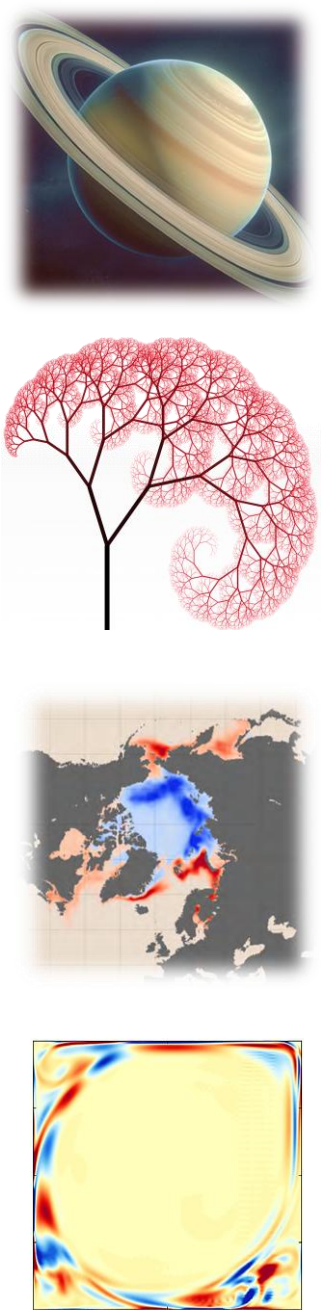




Adversarial Dynamics and Barriers in Operator Learning

Matthew Colbrook
12/5/2025



UNIVERSITY OF
CAMBRIDGE

“To classify is to bring order into chaos.” - George Pólya

Data-driven dynamical systems

- Compact metric space $(\mathcal{X}, d)$ – the state space
 - $x \in \mathcal{X}$ – the state
 - cts $F: \mathcal{X} \rightarrow \mathcal{X}$ – the dynamics: $x_{n+1} = F(x_n)$
- Dynamics (geometry)
19th century
- Borel measure ω on $\mathcal{X}$
 - Function space $L^2 = L^2(\mathcal{X}, \omega)$ (elements g called “observables”)
 - Koopman operator $\mathcal{K}_F: L^2 \rightarrow L^2; [\mathcal{K}_F g](x) = g(F(x))$
 - **Available** snapshot data: $\left\{ \left(x^{(m)}, y^{(m)} = F(x^{(m)}) \right) : m = 1, \dots, M \right\}$

NB: Pointwise definition of $\mathcal{K}_F$ needs $F\#\omega \ll \omega$ – this will hold throughout.

NB: $\mathcal{K}_F$ bounded equivalent to $dF\#\omega/d\omega \in L^\infty$ – this will hold throughout (can be dropped).

Data-driven dynamical systems

- Compact metric space $(\mathcal{X}, d)$ – the state space

- $x \in \mathcal{X}$ – the state

cts $F: \mathcal{X} \rightarrow \mathcal{X}$ – the dynamics: $x_{n+1} = F(x_n)$

Dynamics (geometry)
19th century

- Borel measure ω on $\mathcal{X}$

- Function space $L^2 = L^2(\mathcal{X}, \omega)$ (elements g called “observables”)

- Koopman operator $\mathcal{K}_F: L^2 \rightarrow L^2$; $[\mathcal{K}_F g](x) = g(F(x))$

Analysis
20th century

- Available snapshot data: $\left\{ \left(x^{(m)}, y^{(m)} = F(x^{(m)}) \right) : m = 1, \dots, M \right\}$

NB: Pointwise definition of $\mathcal{K}_F$ needs $F\#\omega \ll \omega$ – this will hold throughout.

NB: $\mathcal{K}_F$ bounded equivalent to $dF\#\omega/d\omega \in L^\infty$ – this will hold throughout (can be dropped).

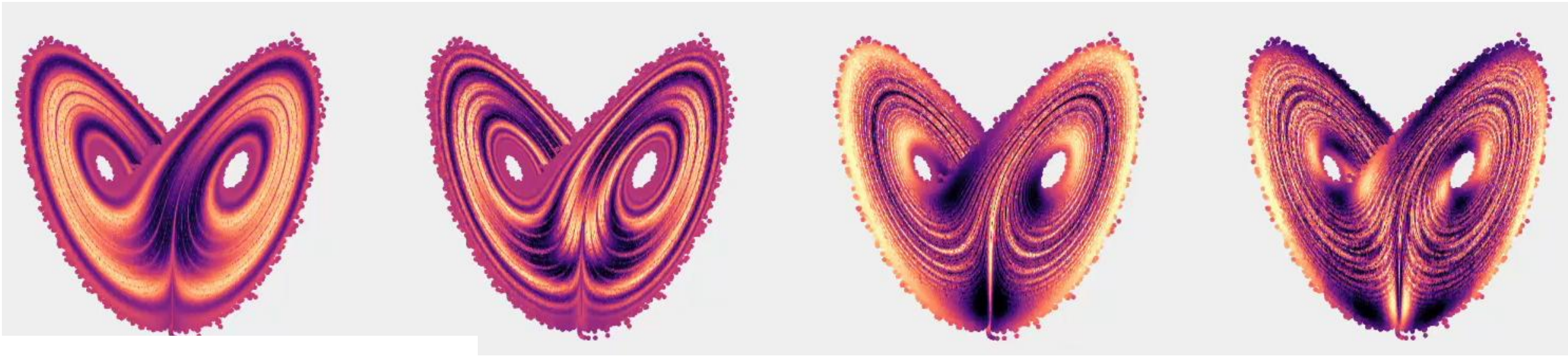
Data-driven dynamical systems

- Compact metric space $(\mathcal{X}, d)$ – the state space
 - $x \in \mathcal{X}$ – the state
 - Unknown cts $F: \mathcal{X} \rightarrow \mathcal{X}$ – the dynamics: $x_{n+1} = F(x_n)$
 - Borel measure ω on $\mathcal{X}$
 - Function space $L^2 = L^2(\mathcal{X}, \omega)$ (elements g called “observables”)
 - Koopman operator $\mathcal{K}_F: L^2 \rightarrow L^2; [\mathcal{K}_F g](x) = g(F(x))$
 - Available snapshot data: $\left\{ \left(x^{(m)}, y^{(m)} = F(x^{(m)}) \right) : m = 1, \dots, M \right\}$
- Dynamics (geometry)
19th century**
- Analysis
20th century**
- Data
21st century**

(DO NOT ASSUME MEASURE-PRESERVING)

Example

If $\|\mathcal{K}g - \lambda g\| \leq \varepsilon$, then $g(x_n) = [\mathcal{K}^n g](x_0) = \lambda^n g(x_0) + \mathcal{O}(n\varepsilon)$



Coherent features!

$$\text{Sp}_{\text{ap},\varepsilon}(\mathcal{K}) = \{z \in \mathbb{C} : \exists g, \|g\| = 1, \|\mathcal{K}g - zg\| \leq \varepsilon\}$$

Trades: Nonlinear, finite-dimensional $\Rightarrow$ Linear, infinite-dimensional.

