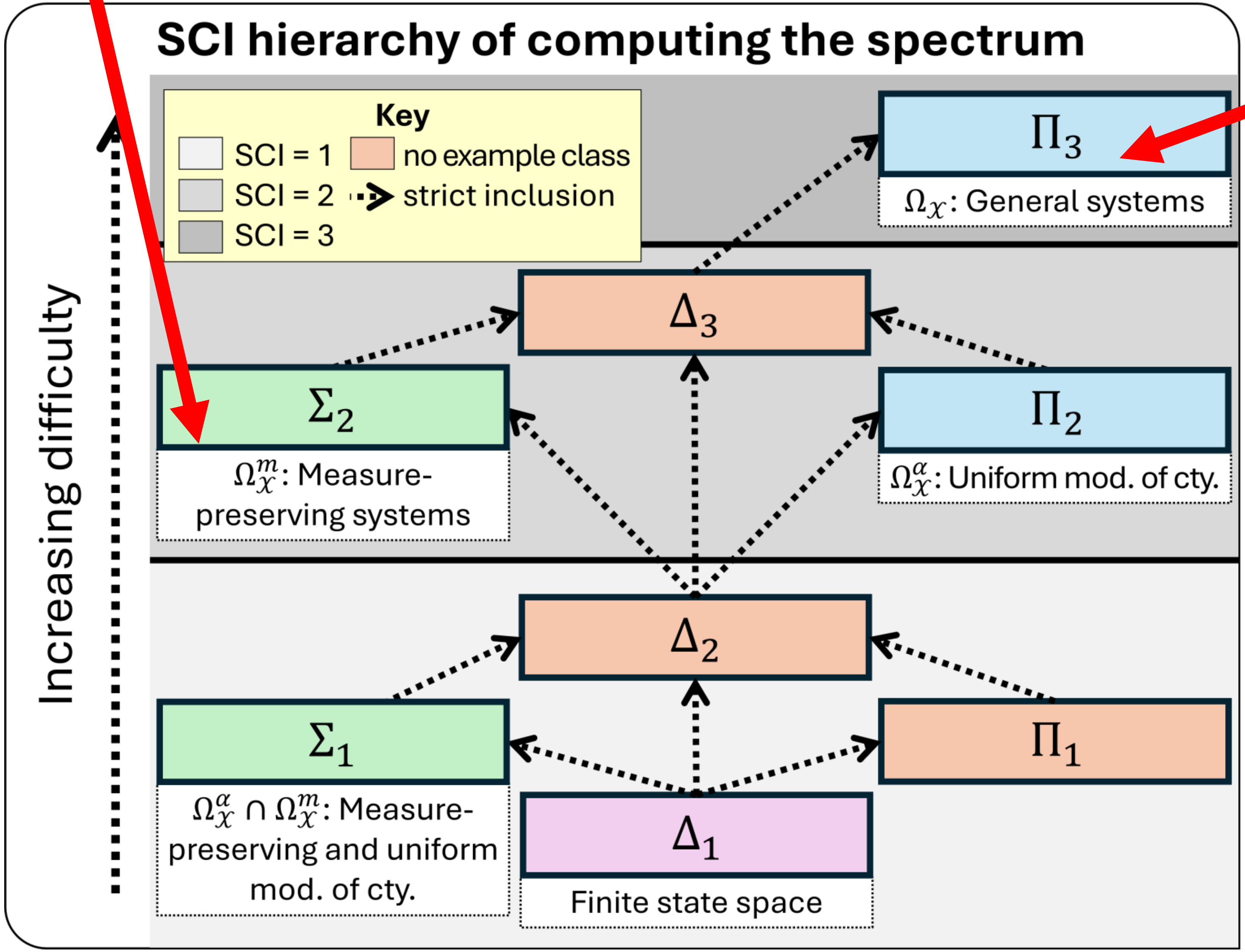
"N/C": method need not converge. "n/a": algorithm not applicable to problem.

Also in Ulam's method for Markov processes, SRB measure computation, control,…

Lower + upper bounds

Classification for Koopman

3 limits needed in general!



Different classes:

$\Omega_{\mathcal{X}} = \{F: \mathcal{X} \to \mathcal{X} \mid F \text{ cts}\}$

$\Omega_{\mathcal{X}}^{m} = \{F: \mathcal{X} \to \mathcal{X} \mid F \text{ cts, m. p.}\}$

$\Omega_{\mathcal{X}}^{\alpha} = \{F: \mathcal{X} \to \mathcal{X} \mid F \text{ mod. cty. } \alpha\}$

$[d_{\mathcal{X}}(F(x), F(y)) \leq \alpha(d_{\mathcal{X}}(x, y))]$

Optimal algorithms and classifications of dynamical systems.

