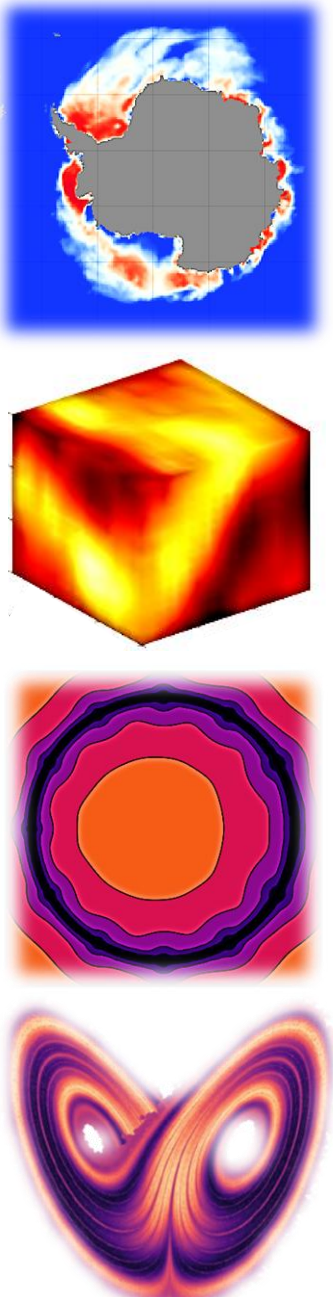




Adversarial Dynamics and the Boundaries of Data-Driven Koopman Spectral Computability

Matthew Colbrook
28th Aug 2025



UNIVERSITY OF
CAMBRIDGE

“To classify is to bring order into chaos.” - George Pólya

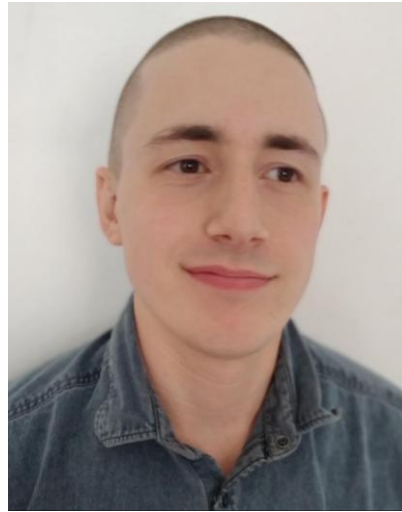
My thanks to a cast of great collaborators!



Alex Townsend
(Cornell)



Igor Mezić
(UC Santa Barbara)



Alexei Stepanenko
(Cam. -> Industry)



Nicolas Boullé
(Imperial)



Gustav Conradie
(Cambridge)

- C., Townsend. *"Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems."* **Communications on Pure and Applied Mathematics**, 2024.
- C., Mezić, Stepanenko, "Adversarial Dynamical Systems Reveal Limits and Rules for Trustworthy Data-Driven Learning." (under revision at **Nature Communications**).
- Boullé, C., Conradie, "Convergent Methods for Koopman Operators on Reproducing Kernel Hilbert Spaces." (**SpecRKHS** - hot off the press: <https://arxiv.org/abs/2506.15782>)

What is a Koopman operator?

- $\mathcal{X}$ – *the state space*
- $\mathcal{X} \ni x$ – *the state*

cts $F: \mathcal{X} \rightarrow \mathcal{X}$ – *the dynamics*: $x_{n+1} = F(x_n)$

Henri Poincaré
(Sorbonne)



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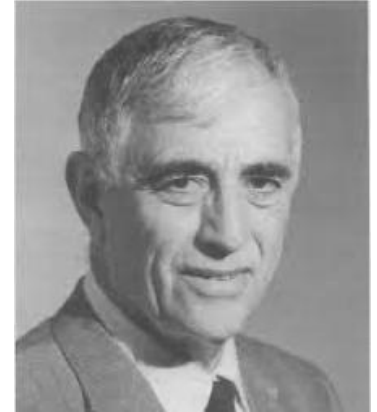
cts $F: \mathcal{X} \rightarrow \mathcal{X}$ – *the dynamics*: $x_{n+1} = F(x_n)$

- Functions $g: \mathcal{X} \rightarrow \mathbb{C}$ a.k.a “observables”
- Koopman operator $\mathcal{K}_F: [\mathcal{K}_F g](x) = g(F(x))$

LINEAR!

Observe g one time step forward

Bernard Koopman
(Columbia)



John von Neumann
(IAS)



- Koopman, “Hamiltonian systems and transformation in Hilbert space,” **Proc. Natl. Acad. Sci. USA**, 1931.
- Koopman, v. Neumann, “Dynamical systems of continuous spectra,” **Proc. Natl. Acad. Sci. USA**, 1932.

What is a Koopman operator?

- $\mathcal{X}$ – the state space
- $\mathcal{X} \ni x$ – the state
- Unknown cts $F: \mathcal{X} \rightarrow \mathcal{X}$ – the dynamics: $x_{n+1} = F(x_n)$
- Functions $g: \mathcal{X} \rightarrow \mathbb{C}$ a.k.a “observables”
- Koopman operator $\mathcal{K}_F: [\mathcal{K}_F g](x) = g(F(x))$ **LINEAR!**
- Available snapshot data: $\left\{ \left(x^{(m)}, y^{(m)} = F(x^{(m)}) \right) : m = 1, \dots, M \right\}$

Can we compute spectral properties from trajectory data?

$$g(x_n) = [\mathcal{K}^n g](x_0)$$

Why?

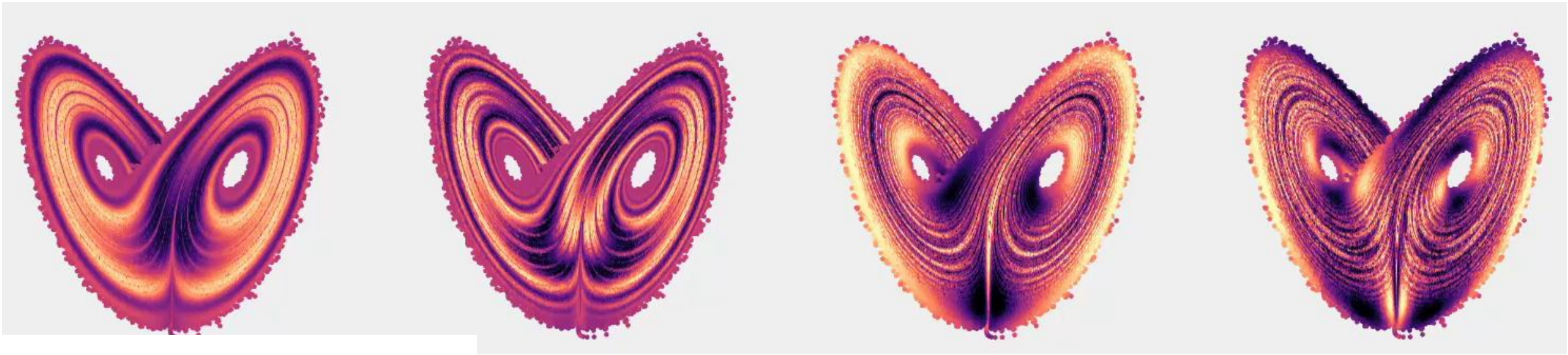
If $\|\mathcal{K}g - \lambda g\| \leq \varepsilon$, then $g(x_n) = [\mathcal{K}^n g](x_0) = \lambda^n g(x_0) + \mathcal{O}(n\varepsilon)$

Trades: Nonlinear, finite-dimensional $\Rightarrow$ Linear, infinite-dimensional.

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Coherent features!

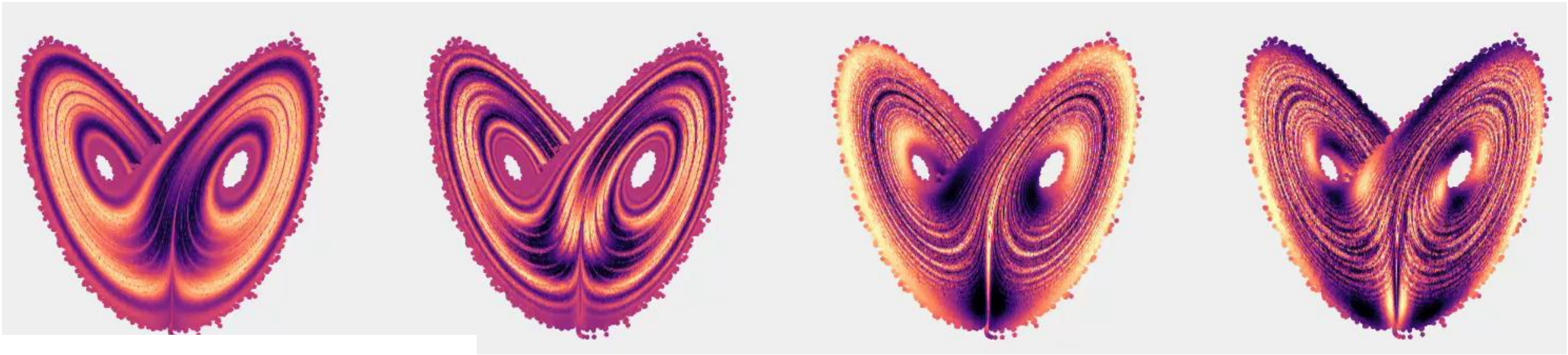
Lorenz attractor

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Coherent features!

$$\text{Sp}_{\text{ap},\varepsilon}(\mathcal{K}) = \{z \in \mathbb{C} : \exists g, \|g\| = 1, \|\mathcal{K}g - zg\| \leq \varepsilon\}$$

Trades: Nonlinear, finite-dimensional $\Rightarrow$ Linear, infinite-dimensional.

Koopman Mode Decomposition

- Find (g_j, λ_j) with $\|\mathcal{K}g_j - \lambda_j g_j\| \leq \varepsilon$
- Expand state:

Verified Eigenfunctions

Koopman modes

$$x \approx \sum_j c_j g_j(x)$$

- Forecasts:

$$x_n = \sum_j \lambda_j^n c_j g_j(x) + \mathcal{O}(n\varepsilon)$$

$$g(x_n) = [\mathcal{K}^n g](x_0)$$

Building a matrix approximation of $\mathcal{K}$: EDMD

Observables $\psi_j: \mathcal{X} \rightarrow \mathbb{C}, j = 1, \dots, N$

$$\{x^{(m)}, y^{(m)} = F(x^{(m)})\}_{m=1}^M$$

quadrature points

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \psi_k(x^{(m)}) = \left[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \dots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \dots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X} \right]^* \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_W \underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \dots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \dots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X} \right]_{jk}$$

quadrature weights

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \underbrace{\psi_k(y^{(m)})}_{[\mathcal{K}\psi_k](x^{(m)})} = \left[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \dots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \dots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X} \right]^* \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_W \underbrace{\begin{pmatrix} \psi_1(y^{(1)}) & \dots & \psi_N(y^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(y^{(M)}) & \dots & \psi_N(y^{(M)}) \end{pmatrix}}_{\Psi_Y} \right]_{jk}$$