Perils of discretization: Warmup on $\ell^2(\mathbb{Z})$

$$\begin{pmatrix} \ddots & & & & \\ & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 & \ddots \end{pmatrix} \xrightarrow{\text{Two-way infinite}} \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 1 & \\ & & & 1 & 0 \\ & & & & 0 \end{pmatrix} \in \mathbb{C}^{N \times N}$$

- Spectrum is unit circle.
- Spectrum is stable.
- Continuous spectra.
- Unitary evolution.

- Spectrum is $\{0\}$.
- Spectrum is unstable.
- Discrete spectra.
- Nilpotent evolution.

Lots of Koopman operators are built up from operators like these!

Extended Dynamic Mode Decomposition (EDMD)

Functions $\psi_j: \mathcal{X} \rightarrow \mathbb{C}, j = 1, \dots, N$

$$\{x^{(m)}, y^{(m)} = F(x^{(m)})\}_{m=1}^M$$

quadrature points

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \psi_k(x^{(m)}) = \left[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \dots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \dots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X} \right]^* \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_W \underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \dots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \dots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X} \right]_{jk}$$

quadrature weights

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \underbrace{\psi_k(y^{(m)})}_{[\mathcal{K}\psi_k](x^{(m)})} = \left[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \dots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \dots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X} \right]^* \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_W \underbrace{\begin{pmatrix} \psi_1(y^{(1)}) & \dots & \psi_N(y^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(y^{(M)}) & \dots & \psi_N(y^{(M)}) \end{pmatrix}}_{\Psi_Y} \right]_{jk}$$

Galerkin
Approximation

$$\mathcal{K} \rightarrow \mathbb{K} = (\Psi_X^* W \Psi_X)^{-1} \Psi_X^* W \Psi_Y = (\sqrt{W} \Psi_X)^\dagger \sqrt{W} \Psi_Y \in \mathbb{C}^{N \times N}$$

- Schmid, "Dynamic mode decomposition of numerical and experimental data," **J. Fluid Mech.**, 2010.
- Rowley, Mezić, Bagheri, Schlatter, Henningson, "Spectral analysis of nonlinear flows," **J. Fluid Mech.**, 2009.
- Williams, Kevrekidis, Rowley "A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition," **J. Nonlinear Sci.**, 2015.

Residual DMD (ResDMD): Approx. $\mathcal{K}$ and $\mathcal{K}^*\mathcal{K}$

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \psi_k(x^{(m)}) = \left[\underbrace{\Psi_X^* W \Psi_X}_G \right]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \underbrace{\psi_k(y^{(m)})}_{[\mathcal{K}\psi_k](x^{(m)})} = \left[\underbrace{\Psi_X^* W \Psi_Y}_{K_1} \right]_{jk}$$

- C., Townsend, “Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems,” **Commun. Pure Appl. Math.**, 2023.
- C., Ayton, Szóke, “Residual Dynamic Mode Decomposition,” **J. Fluid Mech.**, 2023.
- Code: <https://github.com/MColbrook/Residual-Dynamic-Mode-Decomposition>

Residual DMD (ResDMD): Approx. $\mathcal{K}$ and $\mathcal{K}^*\mathcal{K}$

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \psi_k(x^{(m)}) = \left[\underbrace{\Psi_X^* W \Psi_X}_G \right]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \underbrace{\psi_k(y^{(m)})}_{[\mathcal{K}\psi_k](x^{(m)})} = \left[\underbrace{\Psi_X^* W \Psi_Y}_{K_1} \right]_{jk}$$

$$\langle \mathcal{K}\psi_k, \mathcal{K}\psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(y^{(m)})} \psi_k(y^{(m)}) = \left[\underbrace{\Psi_Y^* W \Psi_Y}_{K_2} \right]_{jk}$$

- C., Townsend, “Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems,” **Commun. Pure Appl. Math.**, 2023.
- C., Ayton, Szóke, “Residual Dynamic Mode Decomposition,” **J. Fluid Mech.**, 2023.
- Code: <https://github.com/MColbrook/Residual-Dynamic-Mode-Decomposition>

Residual DMD (ResDMD): Approx. $\mathcal{K}$ and $\mathcal{K}^*\mathcal{K}$

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \psi_k(x^{(m)}) = \left[\underbrace{\Psi_X^* W \Psi_X}_G \right]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \underbrace{\psi_k(y^{(m)})}_{[\mathcal{K}\psi_k](x^{(m)})} = \left[\underbrace{\Psi_X^* W \Psi_Y}_{K_1} \right]_{jk}$$

$$\langle \mathcal{K}\psi_k, \mathcal{K}\psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(y^{(m)})} \psi_k(y^{(m)}) = \left[\underbrace{\Psi_Y^* W \Psi_Y}_{K_2} \right]_{jk}$$



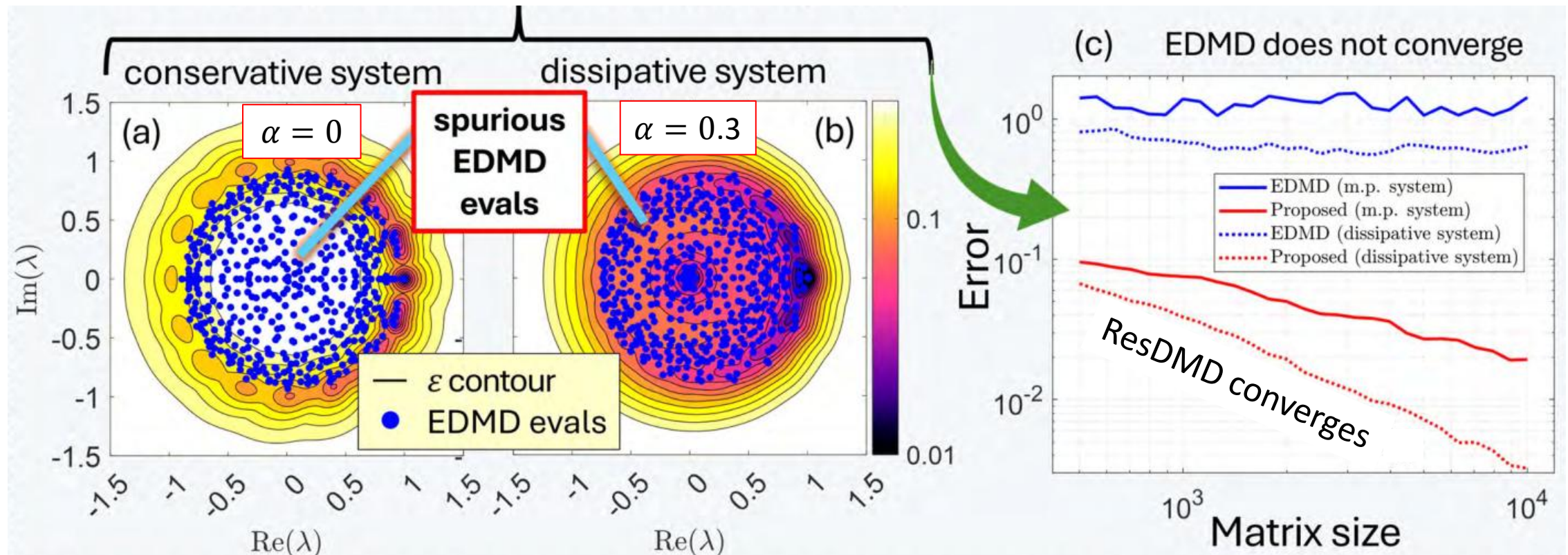
Residuals: $g = \sum_{j=1}^N \mathbf{g}_j \psi_j$, $\|\mathcal{K}g - \lambda g\|^2 = \lim_{M \rightarrow \infty} \mathbf{g}^* [K_2 - \lambda K_1^* - \bar{\lambda} K_1 + |\lambda|^2 G] \mathbf{g}$