- C., Mezić, Stepanenko, "Adversarial Dynamical Systems Reveal Limits and Rules for Trustworthy Data-Driven Learning," **preprint**, 2025.

Peter Lax:

"The trick of the successful mathematician is to turn the question being asked into one he knows how to answer."

Johann Wolfgang von Goethe:

"Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different."

Let's perform this trick by changing the space…

Reproducing kernel Hilbert space (RKHS)

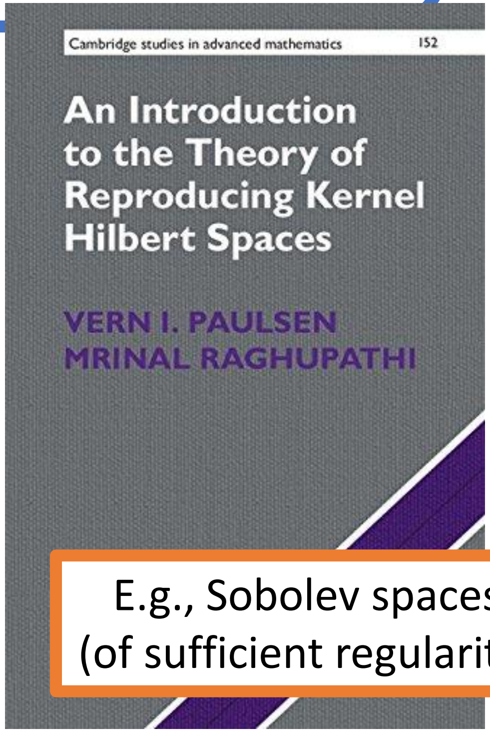
Hilbert space of functions on $\mathcal{X}$ s.t. $g \mapsto g(x)$ bounded $\forall x \in \mathcal{X}$.

Generated by a kernel $\mathfrak{K}\colon \mathcal{X} \times \mathcal{X} \to \mathbb{C}$

$$g(x) = \langle g, \mathfrak{K}_x \rangle, \qquad \mathfrak{K}(x,y) = \langle \mathfrak{K}_x, \mathfrak{K}_y \rangle = \mathfrak{K}_x(y)$$

Advantages over $L^2(\mathcal{X}, \omega)$:

- Forecasts: space bounds ⇒ pointwise bounds.
- High-dimensional systems practical through kernel trick.
- Fast methods for evaluating $\mathfrak{K}$.
- Different $\mathfrak{K}$ ⇒ different $\mathcal{K}$! Can be tailored to application. (This is where the community is currently heading.)
- Leads to fundamental “possibility” gains…



E.g., Sobolev spaces (of sufficient regularity)

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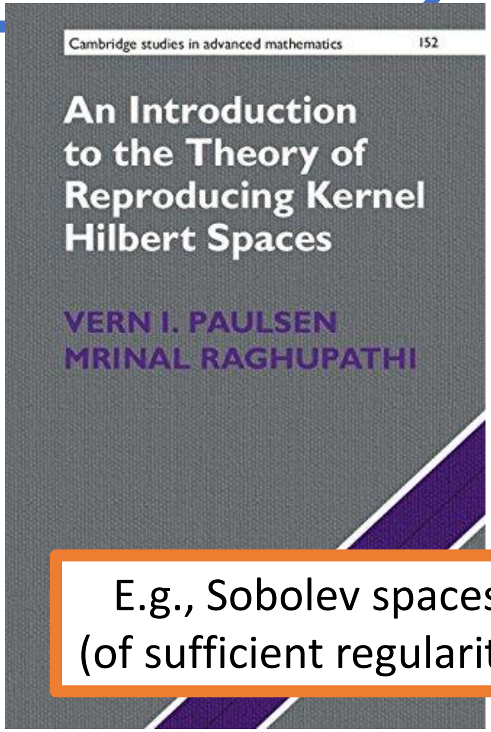

E.g., Sobolev spaces (of sufficient regularity)

SpecRKHS: Avoiding large data limit $M \to \infty$

Look at "Left eigenpairs" through $\mathcal{K}^*$: $\mathcal{K}^*\mathfrak{K}_x = \mathfrak{K}_{F(x)}$

Evolution of functionals.