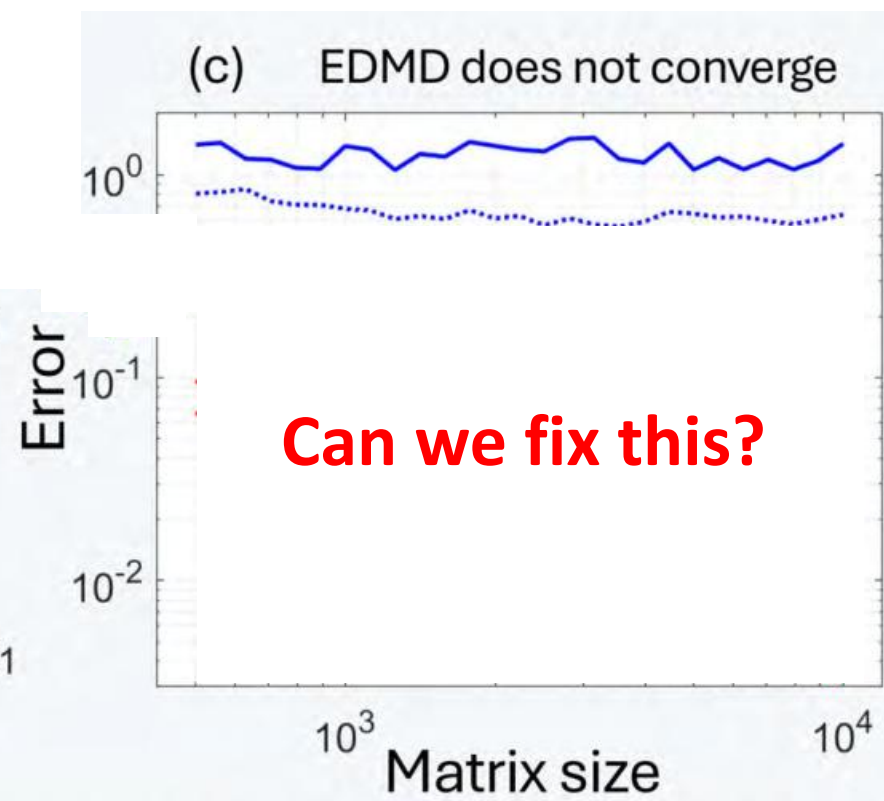
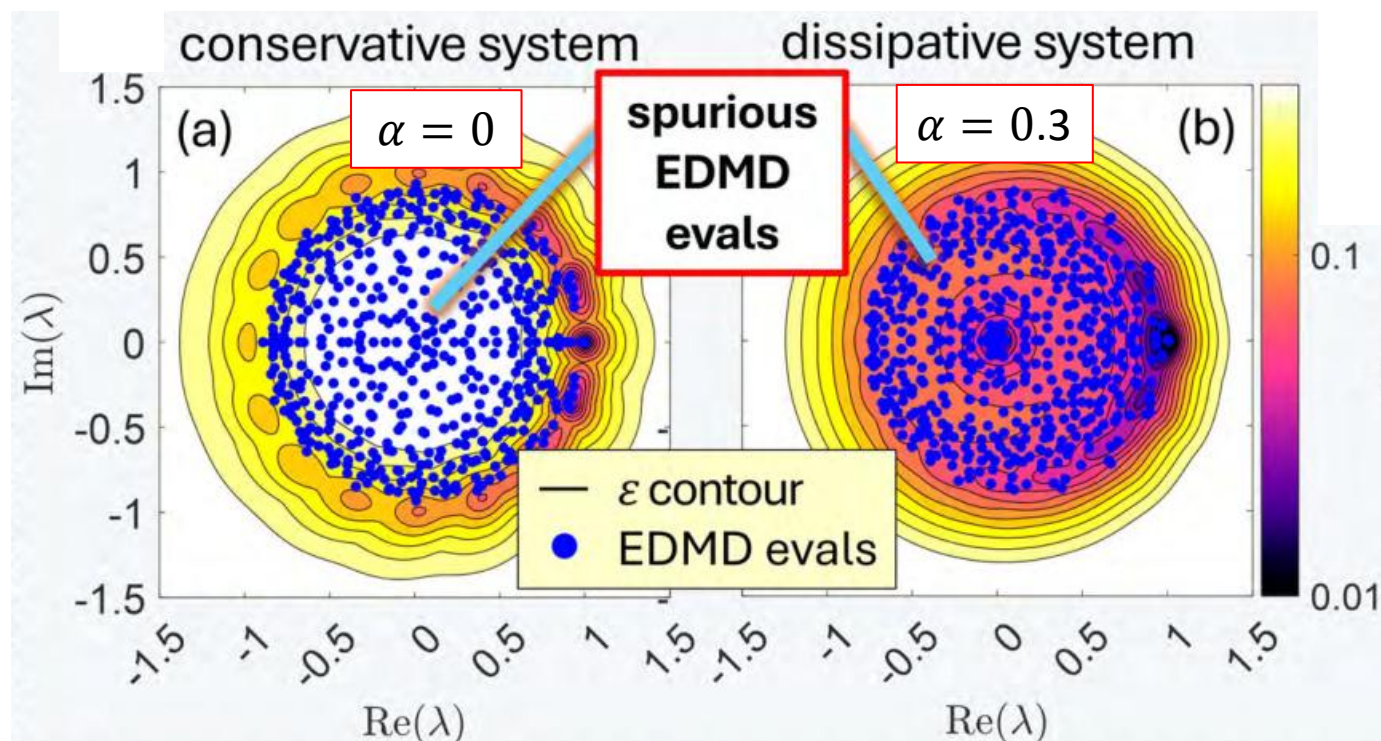
Galerkin
Approximation

$$\mathcal{K} \rightarrow (\Psi_X^* W \Psi_X)^{-1} \Psi_X^* W \Psi_Y \in \mathbb{C}^{N \times N}$$

- Schmid, "Dynamic mode decomposition of numerical and experimental data," **J. Fluid Mech.**, 2010.
- Rowley, Mezić, Bagheri, Schlatter, Henningson, "Spectral analysis of nonlinear flows," **J. Fluid Mech.**, 2009.
- Williams, Kevrekidis, Rowley "A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition," **J. Nonlinear Sci.**, 2015.

EDMD doesn't converge!

- Duffing oscillator: $\dot{x} = y$, $\dot{y} = -\alpha y + x(1 - x^2)$, sampled $\Delta t = 0.3$.
- Gaussian radial basis functions, Monte Carlo integration ($M = 50000$)



The fix: Residual DMD (ResDMD)

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \psi_k(x^{(m)}) = \left[\underbrace{\Psi_X^* W \Psi_X}_G \right]_{jk}$$

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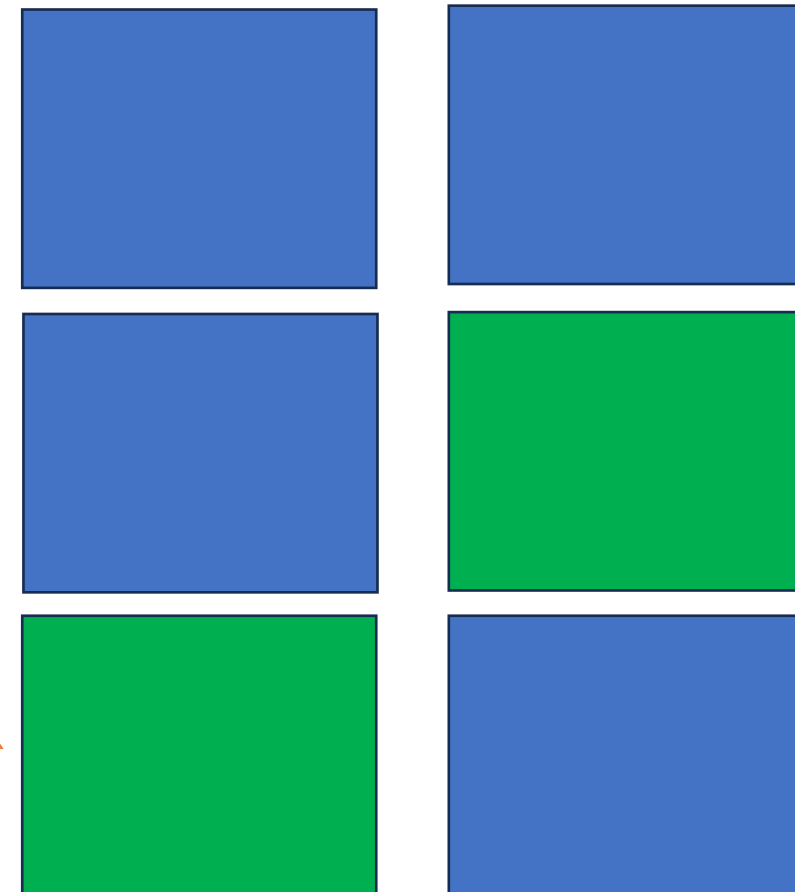
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adjoint



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- C., Townsend, “Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems,” **Commun. Pure Appl. Math.**, 2023.
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Residuals: $g = \sum_{j=1}^N \mathbf{g}_j \psi_j$, $\|\mathcal{K}g - \lambda g\|^2 = \langle \mathcal{K}g - \lambda g, \mathcal{K}g - \lambda g \rangle$

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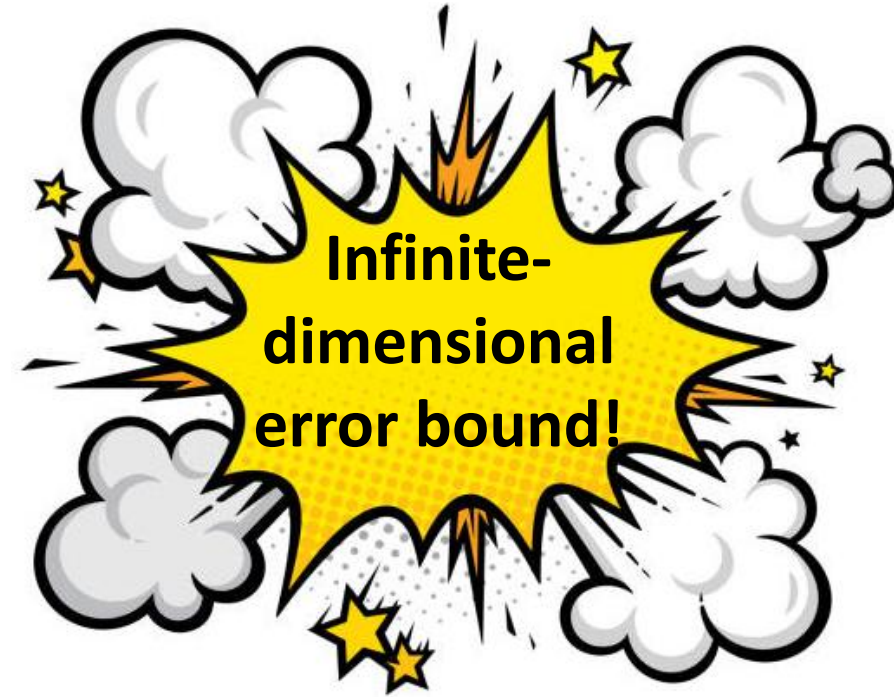
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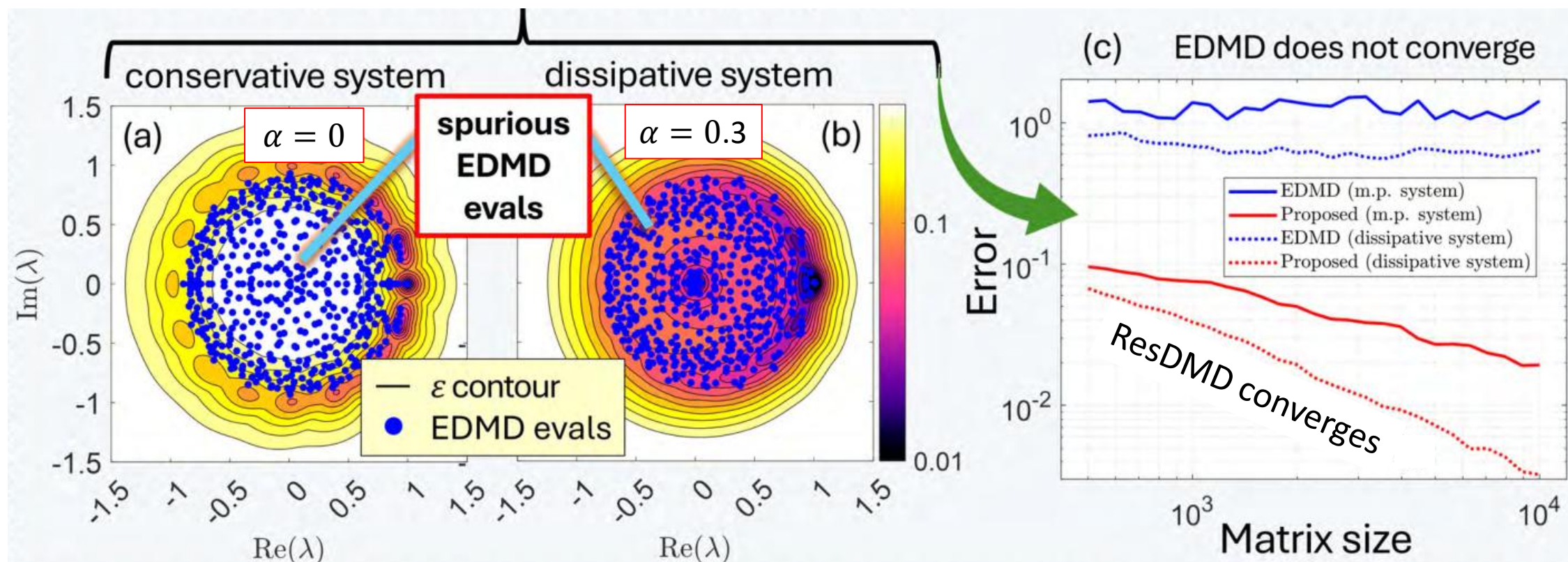
Residuals: $g = \sum_{j=1}^N \mathbf{g}_j \psi_j$, $\|\mathcal{K}g - \lambda g\|^2 = \lim_{M \rightarrow \infty} \mathbf{g}^* [K_2 - \lambda K_1^* - \bar{\lambda} K_1 + |\lambda|^2 G] \mathbf{g}$