- C., Townsend, “Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems,” **Commun. Pure Appl. Math.**, 2023.
- C., Ayton, Szóke, “Residual Dynamic Mode Decomposition,” **J. Fluid Mech.**, 2023.
- Code: <https://github.com/MColbrook/Residual-Dynamic-Mode-Decomposition>

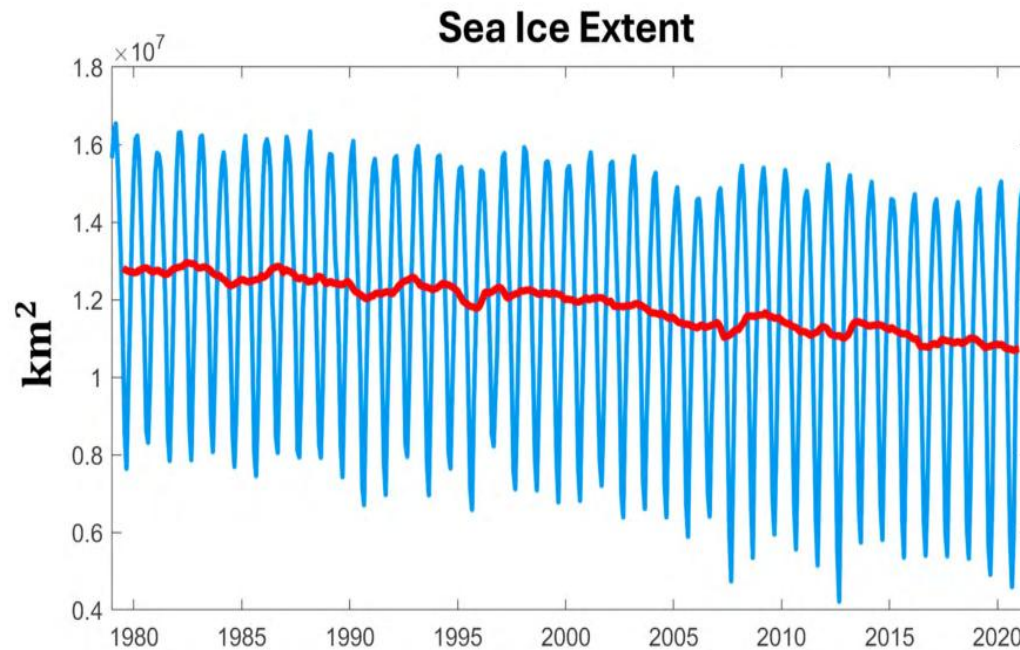
Example: EDMD doesn't converge

- Duffing oscillator: $\dot{x} = y$, $\dot{y} = -\alpha y + x(1 - x^2)$, sampled $\Delta t = 0.3$.
- Gaussian radial basis functions, Monte Carlo integration ($M = 50000$)

Compute $\text{Sp}_{\text{ap},\varepsilon}(\mathcal{K})$, local adaptive control on $\varepsilon \downarrow 0$



Practical Gains: Arctic Sea Ice Forecasting

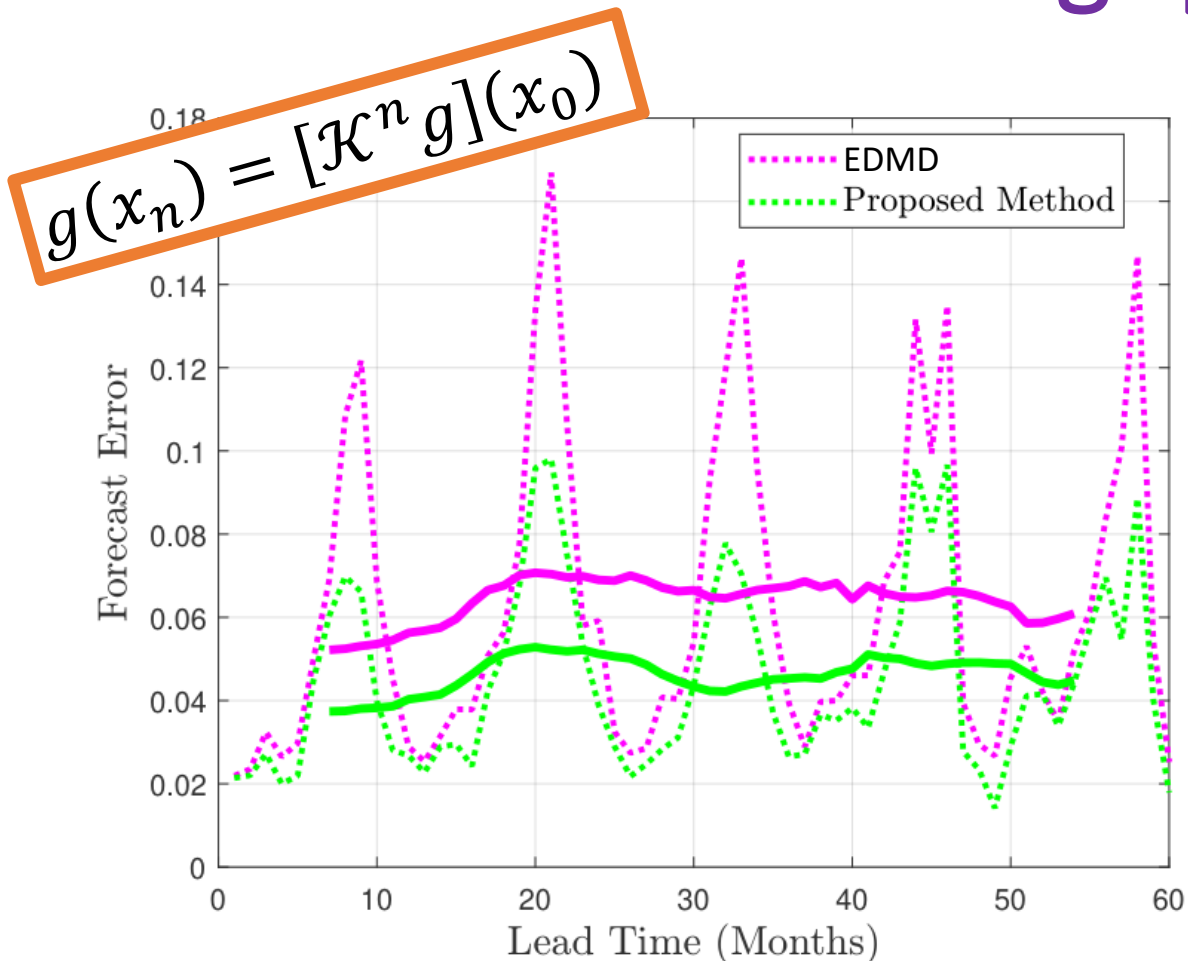


Monthly average from
satellite passive
microwave sensors.