$g(x) = \langle g, \mathfrak{K}_x \rangle_{\mathcal{H}}$

No quadrature needed:

$$G_{jk} = \langle \mathfrak{K}_{x^{(k)}}, \mathfrak{K}_{x^{(j)}} \rangle = \mathfrak{K}(x^{(k)}, x^{(j)})$$

$$A_{jk} = \langle \mathcal{K}^*\mathfrak{K}_{x^{(k)}}, \mathfrak{K}_{x^{(j)}} \rangle = \left\langle \mathfrak{K}_{y^{(k)}}, \mathfrak{K}_{x^{(j)}} \right\rangle = \mathfrak{K}(y^{(k)}, x^{(j)})$$

$$R_{jk} = \langle \mathcal{K}^*\mathfrak{K}_{x^{(k)}}, \mathcal{K}^*\mathfrak{K}_{x^{(j)}} \rangle = \left\langle \mathfrak{K}_{y^{(k)}}, \mathfrak{K}_{y^{(j)}} \right\rangle = \mathfrak{K}(y^{(k)}, y^{(j)})$$

$$g = \sum_{m=1}^{M} \mathbf{g}_m \mathfrak{K}_{x^{(m)}}, \qquad \|\mathcal{K}^*g - \lambda g\|_{\mathcal{H}}^2 = \mathbf{g}^*(R - \lambda A^* - \bar{\lambda} A + G)\mathbf{g}$$

- Boullé, C., Conradie, *"Convergent Methods for Koopman Operators on Reproducing Kernel Hilbert Spaces,"* **preprint**, 2025.

SpecRKHS: Example algorithm

$$\mathrm{res}^*(\lambda, \mathbf{g})^2 = \frac{\|\mathcal{K}^* g - \lambda g\|_{\mathcal{H}}^2}{\|g\|_{\mathcal{H}}^2} = \frac{\mathbf{g}^*\left[R - \lambda A^* - \bar{\lambda} A + G\right]\mathbf{g}}{\mathbf{g}^* G \mathbf{g}}$$

1. Compute $G, A, R \in \mathbb{C}^{N\times N}$ $(N = M)$
2. For z_k in grid, compute $\tau_k = \min_{g=\sum_{m=1}^{N} \mathbf{g}_m \mathfrak{K}_{x^{(m)}}} \mathrm{res}^*(z_k, \mathbf{g})$, corresponding g_k (gen. SVD).
3. **Output:** $\{z_k : \tau_k < \varepsilon\}$, $\{g_k : \tau_k < \varepsilon\}$ (ε-pseudoeigenfunctions).

First convergent method for general $\mathcal{K}$

Theorem:
- **Error control:** $\{z_k : \tau_k < \varepsilon\} \subseteq \mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}^*)$
- **Convergence:** Converges locally uniformly to $\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}^*)$ (as $N \to \infty$)

$$\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}^*) = \{z \in \mathbb{C} : \exists g, \|g\|_{\mathcal{H}} = 1, \|\mathcal{K}^* g - z g\|_{\mathcal{H}} \leq \varepsilon\}$$