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ResDMD does converge!

- Duffing oscillator: $\dot{x} = y$, $\dot{y} = -\alpha y + x(1 - x^2)$, sampled $\Delta t = 0.3$.
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Compute $\text{Sp}_{\text{ap},\varepsilon}(\mathcal{K})$, local adaptive control on $\varepsilon \downarrow 0$



Can maths help guide the way?

Consider space of observables with finite energy: $L^2(\mathcal{X}, \omega)$

Theorem: There **exists** algorithms $\Gamma_{N,M}$ using snapshots such that

$$\lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} \Gamma_{N,M}(F) = \text{Sp}_{\text{ap},\varepsilon}(\mathcal{K}_F)$$

for all systems.



N = size of basis, M = amount of data (quadrature)

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Double limit $\lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty}$

Can we do better?

Can maths help guide the way?

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for all systems.

Answer: *No! Even for smooth “nice” systems on a disc with unlimited data and accuracy, cannot converge in one limit by any algorithm with probability $>1/2$.*



What about different spaces of observables?

Reproducing kernel Hilbert space (RKHS)

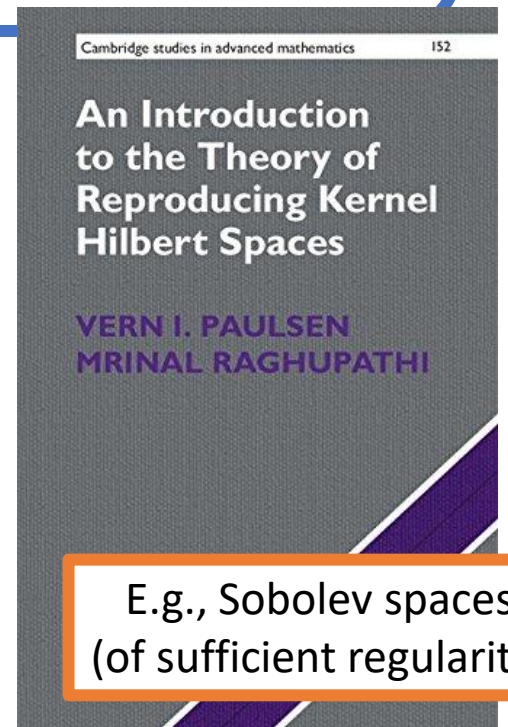
Hilbert space of functions on $\mathcal{X}$ s.t. $g \mapsto g(x)$ bounded $\forall x \in \mathcal{X}$.

Generated by a kernel $\mathfrak{K}: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{C}$

$$g(x) = \langle g, \mathfrak{K}_x \rangle, \quad \mathfrak{K}(x, y) = \langle \mathfrak{K}_x, \mathfrak{K}_y \rangle = \mathfrak{K}_x(y)$$

Advantages over $L^2(\mathcal{X}, \omega)$:

- Forecasts: space bounds $\Rightarrow$ pointwise bounds.
- High-dimensional systems practical through kernel trick.
- Fast methods for evaluating $\mathfrak{K}$.
- Different $\mathfrak{K} \Rightarrow$ different $\mathcal{K}$! Can be tailored to application.
(This is where the community is currently heading.)
- Leads to fundamental “possibility” gains...



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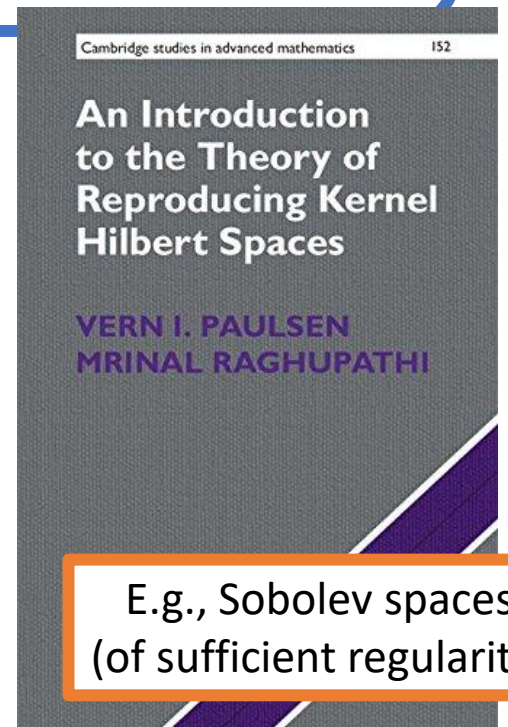
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SpecRKHS: Avoiding large data limit $M \rightarrow \infty$

Look at “Left eigenpairs” through $\mathcal{K}^*$:

$$\mathcal{K}^* \mathfrak{K}_x = \mathfrak{K}_{F(x)}$$

Evolution of functionals.

$$g(x) = \langle g, \mathfrak{K}_x \rangle_{\mathcal{H}}$$

No quadrature needed:

$$G_{jk} = \langle \mathfrak{K}_{x^{(k)}}, \mathfrak{K}_{x^{(j)}} \rangle = \mathfrak{K}(x^{(k)}, x^{(j)})$$

$$A_{jk} = \langle \mathcal{K}^* \mathfrak{K}_{x^{(k)}}, \mathfrak{K}_{x^{(j)}} \rangle = \langle \mathfrak{K}_{y^{(k)}}, \mathfrak{K}_{x^{(j)}} \rangle = \mathfrak{K}(y^{(k)}, x^{(j)})$$

$$R_{jk} = \langle \mathcal{K}^* \mathfrak{K}_{x^{(k)}}, \mathcal{K}^* \mathfrak{K}_{x^{(j)}} \rangle = \langle \mathfrak{K}_{y^{(k)}}, \mathfrak{K}_{y^{(j)}} \rangle = \mathfrak{K}(y^{(k)}, y^{(j)})$$

$$g = \sum_{m=1}^M \mathbf{g}_m \mathfrak{K}_{x^{(m)}}, \quad \|\mathcal{K}^* g - \lambda g\|_{\mathcal{H}}^2 = \mathbf{g}^* (R - \lambda A^* - \bar{\lambda} A + G) \mathbf{g}$$

SpecRKHS: Example algorithm

$$\text{res}^*(\lambda, \mathbf{g})^2 = \frac{\|\mathcal{K}^* g - \lambda g\|_{\mathcal{H}}^2}{\|g\|_{\mathcal{H}}^2} = \frac{\mathbf{g}^* [R - \lambda A^* - \bar{\lambda} A + G] \mathbf{g}}{\mathbf{g}^* G \mathbf{g}}$$

1. Compute $G, A, R \in \mathbb{C}^{N \times N}$ ($N = M$)
2. For z_k in grid, compute $\tau_k = \min_{g = \sum_{m=1}^N \mathbf{g}_m \mathfrak{K}_{x(m)}}$ $\text{res}^*(z_k, \mathbf{g})$, corresponding g_k (gen. SVD).
3. **Output:** $\{z_k: \tau_k < \varepsilon\}, \{g_k: \tau_k < \varepsilon\}$ (ε -pseudoeigenfunctions).

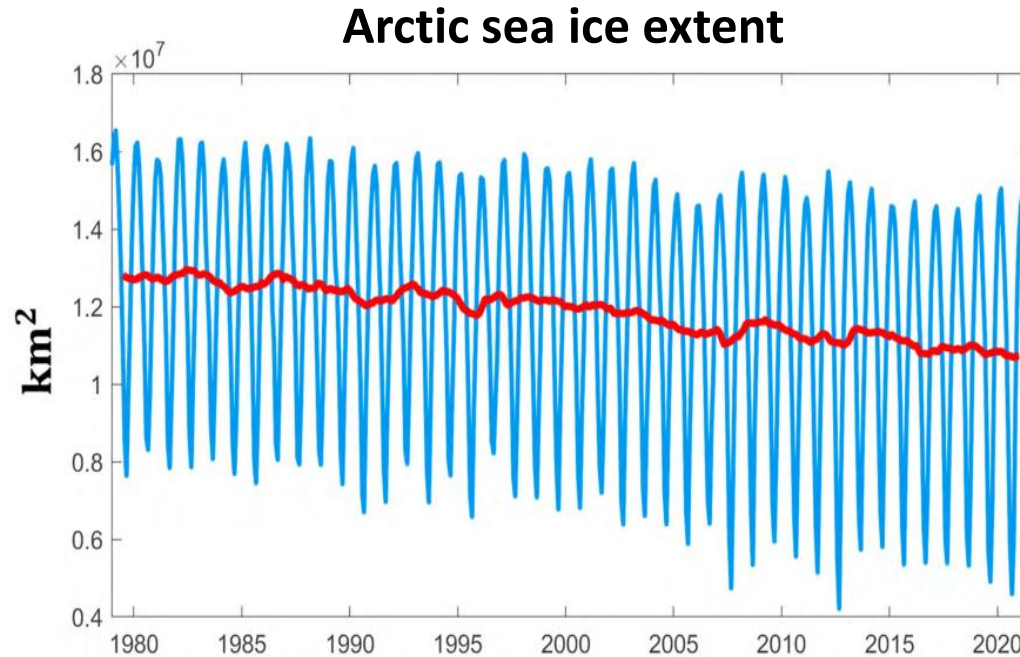
First convergent method for general $\mathcal{K}$