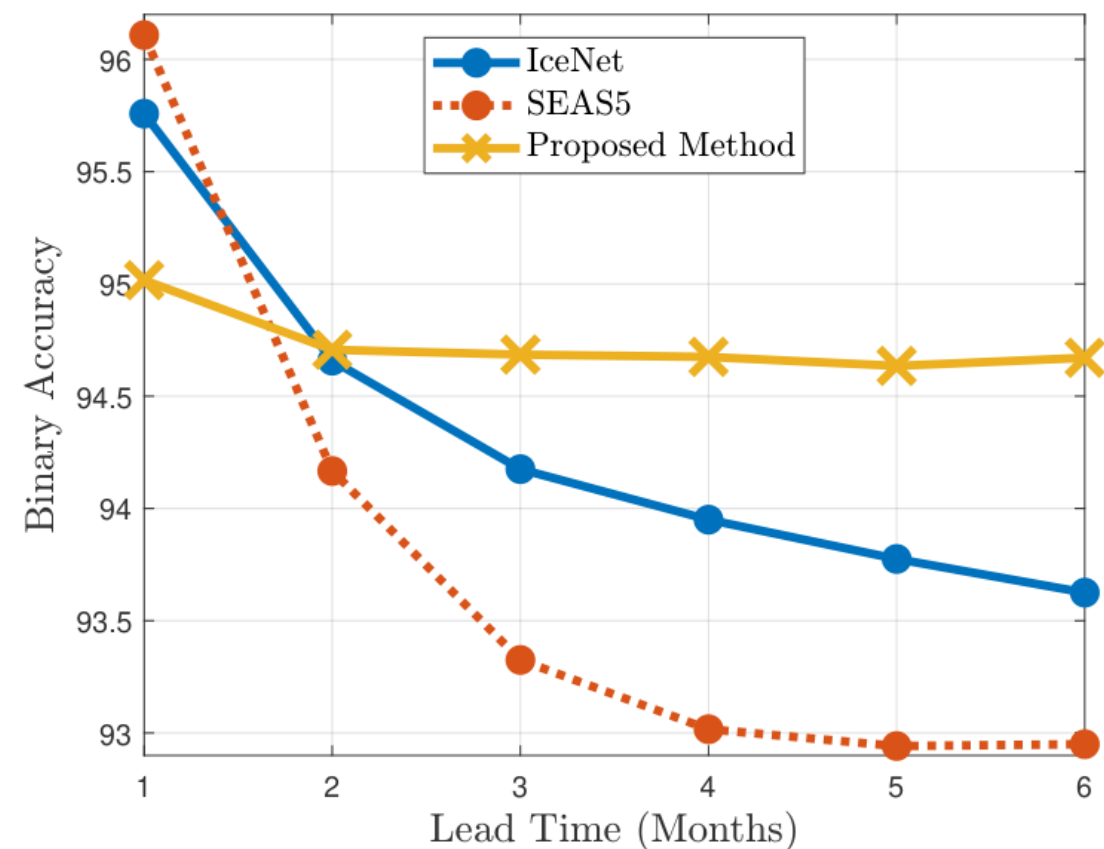
Motivation: Arctic amplification, polar bears, local communities, effect on extreme weather in Northern hemisphere,...

Problem: Very hard to predict more than two months in advance.

Arctic case: Avoiding spurious eigenvalues helps!



Relative mean squared error over 2016-2020. Model built from 2005-2015 data. (Solid lines moving 12-month mean.)



Mean binary accuracy over test years 2012-2020. (*IceNet*: Andersson et al, “Seasonal Arctic sea ice forecasting with probabilistic deep learning.” *Nature Communications*, 2021.)

Good news!

Theorem A: There **exists** *deterministic* algorithms $\{\Gamma_{N,M}\}$ using snapshots such that $\lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} \Gamma_{N,M}(F) = \text{Sp}_{\text{ap},\varepsilon}(\mathcal{K}_F)$ for all systems.

N = size of basis, M = amount of data (quadrature)

Question?

Theorem A: There **exists** *deterministic* algorithms $\{\Gamma_{N,M}\}$ using snapshots such that $\lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} \Gamma_{N,M}(F) = \text{Sp}_{\text{ap},\varepsilon}(\mathcal{K}_F)$ for all systems.

N = size of basis, M = amount of data (quadrature)

Double limit $\lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty}$

Can we do better?

Adversaries: Double limit is necessary!

Implies $\mathcal{K}$ is unitary

Class of systems: $\Omega_{\mathbb{D}} = \{F: \bar{\mathbb{D}} \rightarrow \bar{\mathbb{D}} \mid F \text{ cts, measure preserving, invertible}\}.$

Data an algorithm can use: $\mathcal{T}_F = \{(x, y_m) \mid x \in \bar{\mathbb{D}}, \|F(x) - y_m\| \leq 2^{-m}\}.$

Theorem B: There **does not exist** any sequence of deterministic algorithms $\{\Gamma_n\}$ using $\mathcal{T}_F$ such that $\lim_{n \rightarrow \infty} \Gamma_n(F) = \text{Sp}_{\text{ap}, \varepsilon}(\mathcal{K}_F) \forall F \in \Omega_{\mathbb{D}}.$

NB:

- n can index anything.
- Universal - any type of algorithm or computational model.
- Similarly, no random algorithms converging with probability $> 1/2$.

Adversaries occur with high probability.

Proof idea: Constructing an adversary

$$F_0: \text{rotation by } \pi, \operatorname{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$$

Phase transition lemma: Let $X = \{x_1, \dots, x_N\}, Y = \{y_1, \dots, y_N\}$ be distinct points in annulus $\mathcal{A} = \{x \in \mathbb{D} \mid 0 < R < \|x\| < r < 1\}$ with $X \cap Y = \emptyset$. There exists a measure-preserving homeomorphism H such that H acts as the identity on $\mathbb{D} \setminus \mathcal{A}$ and $H(y_j) = F_0(H(x_j)), j = 1, \dots, N$.

Conjugacy of data ($x_j \rightarrow y_j$) with F_0

Idea: Use lemma to trick any algorithm into oscillating between spectra.

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n \rightarrow \infty} \Gamma_n(F) = \text{Sp}(\mathcal{K}_F) \quad \forall F \in \Omega_{\mathbb{D}}$.