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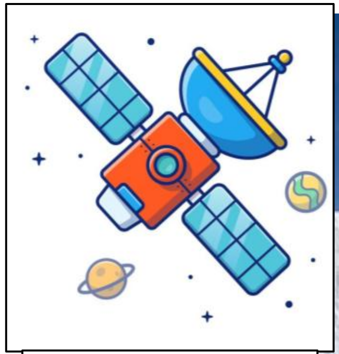
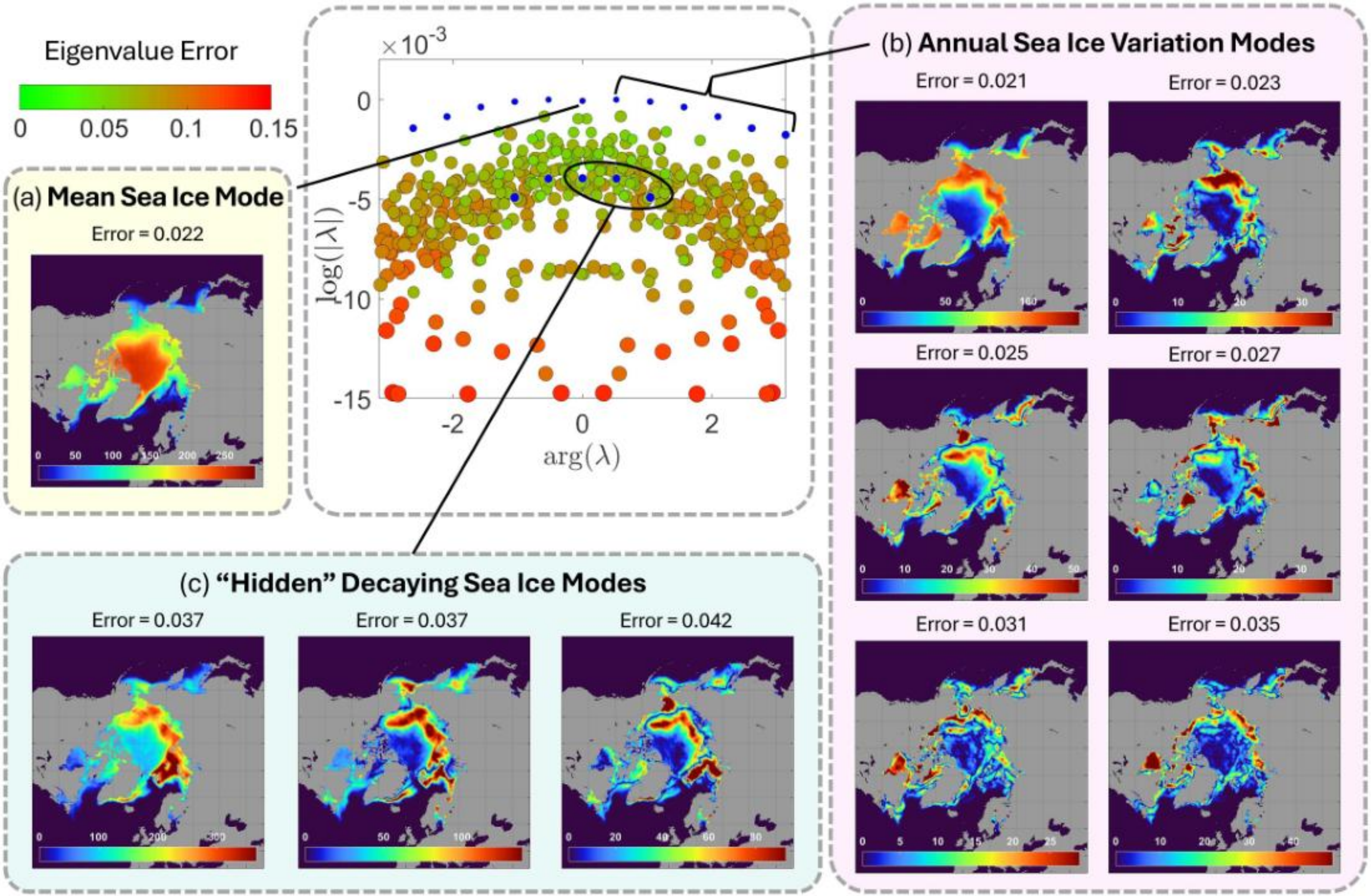
Practical gains: Sea ice forecasting



Satellite data

Motivation: Arctic amplification, polar bears, local communities, effect on extreme weather in Northern hemisphere,…

Problems: 1. Very hard to locate geographical significant regions.
2. Very hard to predict more than two months in advance.