Theorem:

- **Error control:** $\{z_k: \tau_k < \varepsilon\} \subseteq \text{Sp}_{\text{ap}, \varepsilon}(\mathcal{K}^*)$
- **Convergence:** Converges locally uniformly to $\text{Sp}_{\text{ap}, \varepsilon}(\mathcal{K}^*)$ (as $N \rightarrow \infty$)

$$\text{Sp}_{\text{ap}, \varepsilon}(\mathcal{K}^*) = \{z \in \mathbb{C}: \exists g, \|g\|_{\mathcal{H}} = 1, \|\mathcal{K}^* g - z g\|_{\mathcal{H}} \leq \varepsilon\}$$

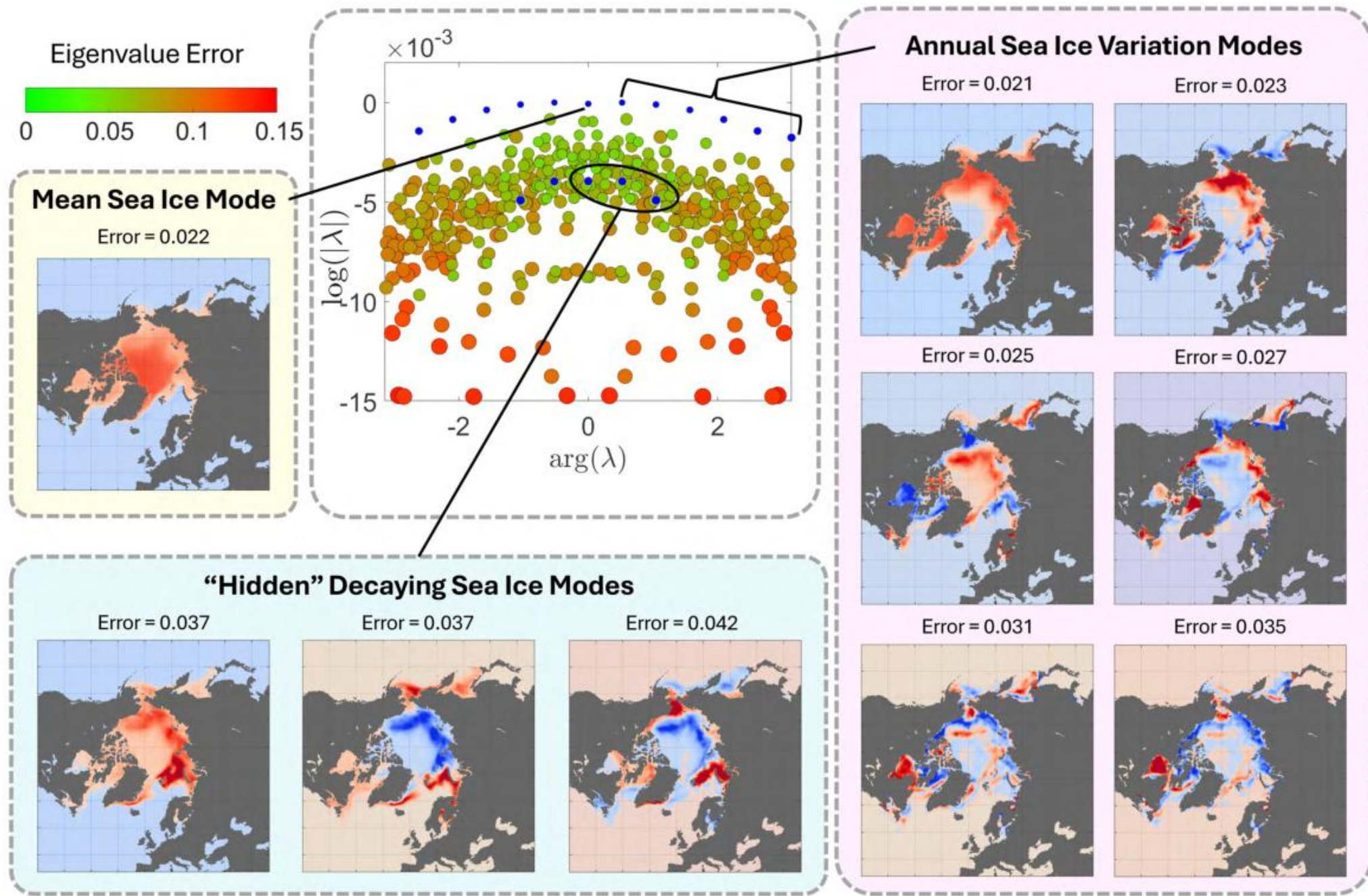
Practical gains: Sea ice forecasting



Monthly average from
satellite passive
microwave sensors.

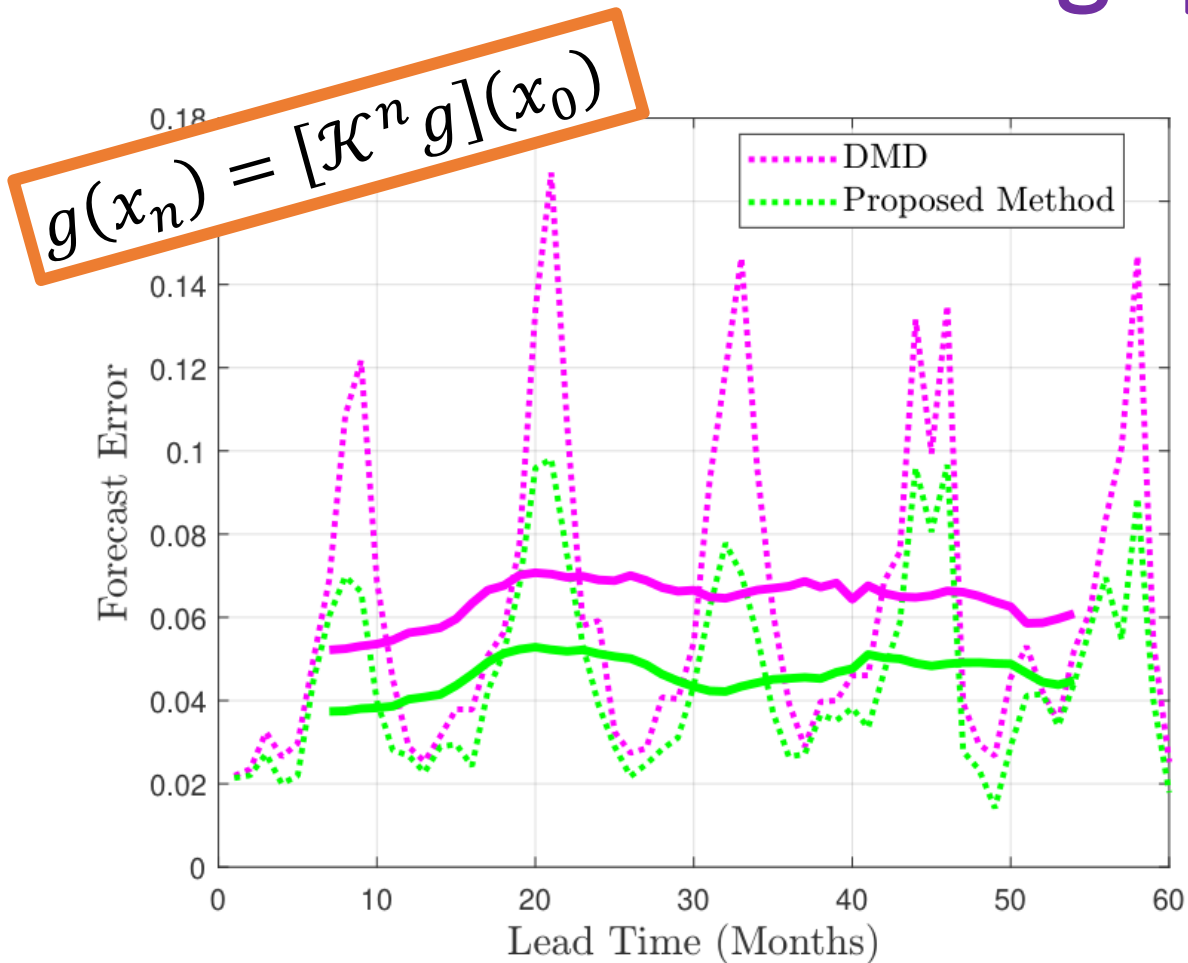
Motivation: Arctic amplification, polar bears, local communities, effect on extreme weather in Northern hemisphere,...

Problem: Very hard to predict more than two months in advance.

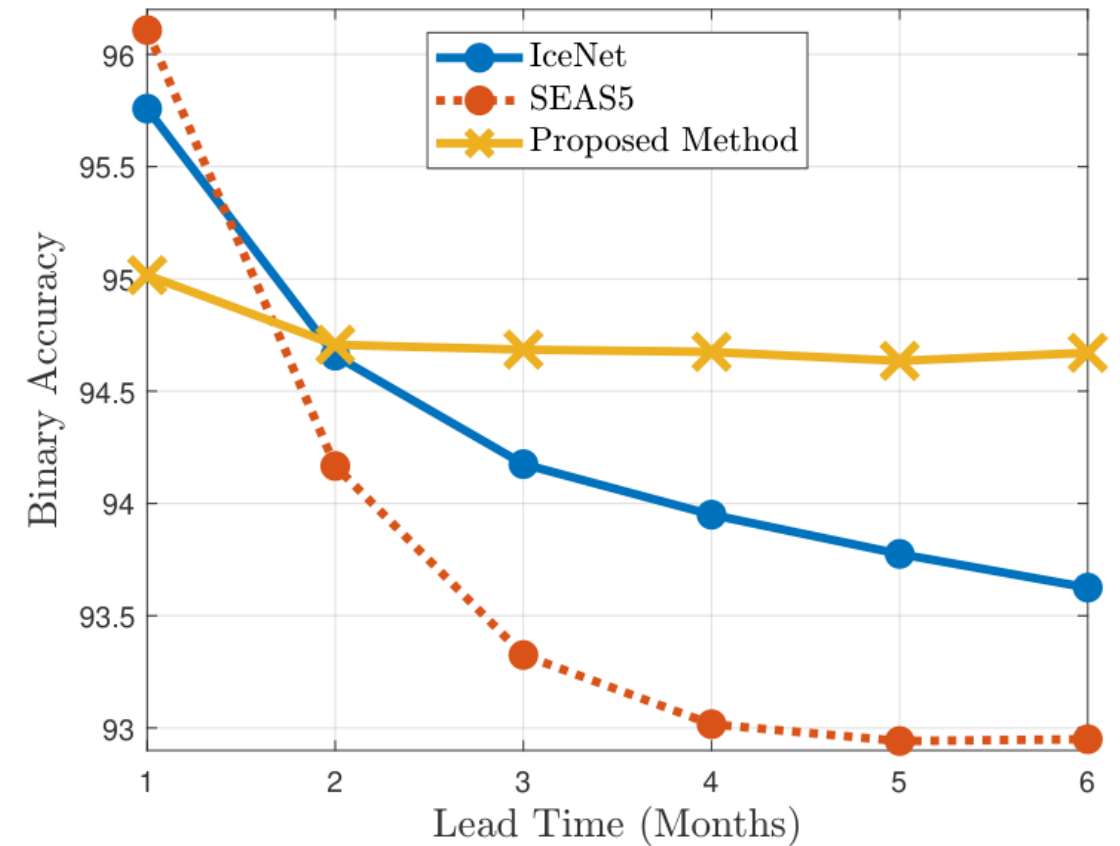


- C., Mezić, Stepanenko, “Adversarial Dynamical Systems Reveal Limits and Rules for Trustworthy Data-Driven Learning,” **preprint**, 2025.