Build an **adversarial** F ...

$$\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$$

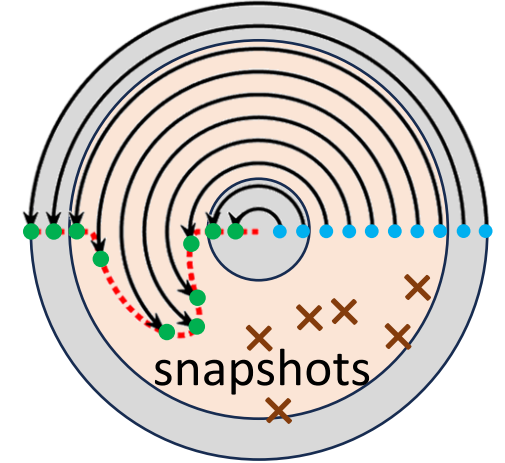
Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n \rightarrow \infty} \Gamma_n(F) = \text{Sp}(\mathcal{K}_F) \quad \forall F \in \Omega_{\mathbb{D}}$.

Build an **adversarial** F ...

$$\widetilde{F}_1(r, \theta) = (r, \theta + \pi + \phi(r)), \text{ supp}(\phi) \subset [1/4, 3/4]$$

$$\text{Sp}(\mathcal{K}_{\widetilde{F}_1}) = \mathbb{T} \text{ (unit circle).}$$



$$\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$$

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n \rightarrow \infty} \Gamma_n(F) = \text{Sp}(\mathcal{K}_F) \quad \forall F \in \Omega_{\mathbb{D}}$.

Build an **adversarial** F ...

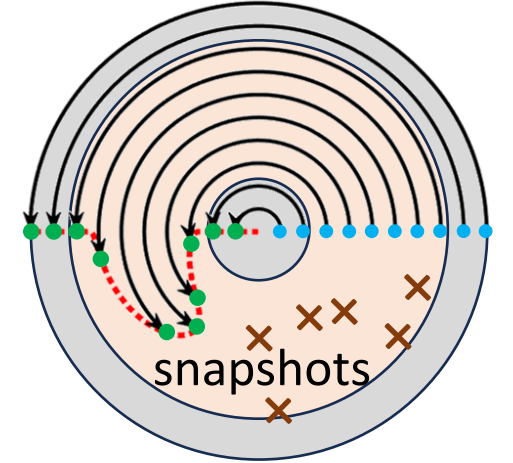
$$\widetilde{F}_1(r, \theta) = (r, \theta + \pi + \phi(r)), \text{ supp}(\phi) \subset [1/4, 3/4]$$

$$\text{Sp}(\mathcal{K}_{\widetilde{F}_1}) = \mathbb{T} \text{ (unit circle).}$$

$$\lim_{n \rightarrow \infty} \Gamma_n(\widetilde{F}_1) = \text{Sp}(\mathcal{K}_{\widetilde{F}_1}) \Rightarrow \exists n_1 \text{ s.t. } \text{dist}(i, \Gamma_{n_1}(\widetilde{F}_1)) \leq 1.$$

BUT Γ_{n_1} uses finite amount of info to output $\Gamma_{n_1}(\widetilde{F}_1)$.

Let X, Y correspond to these snapshots.



$$\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$$

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n \rightarrow \infty} \Gamma_n(F) = \text{Sp}(\mathcal{K}_F) \quad \forall F \in \Omega_{\mathbb{D}}$.

Build an **adversarial** F ...

$$\widetilde{F}_1(r, \theta) = (r, \theta + \pi + \phi(r)), \text{supp}(\phi) \subset [1/4, 3/4]$$

$$\text{Sp}(\mathcal{K}_{\widetilde{F}_1}) = \mathbb{T} \text{ (unit circle).}$$

$$\lim_{n \rightarrow \infty} \Gamma_n(\widetilde{F}_1) = \text{Sp}(\mathcal{K}_{\widetilde{F}_1}) \Rightarrow \exists n_1 \text{ s.t. } \text{dist}(i, \Gamma_{n_1}(\widetilde{F}_1)) \leq 1.$$

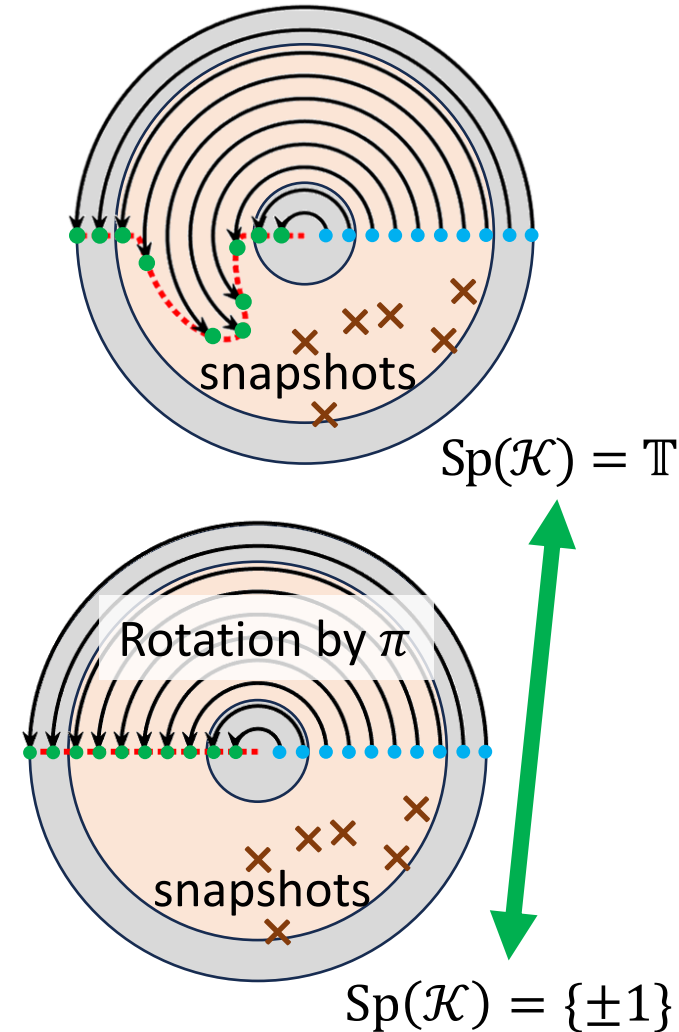
BUT Γ_{n_1} uses finite amount of info to output $\Gamma_{n_1}(\widetilde{F}_1)$.

Let X, Y correspond to these snapshots.

Lemma: $F_1 = H_1^{-1} \circ F_0 \circ H_1$ on annulus $\mathcal{A}_1$.