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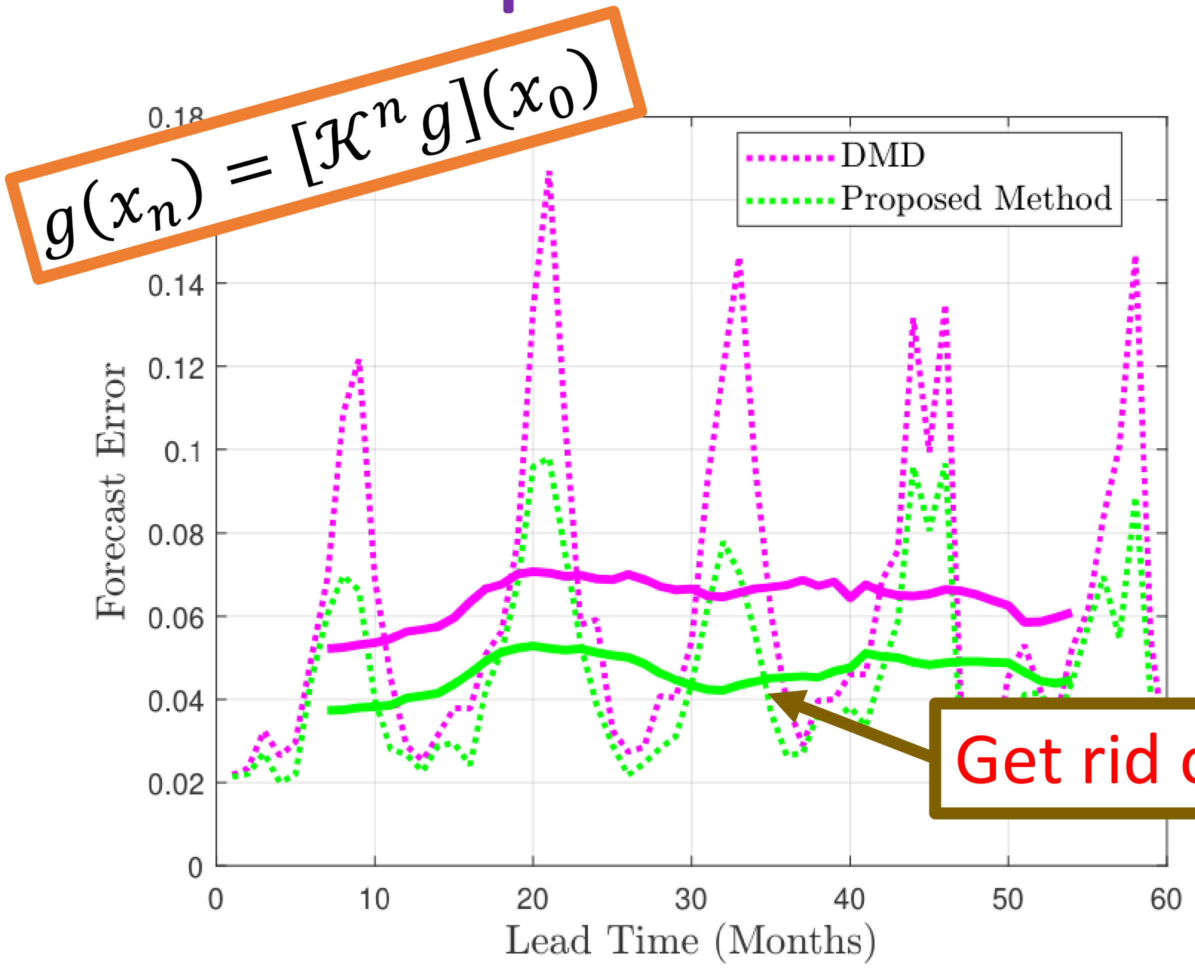
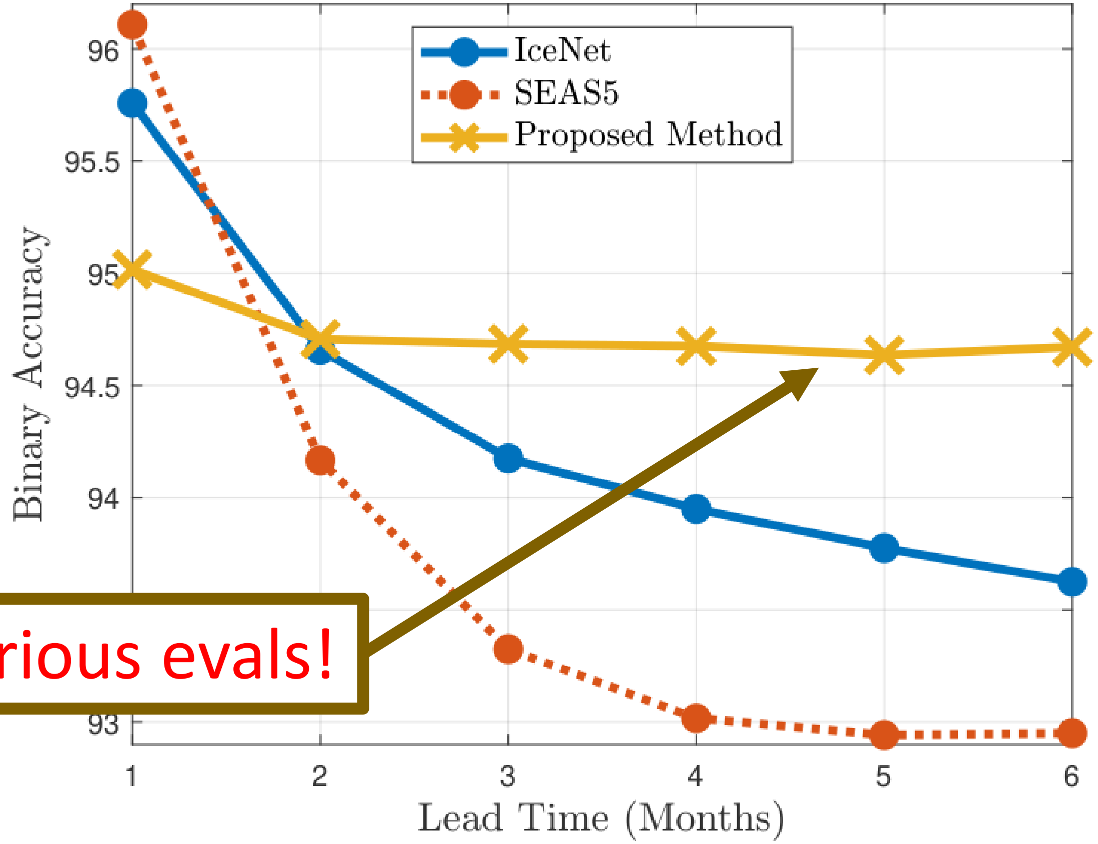
Avoid spurious evals ⇒ State-of-the-art forecasts





Relative mean squared error over 2016-2020. Model built from 2005-2015 data. (Solid lines moving 12-month mean.)