Arctic case: Avoiding spurious eigenvalues helps!



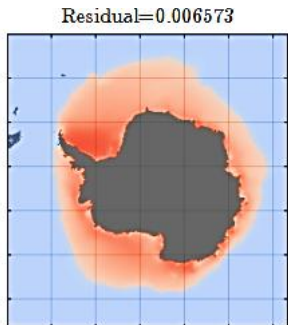
Relative mean squared error over 2016-2020. Model built from 2005-2015 data. (Solid lines moving 12-month mean.)



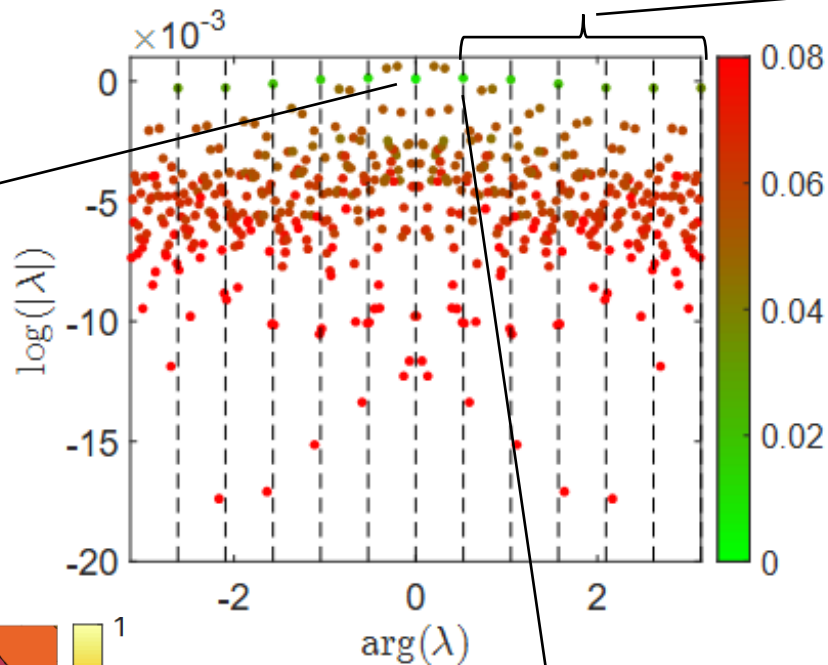
Mean binary accuracy over test years 2012-2020. (*IceNet*: Andersson et al, “Seasonal Arctic sea ice forecasting with probabilistic deep learning.” *Nature Communications*, 2021.)

Antarctic case

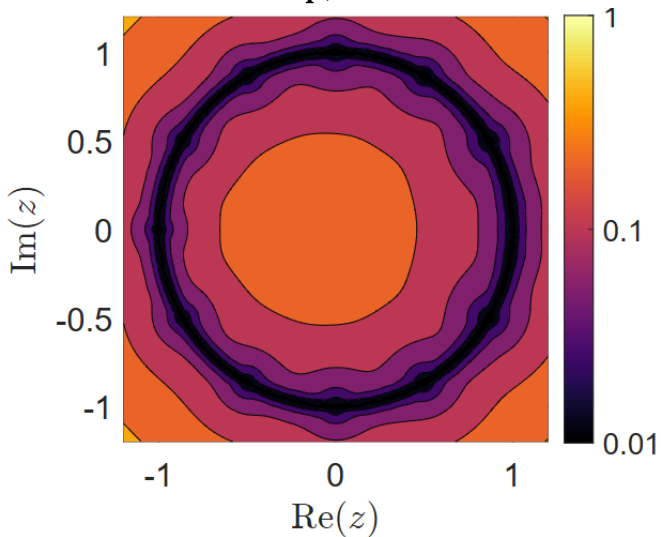
Mean Sea Ice Mode



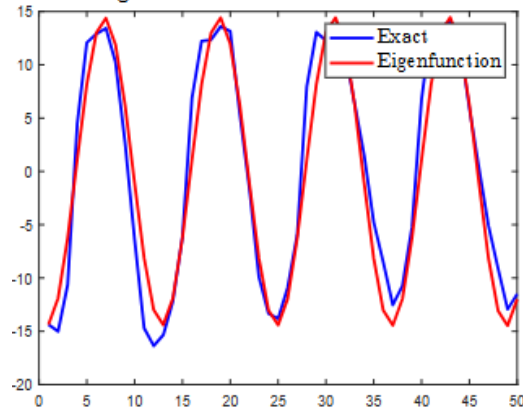
Residuals for Antarctic sea ice data



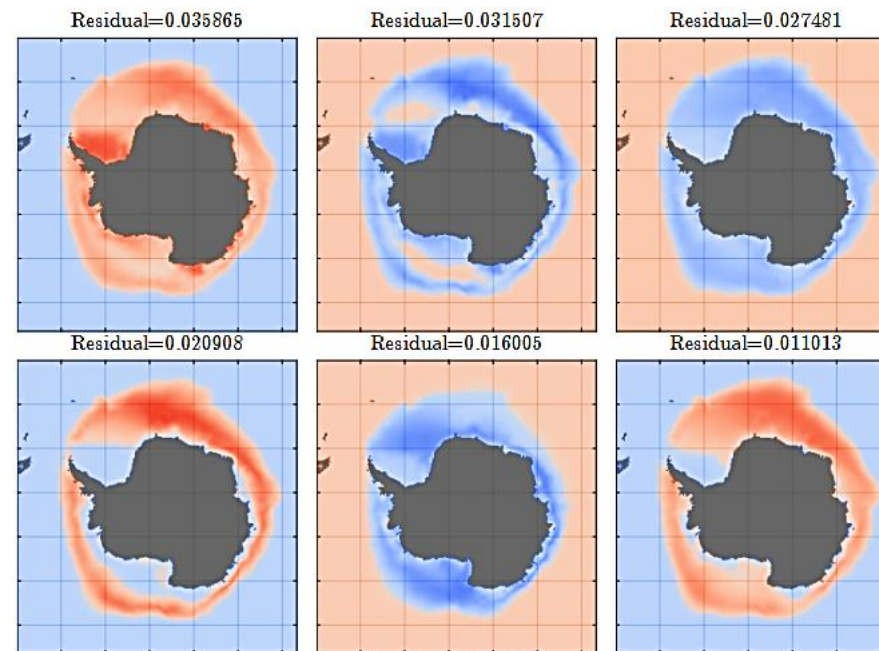
$\text{Sp}_{\text{ap},\varepsilon}(\mathcal{K}^*)$



Eigenvalue=0.866173+0.500016i



Annual Sea Ice Variation Modes



State space dimension: 82907

$N = M = 550$

RKHS: Sobolev space $H^{41454}(\mathbb{R}^{82907})$

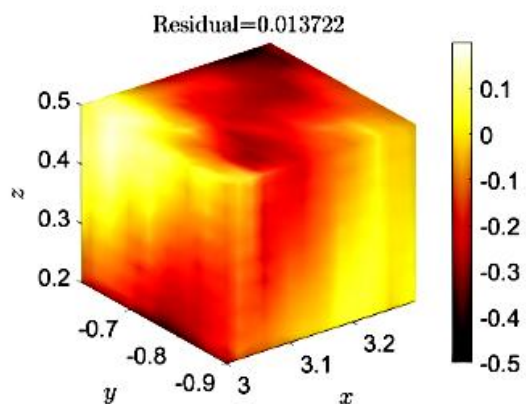
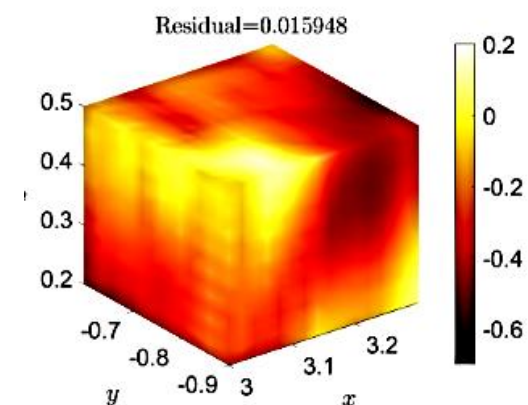
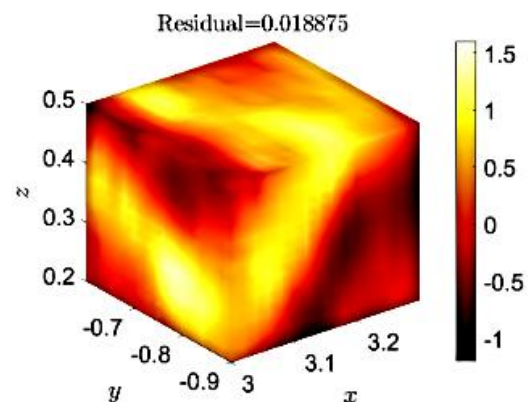
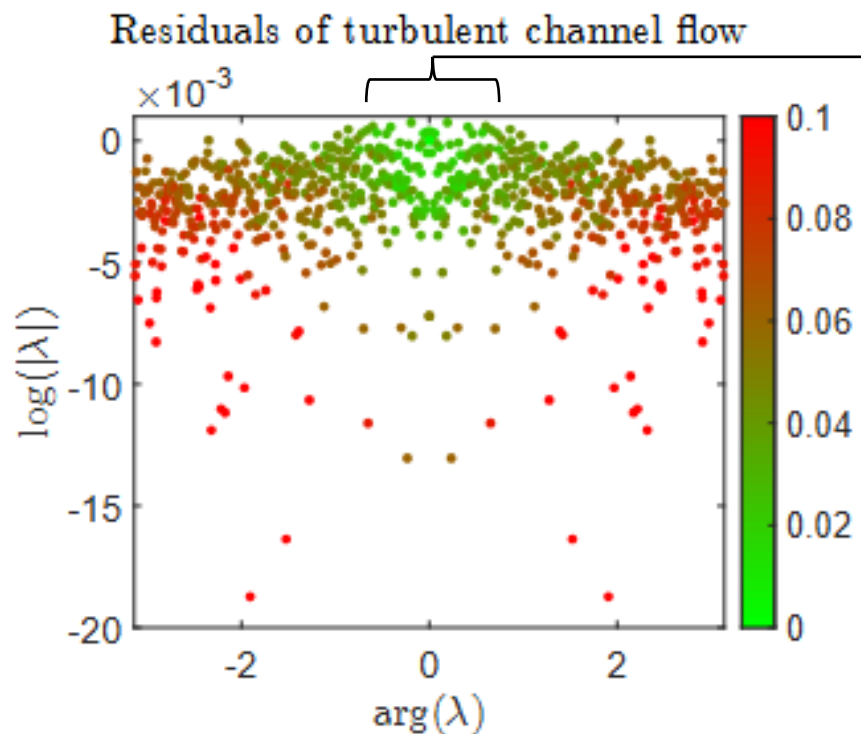
Matérn kernel