Consistent data $\Rightarrow \Gamma_{n_1}(F_1) = \Gamma_{n_1}(\widetilde{F}_1)$, $\text{dist}(i, \Gamma_{n_1}(F_1)) \leq 1$

BUT $\text{Sp}(\mathcal{K}_{F_1}) = \text{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$



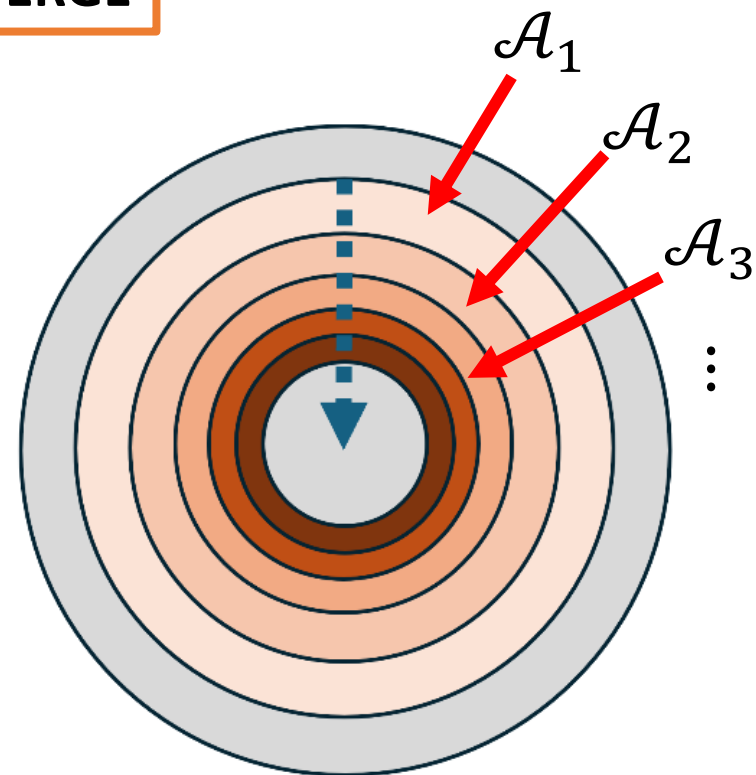
Proof idea: Constructing an adversary

Inductive step: Repeat on annuli, $F_k = H_k^{-1} \circ F_0 \circ H_k$ on $\mathcal{A}_k$. $F = \lim_{k \rightarrow \infty} F_k$

Consistent data $\Rightarrow \Gamma_{n_k}(F) = \Gamma_{n_k}(\widetilde{F_k})$, $\text{dist}(i, \Gamma_{n_k}(F)) \leq 1$, $n_k \rightarrow \infty$

BUT $\text{Sp}(\mathcal{K}_F) = \text{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

CANNOT CONVERGE



Cascade of disks

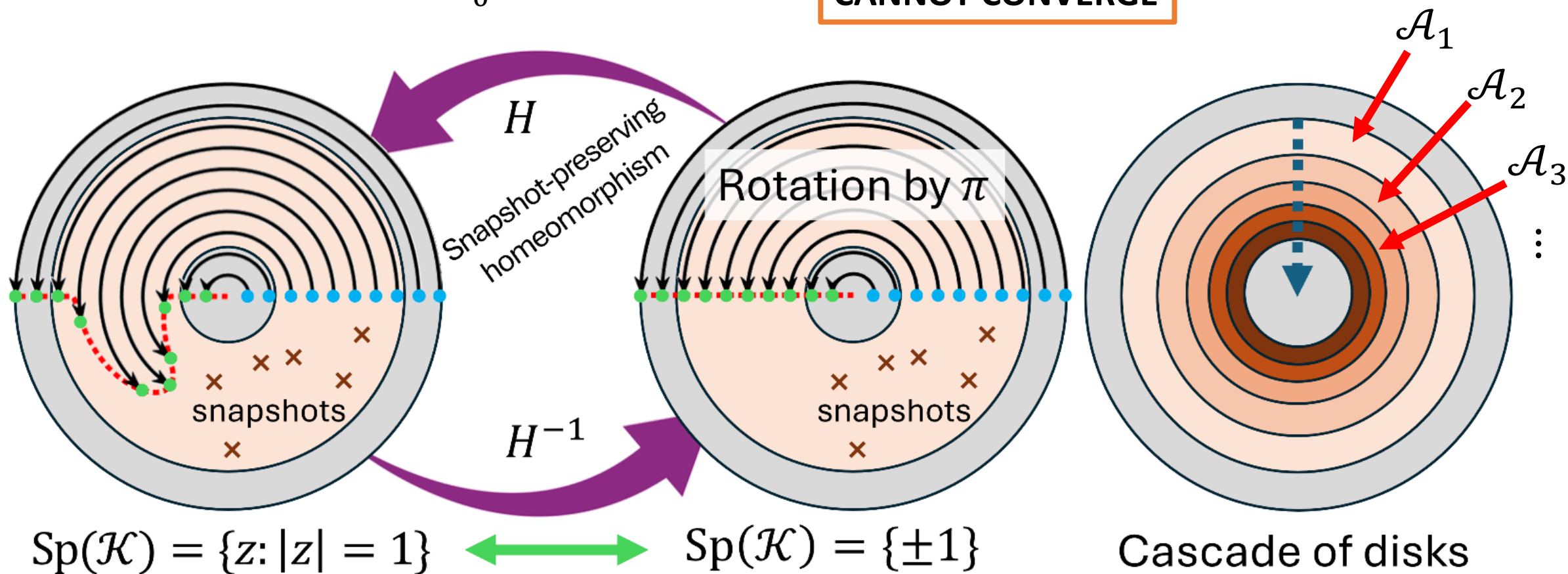
Proof idea: Constructing an adversary

Inductive step: Repeat on annuli, $F_k = H_k^{-1} \circ F_0 \circ H_k$ on $\mathcal{A}_k$. $F = \lim_{k \rightarrow \infty} F_k$

Consistent data $\Rightarrow \Gamma_{n_k}(F) = \Gamma_{n_k}(\widetilde{F}_k)$, $\text{dist}(i, \Gamma_{n_k}(F)) \leq 1$, $n_k \rightarrow \infty$

BUT $\text{Sp}(\mathcal{K}_F) = \text{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

CANNOT CONVERGE



Classifications: *Solvability Complexity Index (SCI)*

SCI: Fewest number of limits needed to solve a computational problem.

- Δ_1 : One limit, full error control. E.g., $d(\Gamma_n(F), \text{Sp}(\mathcal{K}_F)) \leq 2^{-n}$.
- Δ_{m+1} : $\text{SCI} \leq m$.
- Σ_m : $\text{SCI} \leq m$, final limit from below.

$$\text{E.g., } \Sigma_1: \sup_{z \in \Gamma_n(F)} \text{dist}(z, \text{Sp}(\mathcal{K}_F)) \leq 2^{-n}.$$

- Π_m : $\text{SCI} \leq m$, final limit from above.

$$\text{E.g., } \Pi_1: \sup_{z \in \text{Sp}(\mathcal{K}_F)} \text{dist}(z, \Gamma_n(F)) \leq 2^{-n}.$$

-
- Hansen, "On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators." **J. Am. Math. Soc.**, 2011.
 - C., "The foundations of infinite-dimensional spectral computations," **PhD diss.**, University of Cambridge, 2020.
 - C., Hansen, "The foundations of spectral computations via the solvability complexity index hierarchy," **J. Eur. Math. Soc.**, 2022.
 - C., Antun, Hansen, "The difficulty of computing stable and accurate neural networks," **Proc. Natl. Acad. Sci. USA**, 2022.
 - Ben-Artzi, C., Hansen, Nevanlinna, Seidel, "On the solvability complexity index hierarchy and towers of algorithms," arXiv, 2020.