Mean binary accuracy over test years 2012-2020. *(IceNet: Andersson et al, "Seasonal Arctic sea ice forecasting with probabilistic deep learning." Nature Communications, 2021.)*

- C., Mezić, Stepanenko, "Adversarial Dynamical Systems Reveal Limits and Rules for Trustworthy Data-Driven Learning," **preprint**, 2025.

Optimal algorithms and classifications of systems

increasing difficulty →

Error control | **1 limit** | **2 limits** | **3 limits**

$$
\begin{array}{ccccccccccccc}
 & & \Pi_1 & & & & \Pi_2 & & & & \Pi_3 & & \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \cdots & \\
\Delta_1 & \subsetneq & \Sigma_1 \cup \Pi_1 & \subsetneq & \Delta_2 & \subsetneq & \Sigma_2 \cup \Pi_2 & \subsetneq & \Delta_3 & \subsetneq & \Sigma_3 \cup \Pi_3 & \cdots & \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \cdots & \\
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\end{array}
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Optimal algorithms and classifications of systems

$\varepsilon = 0$ →

increasing difficulty →

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$L^2(\mathcal{X},\omega)$

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$\Omega_{\mathcal{X}}^{m} = \{F : F\ \mathrm{cts,\ m.\ p.}\}$

$\Omega_{\mathcal{X}} = \{F : F\ \mathrm{cts}\}$

$\Omega_{\mathcal{X}}^{m} \cap \Omega_{\mathcal{X}}^{\alpha}$

$$\Pi_1 \qquad \Pi_2 \qquad \Pi_3$$

$$\Delta_1 \subsetneq \Sigma_1 \cup \Pi_1 \subsetneq \Delta_2 \subsetneq \Sigma_2 \cup \Pi_2 \subsetneq \Delta_3 \subsetneq \Sigma_3 \cup \Pi_3 \cdots$$

$$\Sigma_1 \qquad \Sigma_2 \qquad \Sigma_3$$

Optimal algorithms and classifications of systems

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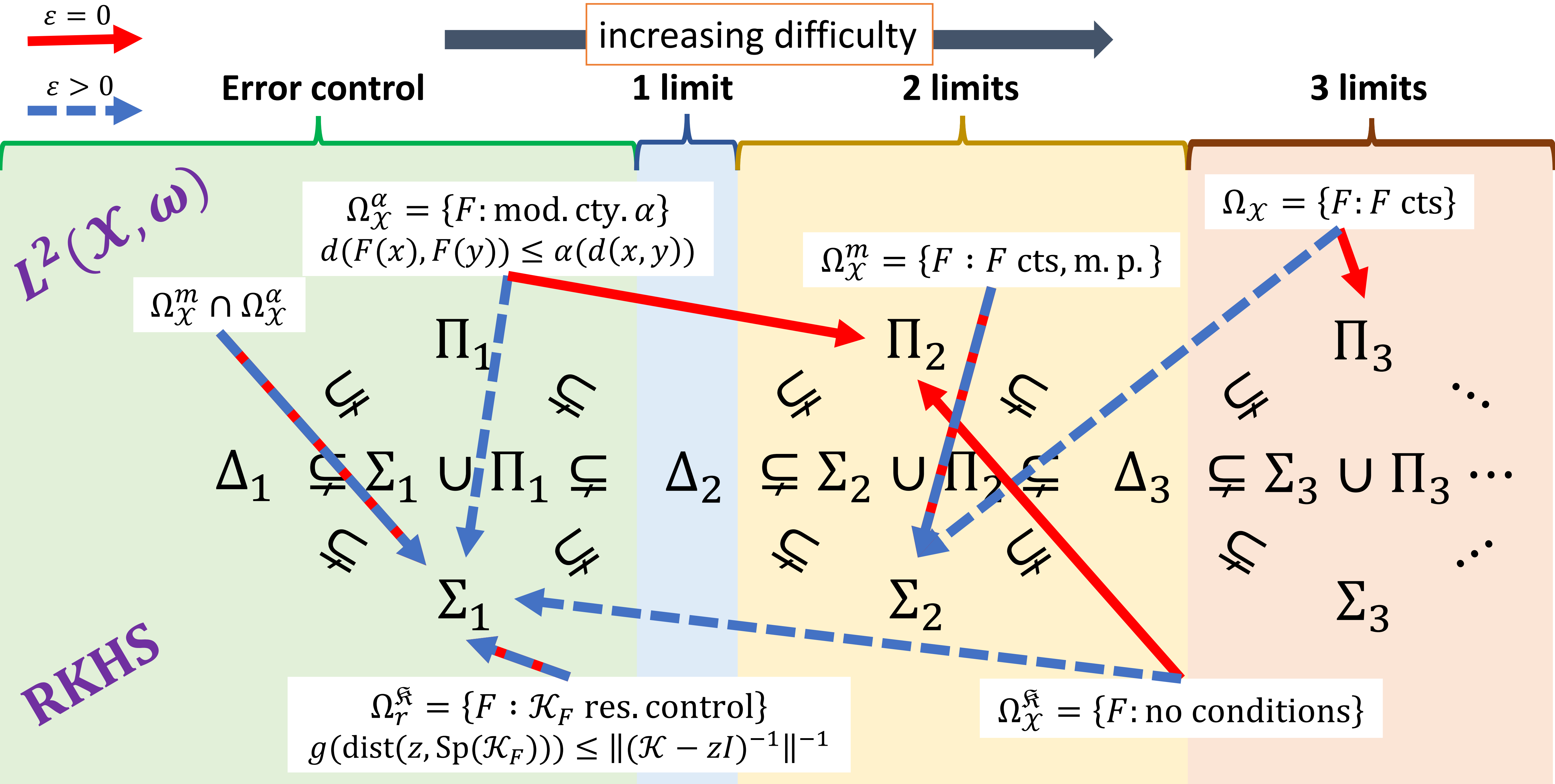
$$\Sigma_1 \qquad \Sigma_2 \qquad \Sigma_3$$