$$\propto (\sigma \|x - y\|_2)^{1/2} K_{-1/2}(\sigma \|x - y\|_2)$$

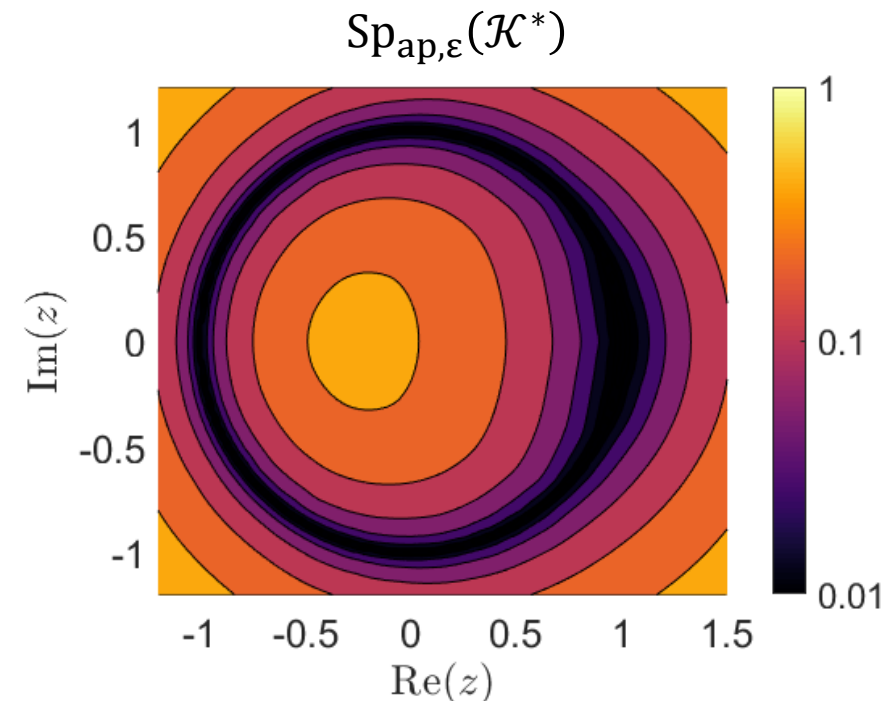
Modified Bessel function of second kind of order -1/2

scaling parameter

3D turbulence ($Re \approx 1000$)



Can handle very non-normal systems!



State space dimension: 4096

$N = M = 800$

RKHS: Sobolev space $H^{2049}(\mathbb{R}^{4096})$

Matérn kernel

$$\propto (\sigma \|x - y\|_2) K_{-1}(\sigma \|x - y\|_2)$$

Example of the method of adversarial dynamical systems

What about $\varepsilon = 0$? (i.e., the spectrum)

If $\mathcal{K}_F$ is normal, $\text{dist}(z, \text{Sp}(\mathcal{K}_F^*)) = \min\{\varepsilon: z \in \text{Sp}_{\text{ap},\varepsilon}(\mathcal{K}_F^*)\}$, adaptive search for local minimisers. Otherwise...



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Input class: $\Omega_{\mathbb{I}} = \{F: (0,1) \rightarrow (0,1) \mid F \text{ smooth bijection}\}$.

Data algorithm can use: $\mathcal{T}_F = \{(x, y) \mid x \in (0,1), y = F(x)\}$

Theorem: Consider Koopman operators on $H^r((0,1))$ ($r \geq 1/2$). There **does not exist** any sequence of deterministic algorithms $\{\Gamma_n\}$ using $\mathcal{T}_F$ such that $\lim_{n \rightarrow \infty} \Gamma_n(F) = \text{Sp}_{\text{ap}}(\mathcal{K}_F^*) \quad \forall F \in \Omega_{\mathbb{I}}$.

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n can index anything. Universal - any type of algorithm or comp. model.

Adversaries occur with high probability.  No random algorithms converging with probability $> 1/2$.

Extends to higher dimensions + other domains by embedding this proof.

Lemma

If $F'(x) = \alpha \in (0,1)$ in a neighborhood of 1,

$F'(x) = 1/\alpha$ in a neighborhood of -1

Then:

$$\alpha^{-1/2} \leq \sup\{|z|: z \in \text{Sp}_{\text{ap}}(\mathcal{K}_F^*)\} \leq \alpha^{1/2-r}$$

Idea: Use lemma to trick any algorithm into oscillating between spectra.

Proof idea

Suppose $\{\Gamma_n\}$ converges.

$$\alpha^{-1/2} \leq \sup\{|z|: z \in \text{Sp}_{\text{ap}}(\mathcal{K}_F^*)\} \leq \alpha^{1/2-r}$$

Proof idea

$$\alpha^{-1/2} \leq \sup\{|z| : z \in \text{Sp}_{\text{ap}}(\mathcal{K}_F^*)\} \leq \alpha^{1/2-r}$$

Suppose $\{\Gamma_n\}$ converges.

Pick $\alpha_1, \alpha_2 \in (0,1)$ and $\tau > 0$ with $(1 + 2\tau)\alpha_1^{1/2-r} < (1 - 2\tau)\alpha_2^{-1/2}$

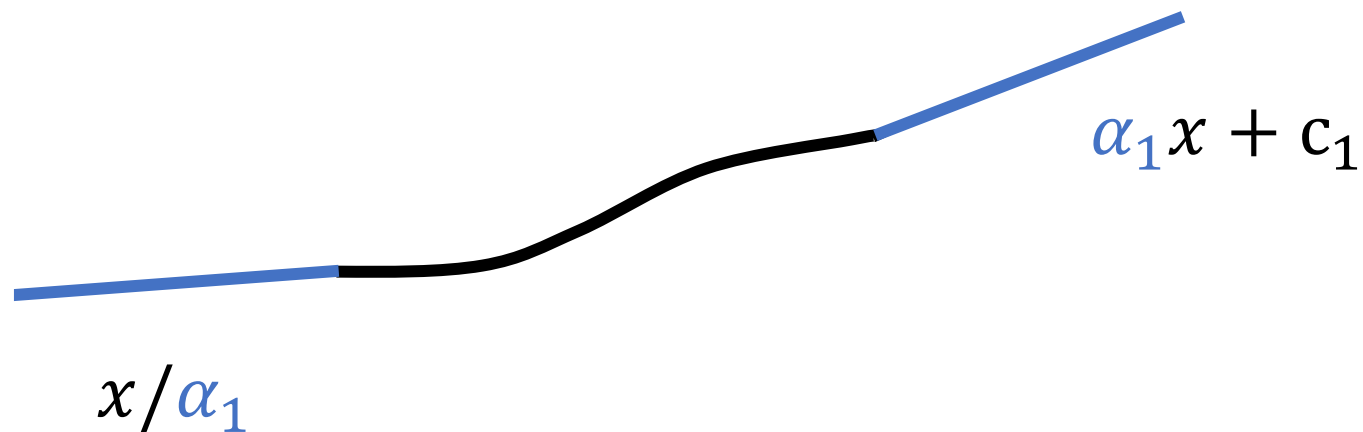
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F_1 :



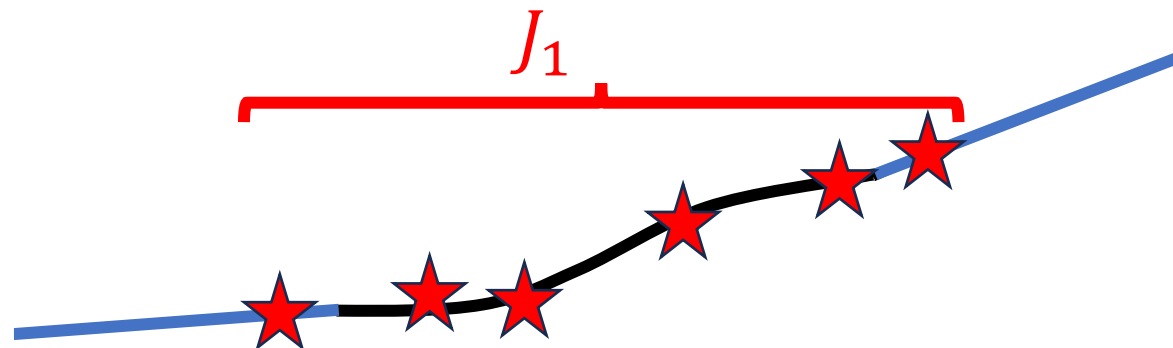
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F_1 :



Lemma $\Rightarrow \exists$ subinterval $J_1 \subset (0,1)$, $n_1 \in \mathbb{N}$ such that, when given input F_1 , Γ_{n_1} only samples from J_1 and $\sup\{|z| : z \in \Gamma_{n_1}(F_1)\} \leq (1 + \tau)\alpha_1^{1/2-r}$

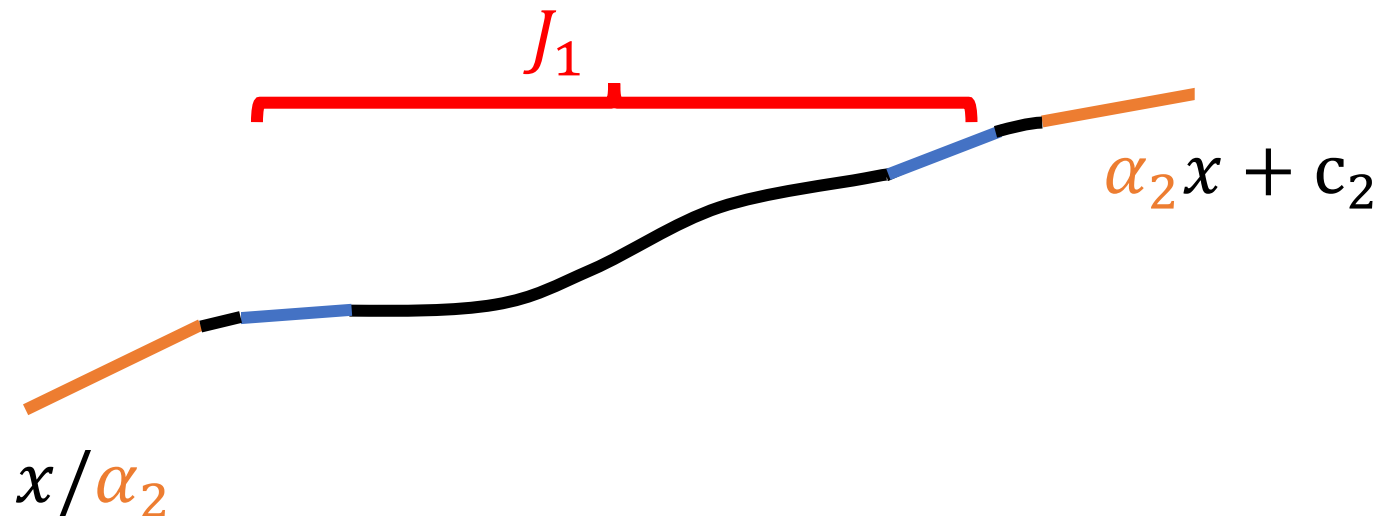
Proof idea

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Pick $\alpha_1, \alpha_2 \in (0,1)$ and $\tau > 0$ with $(1 + 2\tau)\alpha_1^{1/2-r} < (1 - 2\tau)\alpha_2^{-1/2}$

F_2 :