Classifications: *Solvability Complexity Index (SCI)*

SCI: Fewest number of limits needed to solve a computational problem.

- Δ_1 : One limit, full error control. E.g., $d(\Gamma_n(F), \text{Sp}(\mathcal{K}_F)) \leq 2^{-n}$.
- Δ_{m+1} : $\text{SCI} \leq m$.
- Σ_m : $\text{SCI} \leq m$, final limit from below.

$$\text{E.g., } \Sigma_1: \sup_{z \in \Gamma_n(F)} \text{dist}(z, \text{Sp}(\mathcal{K}_F)) \leq 2^{-n}.$$

- Π_m : $\text{SCI} \leq m$, final limit from above.

$$\text{E.g., } \Pi_1: \sup_{z \in \text{Sp}(\mathcal{K}_F)} \text{dist}(z, \Gamma_n(F)) \leq 2^{-n}.$$

trust output

verification

covers spectrum

- Hansen, "On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators." *J. Am. Math. Soc.*, 2011.
- C., "The foundations of infinite-dimensional spectral computations," **PhD diss.**, University of Cambridge, 2020.
- C., Hansen, "The foundations of spectral computations via the solvability complexity index hierarchy," *J. Eur. Math. Soc.*, 2022.
- C., Antun, Hansen, "The difficulty of computing stable and accurate neural networks," *Proc. Natl. Acad. Sci. USA*, 2022.
- Ben-Artzi, C., Hansen, Nevanlinna, Seidel, "On the solvability complexity index hierarchy and towers of algorithms," arXiv, 2020.

Lots of SCI upper bounds lurking in Koopman literature!

SCI: Fewest number of limits needed to solve a computational problem.

Algorithm	Comments/Assumptions	Spectral Problem's Corresponding SCI Upper Bound			
		<i>KMD</i>	<i>Spectrum</i>	<i>Spectral Measure (if m.p.)</i>	<i>Spectral Type (if m.p.)</i>
Extended DMD [47]	general L^2 spaces	$\text{SCI} \leq 2^*$	N/C	N/C	n/a
Residual DMD [44]	general L^2 spaces	$\text{SCI} \leq 2^*$	$\text{SCI} \leq 3^*$	$\text{SCI} \leq 2^*$	varies, see [84] e.g., a.c. density: $\text{SCI} \leq 2^*$
Measure-preserving EDMD [45]	m.p. systems	$\text{SCI} \leq 1$	N/C	$\text{SCI} \leq 2^*$ (general) $\text{SCI} \leq 1$ (delay-embedding)	n/a
Hankel DMD [85]	m.p. ergodic systems	$\text{SCI} \leq 2^*$	N/C	N/C	n/a
Periodic approximations [86]	m.p. + ω a.c.	$\text{SCI} \leq 2$	N/C	$\text{SCI} \leq 2$ (see [87])	a.c. density: $\text{SCI} \leq 3$
Christoffel–Darboux kernel [40]	m.p. ergodic systems	$\text{SCI} \leq 3$	n/a	$\text{SCI} \leq 2$	e.g., a.c. density: $\text{SCI} \leq 2$
Generator EDMD [88]	cts.-time, samples ∇F (otherwise additional limit)	$\text{SCI} \leq 2$	N/C	$\text{SCI} \leq 2$ (see [89])	n/a
Compactification [42]	cts.-time, m.p. ergodic systems	$\text{SCI} \leq 4$	N/C	$\text{SCI} \leq 4$	n/a
Resolvent compactification [43]	cts.-time, m.p. ergodic systems	$\text{SCI} \leq 5$	N/C	$\text{SCI} \leq 5$	n/a
Diffusion maps [90] (see also [10])	cts.-time, m.p. ergodic systems	$\text{SCI} \leq 3$	n/a	n/a	n/a

Are these sharp?

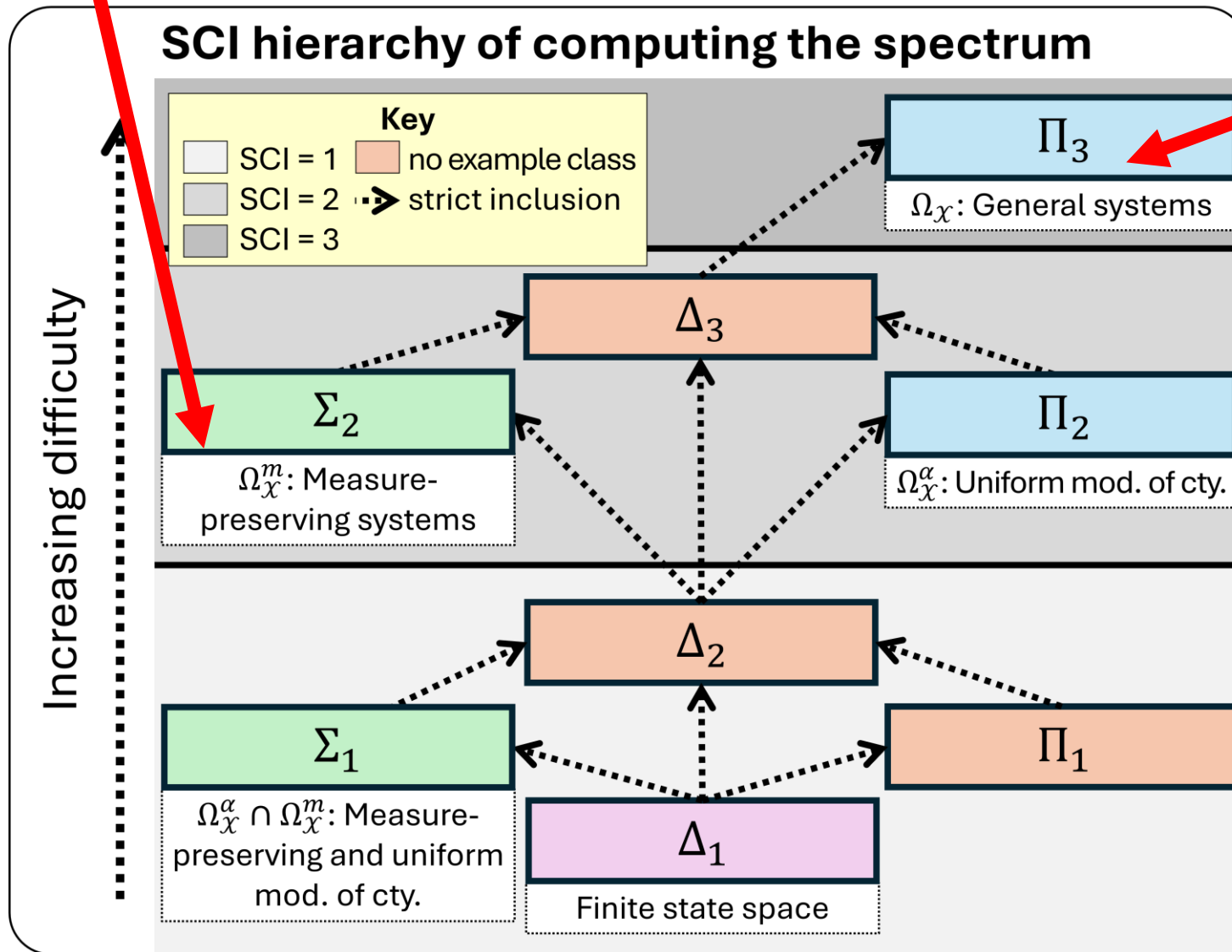
Previous techniques prove upper bounds on SCI.