RKHS

$\Omega_{r}^{\mathfrak{K}} = \{F: \mathcal{K}_F\ \mathrm{res.\,control}\}$
$g(\mathrm{dist}(z,\mathrm{Sp}(\mathcal{K}_F))) \leq \|(\mathcal{K} - zI)^{-1}\|^{-1}$

$\Omega_{\mathcal{X}}^{\mathfrak{K}} = \{F: \mathrm{no\ conditions}\}$

Optimal algorithms and classifications of systems



Pointers

1. Data-driven spectral problems for Koopman operators are hugely popular.
   BUT: Standard truncation methods often fail.

2. **General method with convergence for spectral properties** (spectra, pseudospectra, spectral measures etc.) of K. operators!
   E.g., Verification of approximate eigenfunctions leads to practical gains.

3. ***SCI hierarchy*** classifies computational problems:
   Lower bounds through method of *adversarial dynamics*.
   Upper bounds $\Longrightarrow$ new "inf.-dim." algorithms. *Rigorous, optimal, practical.*

$\longrightarrow$ We now have a near complete picture for Koopman on $L^2(\mathcal{X}, \omega)$ and RKHS!

NB: *Similar picture has emerged for spectral measures, dealing with continuous spectra (versus eigenvalues) and spectral type (different flavors of dynamics).*

Shameless plug…

Upcoming book with CUP:

INFINITE-DIMENSIONAL SPECTRAL COMPUTATIONS

Foundations, Algorithms, and Modern Applications

100s of: classifications, algorithms, examples (webpage: full code), figures, exercises (webpage: full solutions).

Out by end of 2025 (hopefully!)…

Contents

If something interests you, please speak to me after.

Some Open Problems

1. To capture nonlinearity, infinite dimensions are crucial! Can we develop infinite-dimensional NLA to tackle these problems? Solve-then-discretize!
2. Other spaces of observables? When is this useful?
3. Data perturbation analysis almost completely missing for DMD type algorithms.
4. Stronger links between dynamical systems classes and complexity?
5. What about partial measurements? I.e., access to $h(x)$ or sketches?
6. What are classifications for control in this domain? (Linear control $\Rightarrow$ convex optimization problems.)
7. Can lower bounds be proven for PDE learning? E.g., hyperbolic PDEs.
8. Links between methods for continuous spectra (not in this talk!), quadrature, and iterative methods.
9. Continuous-time systems.
10. Links between Markov chains and LLMs - can ChatGPT be studied as a big Koopman operator?

To get started in Koopman (from a data-driven NA perspective):

- *C. "The multiverse of dynamic mode decomposition algorithms." Handbook of Numerical Analysis, 2024.*
- ***Out soon:** C., Drmač, Horning, "An Introductory Guide to Computations with Koopman Operators"*

References

[1] Colbrook, Matthew J., and Alex Townsend. "Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems." Communications on Pure and Applied Mathematics 77.1 (2024): 221-283.