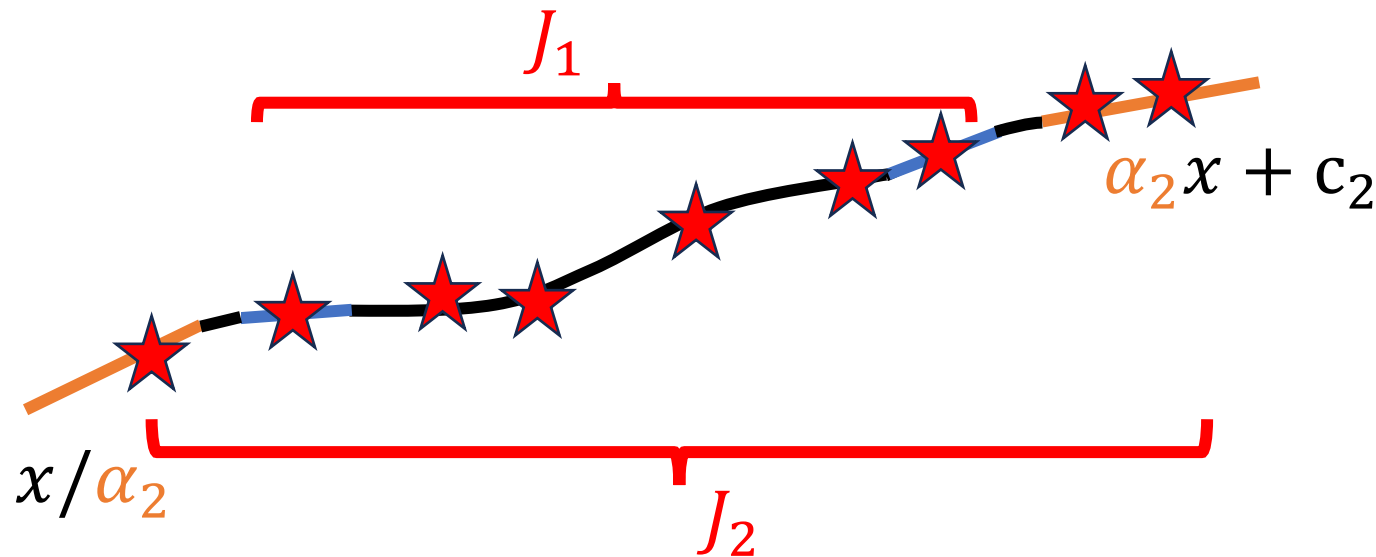
Proof idea

$$\alpha^{-1/2} \leq \sup\{|z|: z \in \text{Sp}_{\text{ap}}(\mathcal{K}_F^*)\} \leq \alpha^{1/2-r}$$

Suppose $\{\Gamma_n\}$ converges.

Pick $\alpha_1, \alpha_2 \in (0,1)$ and $\tau > 0$ with $(1 + 2\tau)\alpha_1^{1/2-r} < (1 - 2\tau)\alpha_2^{-1/2}$

F_2 :



Lemma $\Rightarrow \exists$ subinterval $J_2 \subset (0,1)$, $n_2 \in \mathbb{N}$ such that, when given input F_2 , Γ_{n_2} only samples from J_2 and $\sup\{|z|: z \in \Gamma_{n_2}(F_2)\} \geq (1 - \tau)\alpha_2^{-1/2}$

Proof idea

$$\alpha^{-1/2} \leq \sup\{|z|: z \in \text{Sp}_{\text{ap}}(\mathcal{K}_F^*)\} \leq \alpha^{1/2-r}$$

Suppose $\{\Gamma_n\}$ converges.

Pick $\alpha_1, \alpha_2 \in (0,1)$ and $\tau > 0$ with $(1 + 2\tau)\alpha_1^{1/2-r} < (1 - 2\tau)\alpha_2^{-1/2}$

Inductively:

$$\sup\{|z|: z \in \Gamma_{n_k}(F_k)\} \begin{cases} \leq (1 + \tau)\alpha_1^{1/2-r}, & k \text{ odd} \\ \geq (1 - \tau)\alpha_2^{-1/2}, & k \text{ even} \end{cases}$$

$F = \lim_{k \rightarrow \infty} F_k$, data consistency implies $\Gamma_{n_k}(F) = \Gamma_{n_k}(F_k)$.

$\therefore \Gamma_n(F)$ cannot converge, contradiction!

Classifications: *Solvability Complexity Index (SCI)*

SCI: Fewest number of limits needed to solve a computational problem.

- Δ_1 : One limit, full error control. E.g., $d(\Gamma_n(F), \text{Sp}(\mathcal{K}_F)) \leq 2^{-n}$.

- Δ_{m+1} : $\text{SCI} \leq m$.

- Σ_m : $\text{SCI} \leq m$, final limit from below.

$$\text{E.g., } \Sigma_1: \sup_{z \in \Gamma_n(F)} \text{dist}(z, \text{Sp}(\mathcal{K}_F)) \leq 2^{-n}.$$

- Π_m : $\text{SCI} \leq m$, final limit from above.

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-
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trust output

verification

covers spectrum

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Lots of SCI upper bounds lurking in Koopman literature!

SCI: Fewest number of limits needed to solve a computational problem.

Algorithm	Comments/Assumptions	Spectral Problem's Corresponding SCI Upper Bound			
		<i>KMD</i>	<i>Spectrum</i>	<i>Spectral Measure (if m.p.)</i>	<i>Spectral Type (if m.p.)</i>
Extended DMD [47]	general L^2 spaces	$\text{SCI} \leq 2^*$	N/C	N/C	n/a
Residual DMD [44]	general L^2 spaces	$\text{SCI} \leq 2^*$	$\text{SCI} \leq 3^*$	$\text{SCI} \leq 2^*$	varies, see [84] e.g., a.c. density: $\text{SCI} \leq 2^*$
Measure-preserving EDMD [45]	m.p. systems	$\text{SCI} \leq 1$	N/C	$\text{SCI} \leq 2^*$ (general) $\text{SCI} \leq 1$ (delay-embedding)	n/a
Hankel DMD [85]	m.p. ergodic systems	$\text{SCI} \leq 2^*$	N/C	N/C	n/a
Periodic approximations [86]	m.p. + ω a.c.	$\text{SCI} \leq 2$	N/C	$\text{SCI} \leq 2$ (see [87])	a.c. density: $\text{SCI} \leq 3$
Christoffel–Darboux kernel [40]	m.p. ergodic systems	$\text{SCI} \leq 3$	n/a	$\text{SCI} \leq 2$	e.g., a.c. density: $\text{SCI} \leq 2$
Generator EDMD [88]	cts.-time, samples ∇F (otherwise additional limit)	$\text{SCI} \leq 2$	N/C	$\text{SCI} \leq 2$ (see [89])	n/a
Compactification [42]	cts.-time, m.p. ergodic systems	$\text{SCI} \leq 4$	N/C	$\text{SCI} \leq 4$	n/a
Resolvent compactification [43]	cts.-time, m.p. ergodic systems	$\text{SCI} \leq 5$	N/C	$\text{SCI} \leq 5$	n/a
Diffusion maps [90] (see also [10])	cts.-time, m.p. ergodic systems	$\text{SCI} \leq 3$	n/a	n/a	n/a

Are these sharp?

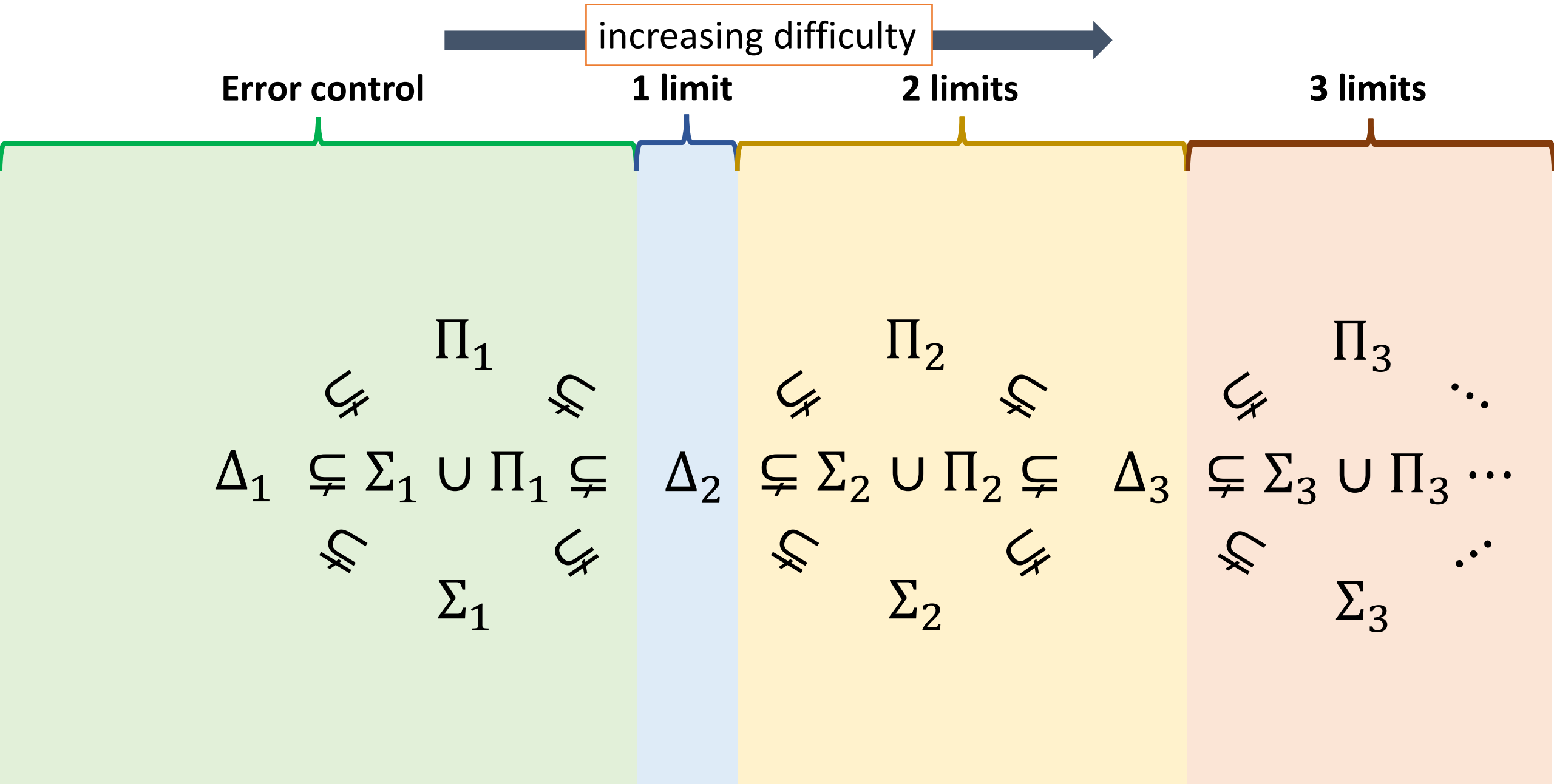
Previous techniques prove upper bounds on SCI.

“N/C”: method need not converge. “n/a”: algorithm not applicable to problem.