“N/C”: method need not converge. “n/a”: algorithm not applicable to problem.

Also in Ulam’s method for Markov processes, SRB measure computation, control,...

Classification for Koopman

3 limits needed in general!



Different classes:

$$\Omega_X = \{F: X \rightarrow X \mid F \text{ cts}\}$$

$$\Omega_X^m = \{F: X \rightarrow X \mid F \text{ cts, m. p.}\}$$

$$\Omega_X^\alpha = \{F: X \rightarrow X \mid F \text{ mod. cty. } \alpha\}$$

$$[d_X(F(x), F(y)) \leq \alpha(d_X(x, y))]$$

Optimal algorithms and classifications of dynamical systems.

Conclusion: FOUNDATIONS $\leftrightarrow$ METHODS

- Data-driven spectral problems for Koopman operators are hugely popular.

BUT: Standard truncation methods often fail.

- *SCI hierarchy* classifies computational problems:

Lower bounds through method of adversarial dynamics.

Upper bounds $\Rightarrow$ new “inf.-dim.” algorithms. Rigorous, optimal, practical.

(spectra, pseudospectra, spectral measures etc.)

E.g., Verification of approximate eigenfunctions leads to practical gains.

$\rightarrow$ **We now have a near complete picture for Koopman on $L^2(\mathcal{X}, \omega)$!**

NB: *Similar story for spectral measures, dealing with continuous spectra (versus eigenvalues) and spectral type (different flavors of dynamics).*

References

- [1] Colbrook, Matthew J., and Alex Townsend. "Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems." *Communications on Pure and Applied Mathematics* 77.1 (2024): 221-283.
- [2] Colbrook, Matthew J., Loma J. Ayton, and Máté Szőke. "Residual dynamic mode decomposition: robust and verified Koopmanism." *Journal of Fluid Mechanics* 955 (2023): A21.
- [3] Colbrook, M. J., Li, Q., Raut, R. V., & Townsend, A. "Beyond expectations: residual dynamic mode decomposition and variance for stochastic dynamical systems." *Nonlinear Dynamics* 112.3 (2024): 2037-2061.
- [4] Colbrook, Matthew J. "The Multiverse of Dynamic Mode Decomposition Algorithms." *Handbook of Numerical Analysis*, vol. 25, pp. 127-230. Elsevier, 2024..
- [5] Colbrook, Matthew J. "The mpEDMD algorithm for data-driven computations of measure-preserving dynamical systems." *SIAM Journal on Numerical Analysis* 61.3 (2023): 1585-1608.
- [6] Colbrook, Matthew J., Catherine Drysdale, and Andrew Horning. "Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators." *SIAM Journal on Applied Dynamical Systems* 24, no. 2 (2025): 1150-1190.
- [7] Boullé, Nicolas, and Matthew J. Colbrook. "Multiplicative Dynamic Mode Decomposition." *SIAM Journal on Applied Dynamical Systems* 24, no. 2 (2025): 1945-1968.
- [8] Boullé, Nicolas and Matthew J. Colbrook, "On the Convergence of Hermitian Dynamic Mode Decomposition" *Physica D: Nonlinear Phenomena*, 472 (2025).
- [9] Colbrook, Matthew J., Andrew Horning, and Tianyiwa Xie. "Computing Generalized Eigenfunctions in Rigged Hilbert Spaces." *arXiv preprint arXiv:2410.08343* (2024).
- [10] Zagli, Niccolò, et al. "Bridging the Gap between Koopmanism and Response Theory: Using Natural Variability to Predict Forced Response." *arXiv preprint arXiv:2410.01622* (2024).
- [11] Colbrook, Matthew J. "Another look at Residual Dynamic Mode Decomposition in the regime of fewer Snapshots than Dictionary Size." *Physica D: Nonlinear Phenomena* 469 (2024).
- [12] Colbrook, Matthew. "The foundations of infinite-dimensional spectral computations." *Diss. University of Cambridge*, 2020.
- [13] Ben-Artzi, J., Colbrook, M. J., Hansen, A. C., Nevanlinna, O., & Seidel, M. (2020). "Computing Spectra—On the Solvability Complexity Index Hierarchy and Towers of Algorithms." *arXiv preprint arXiv:1508.03280*.
- [14] Colbrook, Matthew J., Vegard Antun, and Anders C. Hansen. "The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale's 18th problem." *Proceedings of the National Academy of Sciences* 119.12 (2022): e2107151119.
- [15] Colbrook, Matthew, Andrew Horning, and Alex Townsend. "Computing spectral measures of self-adjoint operators." *SIAM review* 63.3 (2021): 489-524.
- [16] Colbrook, Matthew J., Bogdan Roman, and Anders C. Hansen. "How to compute spectra with error control." *Physical Review Letters* 122.25 (2019): 250201.
- [17] Colbrook, Matthew J., and Anders C. Hansen. "The foundations of spectral computations via the solvability complexity index hierarchy." *Journal of the European Mathematical Society* (2022).
- [18] Colbrook, Matthew J. "Computing spectral measures and spectral types." *Communications in Mathematical Physics* 384 (2021): 433-501.
- [19] Colbrook, Matthew J., and Anders C. Hansen. "On the infinite-dimensional QR algorithm." *Numerische Mathematik* 143 (2019): 17-83.
- [20] Colbrook, Matthew J. "On the computation of geometric features of spectra of linear operators on Hilbert spaces." *Foundations of Computational Mathematics* (2022): 1-82.
- [21] Brunton, Steven L., and Matthew J. Colbrook. "Resilient Data-driven Dynamical Systems with Koopman: An Infinite-dimensional Numerical Analysis Perspective."
- [22] Colbrook, Matthew J., Igor Mezić, and Alexei Stepanenko. "Limits and Powers of Koopman Learning." *arXiv preprint arxiv:2407.06312* (2024).
- [23] Herwig, April, Matthew J. Colbrook, Oliver Junge, Péter Koltai, and Julia Slipantschuk. "Avoiding spectral pollution for transfer operators using residuals." *arXiv preprint arXiv:2507.16915* (2025).
- [24] Boullé, Nicolas, Matthew J. Colbrook, and Gustav Conradie. "Convergent Methods for Koopman Operators on Reproducing Kernel Hilbert Spaces." *arXiv preprint arXiv:2506.15782* (2025).