[2] Colbrook, Matthew J., Lorna J. Ayton, and Máté Szőke. "Residual dynamic mode decomposition: robust and verified Koopmanism." Journal of Fluid Mechanics 955 (2023): A21.

[3] Colbrook, M. J., Li, Q., Raut, R. V., & Townsend, A. "Beyond expectations: residual dynamic mode decomposition and variance for stochastic dynamical systems." Nonlinear Dynamics 112.3 (2024): 2037-2061.

[4] Colbrook, Matthew J. "The Multiverse of Dynamic Mode Decomposition Algorithms." Handbook of Numerical Analysis, vol. 25, pp. 127-230. Elsevier, 2024..

[5] Colbrook, Matthew J. "The mpEDMD algorithm for data-driven computations of measure-preserving dynamical systems." SIAM Journal on Numerical Analysis 61.3 (2023): 1585-1608.

[6] Colbrook, Matthew J., Catherine Drysdale, and Andrew Horning. "Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators." SIAM Journal on Applied Dynamical Systems 24, no. 2 (2025): 1150-1190.

[7] Boullé, Nicolas, and Matthew J. Colbrook. "Multiplicative Dynamic Mode Decomposition." SIAM Journal on Applied Dynamical Systems 24, no. 2 (2025): 1945-1968.

[8] Boullé, Nicolas and Matthew J. Colbrook, "On the Convergence of Hermitian Dynamic Mode Decomposition" Physica D: Nonlinear Phenomena, 472 (2025).

[9] Colbrook, Matthew J., Andrew Horning, and Tianyiwa Xie. "Computing Generalized Eigenfunctions in Rigged Hilbert Spaces." arXiv preprint arXiv:2410.08343 (2024).

[10] Zagli, Niccolò, et al. "Bridging the Gap between Koopmanism and Response Theory: Using Natural Variability to Predict Forced Response." arXiv preprint arXiv:2410.01622 (2024).

[11] Colbrook, Matthew J. "Another look at Residual Dynamic Mode Decomposition in the regime of fewer Snapshots than Dictionary Size." Physica D: Nonlinear Phenomena 469 (2024).

[12] Colbrook, Matthew. "The foundations of infinite-dimensional spectral computations." Diss. University of Cambridge, 2020.

[13] Ben-Artzi, J., Colbrook, M. J., Hansen, A. C., Nevanlinna, O., & Seidel, M. (2020). "Computing Spectra--On the Solvability Complexity Index Hierarchy and Towers of Algorithms." arXiv preprint arXiv:1508.03280.

[14] Colbrook, Matthew J., Vegard Antun, and Anders C. Hansen. "The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale's 18th problem." Proceedings of the National Academy of Sciences 119.12 (2022): e2107151119.

[15] Colbrook, Matthew, Andrew Horning, and Alex Townsend. "Computing spectral measures of self-adjoint operators." SIAM review 63.3 (2021): 489-524.

[16] Colbrook, Matthew J., Bogdan Roman, and Anders C. Hansen. "How to compute spectra with error control." Physical Review Letters 122.25 (2019): 250201.

[17] Colbrook, Matthew J., and Anders C. Hansen. "The foundations of spectral computations via the solvability complexity index hierarchy." Journal of the European Mathematical Society (2022).

[18] Colbrook, Matthew J. "Computing spectral measures and spectral types." Communications in Mathematical Physics 384 (2021): 433-501.

[19] Colbrook, Matthew J., and Anders C. Hansen. "On the infinite-dimensional QR algorithm." Numerische Mathematik 143 (2019): 17-83.

[20] Colbrook, Matthew J. "On the computation of geometric features of spectra of linear operators on Hilbert spaces." Foundations of Computational Mathematics (2022): 1-82.

[21] Brunton, Steven L., and Matthew J. Colbrook. "Resilient Data-driven Dynamical Systems with Koopman: An Infinite-dimensional Numerical Analysis Perspective."

[22] Colbrook, Matthew J., Igor Mezić, and Alexei Stepanenko. "Limits and Powers of Koopman Learning." arXiv preprint arxiv:2407.06312 (2024).

[23] Herwig, April, Matthew J. Colbrook, Oliver Junge, Péter Koltai, and Julia Slipantschuk. "Avoiding spectral pollution for transfer operators using residuals." arXiv preprint arXiv:2507.16915 (2025).

[24] Boullé, Nicolas, Matthew J. Colbrook, and Gustav Conradie. "Convergent Methods for Koopman Operators on Reproducing Kernel Hilbert Spaces." arXiv preprint arXiv:2506.15782 (2025).