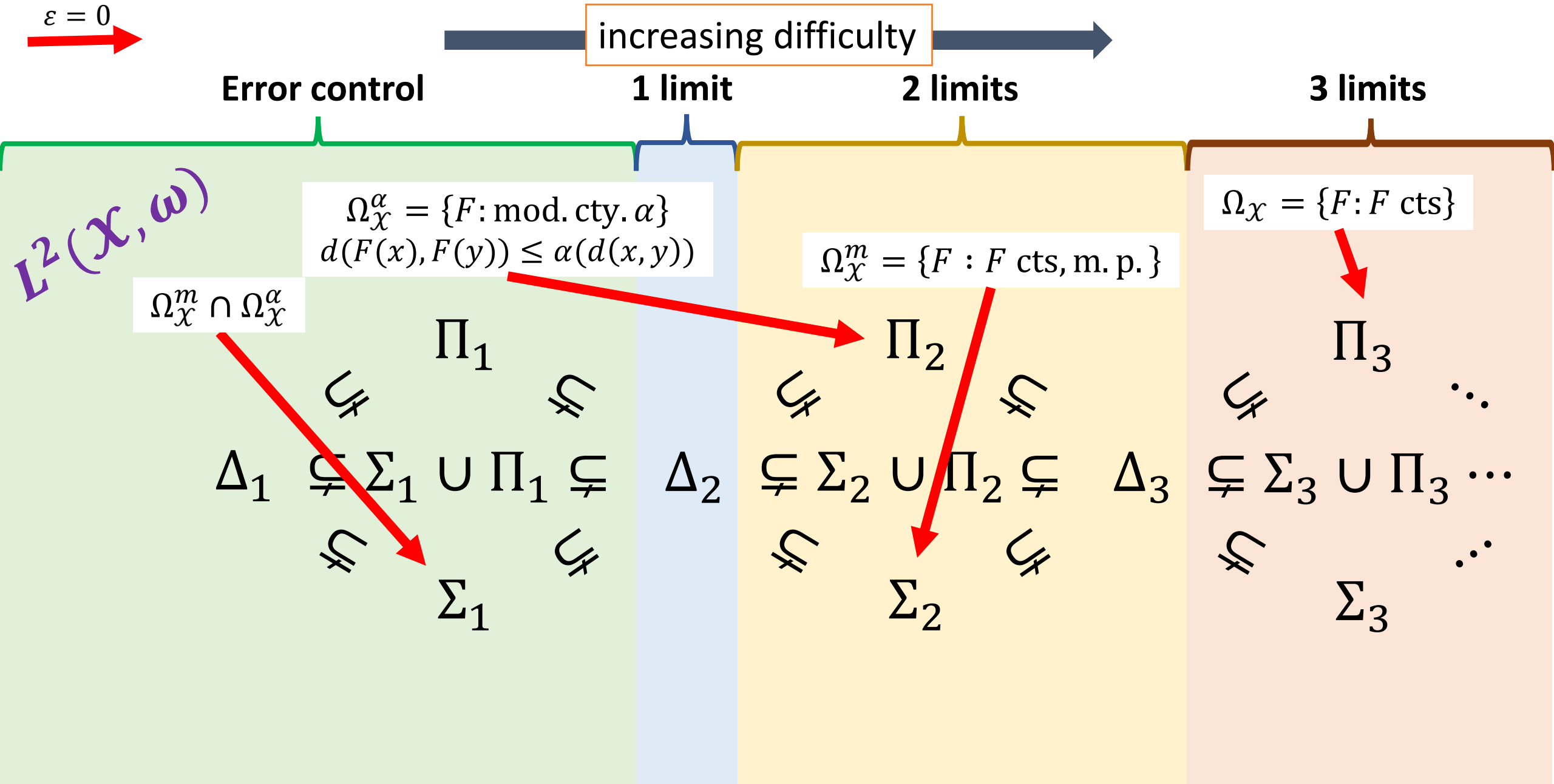
Also in Ulam’s method for Markov processes, SRB measure computation, control,...

Optimal algorithms and classifications of systems

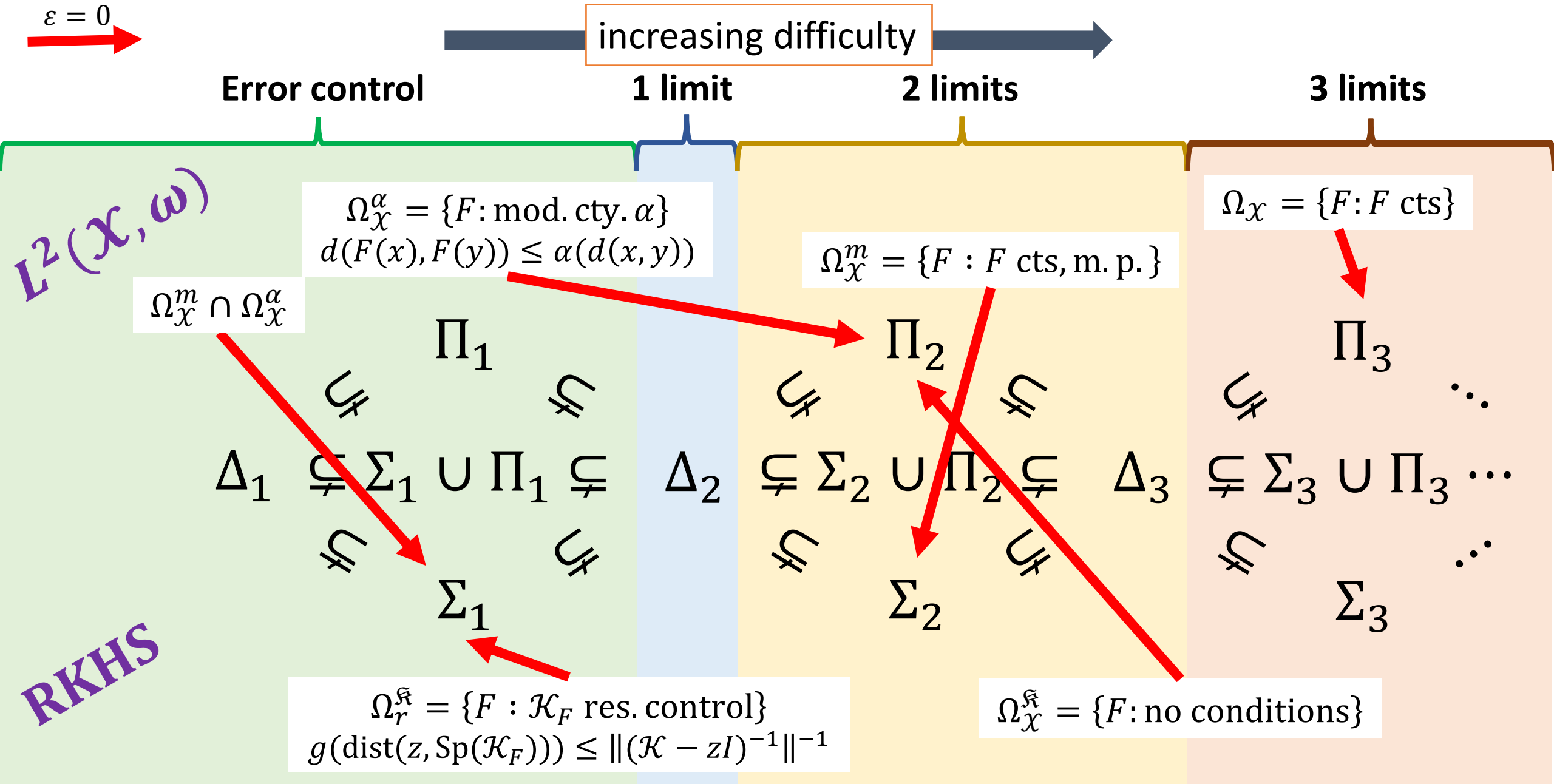
47



Optimal algorithms and classifications of systems

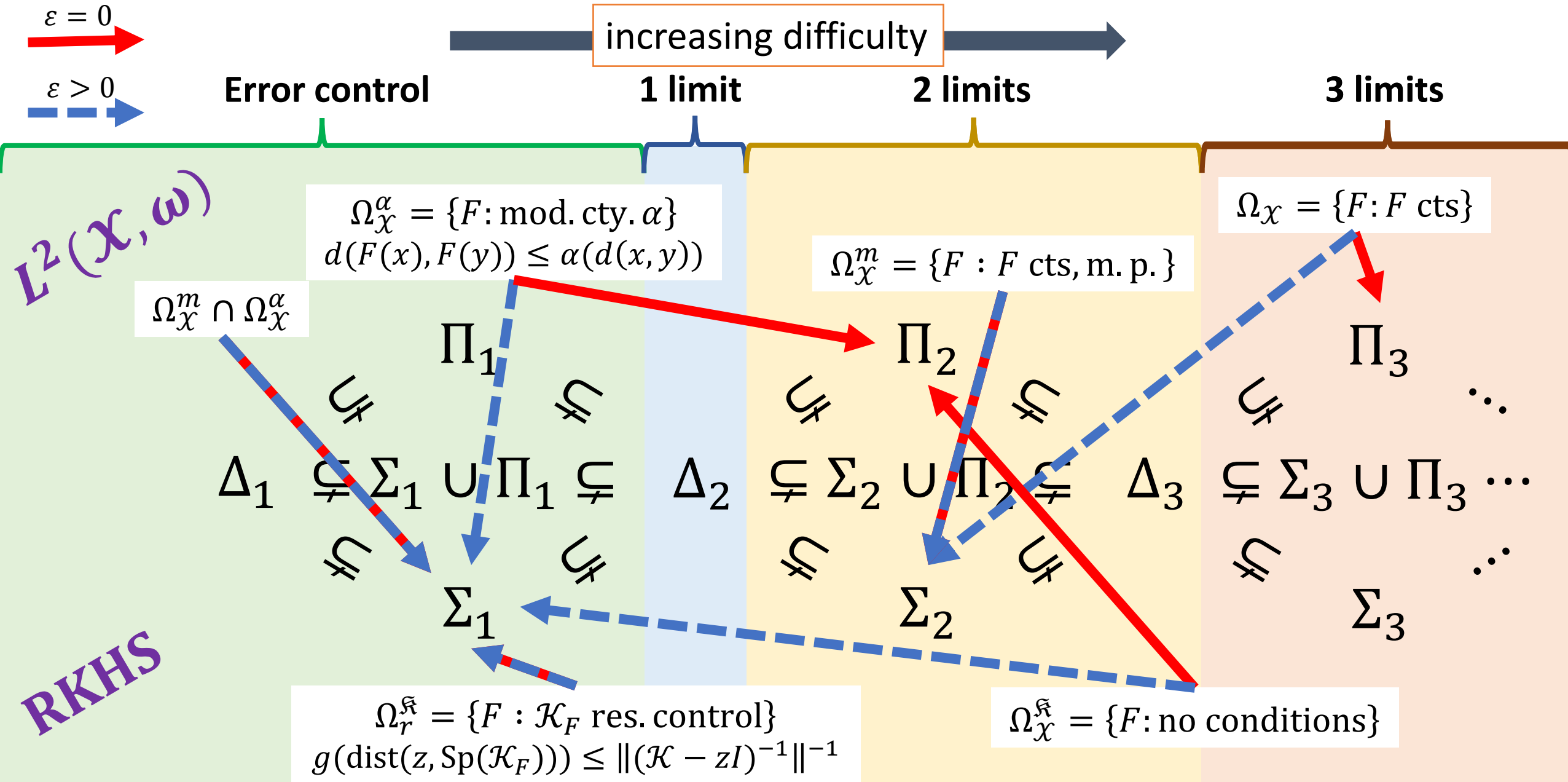


Optimal algorithms and classifications of systems



Optimal algorithms and classifications of systems

50



Conclusion: MATHS $\leftrightarrow$ METHODS

1. Data-driven spectral problems for Koopman operators are hugely popular.

BUT: Standard truncation methods often fail.

2. General method with convergence for spectral properties

(spectra, pseudospectra, spectral measures etc.) of K. operators on RKHS!

$\mathcal{K}^* \mathfrak{K}_x = \mathfrak{K}_{F(x)}$ E.g., Verification of approximate eigenfunctions leads to practical gains.

3. *SCI hierarchy* classifies computational problems:

Lower bounds through method of adversarial dynamics.

Upper bounds $\Rightarrow$ new “inf.-dim.” algorithms. Rigorous, optimal, practical.

$\rightarrow$ We now have a near complete picture for Koopman on $L^2(\mathcal{X}, \omega)$ and RKHS!

NB: Similar picture has emerged for spectral measures, dealing with continuous spectra (versus eigenvalues) and spectral type (different flavors of dynamics).

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