On the Mathematical Foundations of Infinite-Dimensional Spectral Computations

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"The infinite! No other question has ever moved so profoundly the spirit of humankind; no other idea has so fruitfully stimulated the intellect; yet no other concept stands in greater need of clarification." – **David Hilbert**

"To classify is to bring order into chaos." – **George Pólya**

For papers and talks, visit: http://www.damtp.cam.ac.uk/user/mjc249/home.html

Motivation: Szegő limit theorems

Beiträge zur Theorie der Toeplitzschen Formen.

(Erste Mitteilung.)

Von

G. Szegő in Budapest.

§ 8.

Ein Satz über die Eigenwerte der Toeplitzschen Formen.

21. Satz XVIII. *Es sei* $f(\theta)$ (L) *integrabel und* $m \leqq f(\theta) \leqq M$. *Es seien ferner*

$$\lambda_0^{(n)}, \lambda_1^{(n)}, \ldots, \lambda_n^{(n)} \qquad (n = 0, 1, 2, \ldots)$$

die zu $f(\theta)$ *gehörigen Eigenwerte. Dann ist*

$$m \leqq \lambda_\varkappa^{(n)} \leqq M \qquad (\varkappa = 0, 1, \ldots, n;\ n = 0, 1, 2, \ldots)$$

und wenn $F(\lambda)$ *eine für* $m \leqq \lambda \leqq M$ *definierte stetige Funktion bezeichnet,*

$$\lim_{n=\infty} \frac{F(\lambda_0^{(n)}) + F(\lambda_1^{(n)}) + \ldots + F(\lambda_n^{(n)})}{n+1} = \frac{1}{2\pi} \int_0^{2\pi} F[f(\theta)]\, d\theta. \text{ [37])}$$

Integrable real-valued function

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, e^{in\Theta}\, d\theta$$

Gábor Szegő (Stanford)

Eigenvalues

$$\begin{pmatrix} c_0 & c_{-1} & c_{-2} & \ldots & c_{-n} \\ c_1 & c_0 & c_{-1} & \ldots & c_{-n+1} \\ c_2 & c_1 & c_0 & \ldots & c_{-n+2} \\ & & & \ldots & . \\ c_n & c_{n-1} & c_{n-2} & \ldots & c_0 \end{pmatrix},$$

Self-adjoint Toeplitz matrix.

Continuous test function.

Asymptotic behavior of spectral measures!

Motivation: Schwinger's approx. of quantum systems

Schwinger: approx. $-\Delta + V$ on $\mathbb{R}$ using periodic finite grids

$$X_N = \left\{ j\sqrt{2\pi/N} : j = 0, \pm 1, \dots, \pm N \right\}, \qquad q_N \text{ multiplication},$$

$$p_N \quad \text{F.T. of} \quad q_N, \qquad H_N = \frac{1}{2}\left(p_N^2 + V(q_N)\right)$$

Julian Schwinger
(Harvard,
Nobel Prize in Physics 1965)

Digernes, Varadarajan, and Varadhan: Schwinger's method converges to spectra of $-\Delta + V$ on $L^2(\mathbb{R}^d)$ for certain families with compact resolvent.

Given a self-adjoint Schrödinger operator $-\Delta + V$ on $\mathbb{R}^d$, can we approximate its spectrum from sampling V?

- Schwinger, *"Unitary operator bases,"* **Proc. Natl. Acad. Sci. USA**, 1960.
- Weyl, *"The theory of groups and quantum mechanics,"* **Dover**, 1931.
- Digernes, Varadarajan, Varadhan, *"Finite approximations to quantum systems,"* **Rev. Math. Phys.**, 1994.

Computational spectral theory

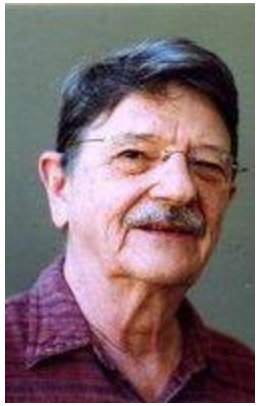
- **Applications:** Quantum mechanics, structural mechanics, optics, acoustics, statistical physics, number theory, matter physics, PDEs, data analysis, neural networks and AI, nuclear scattering, optics, computational chemistry, …
- **Rich history:** D. Arnold (Minnesota), W. Arveson (Berkeley), A. Böttcher (Chemnitz), W. Dahmen (South Carolina), E. B. Davies (KCL), P. Deift (NYU), L. Demanet (MIT), M. Embree (Virginia Tech), C. Fefferman (Princeton), G. Golub (Stanford), A. Iserles (Cambridge), I. Ipsen (NCS), S. Jitomirskaya (UCI), A. Laptev (Imperial), L. Lin (Berkeley) M. Luskin (Minnesota), S. Mayboroda (Minnesota), W. Schlag (Yale), E. Schrödinger (DIAS), J. Schwinger (Harvard), N. Trefethen (Oxford), V. Varadarajan (UCLA), S. Varadhan (NYU), J. von Neumann (IAS), M. Zworski (Berkeley), ...
- **Verified computations:** Many **computer-assisted proofs** involve spectra. E.g., $E(Z) = \inf_{N\geq 1} \inf\{z \in \mathrm{Sp}(H_{N,Z})\}$, $H_{N,Z} = \sum_{k=1}^{N}\left(-\Delta_{x_k} - Z|x_k|^{-1}\right) + \sum_{j\leq k}|x_j - x_k|^{-1}$.

  Dirac-Schwinger conjecture: asymptotics of $E(Z)$ as $Z \to \infty$ (Fefferman, Seco 1996)
- **Foundations:** What is computationally possible? Beyond spectra etc.

Arveson's work on C^*-algebras and finite sections of bounded operators

$$A \text{"="} \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad A\left(\sum_{k=1}^{\infty} x_k e_k\right) = \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{jk} x_k\right) e_j$$

Canonical basis vectors of $l^2(\mathbb{N})$



William Arveson (Berkeley)

$$\mathrm{Sp}(A) = \{z \in \mathbb{C} : A - zI \ \text{ is not invertible}\}$$

Arveson's work on C^*-algebras and finite sections of bounded operators

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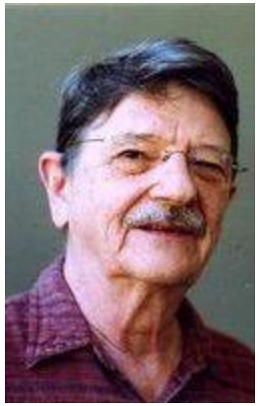
n

A_n

Canonical basis vectors of $l^2(\mathbb{N})$



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When does $\mathrm{Sp}(A_n)$ converge? In what sense?

Can we compute $\mathrm{Sp}(A)$ from matrix entries?

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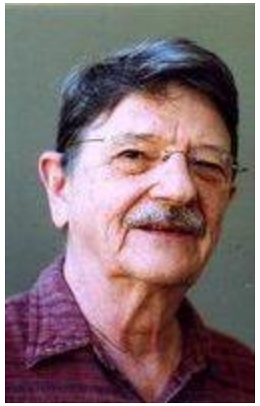

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"Most operators that arise in practice are not presented in a representation in which they are diagonalized, and it is often very hard to locate even a single point in the spectrum. Thus, one often has to settle for numerical approximations. Unfortunately, there is a dearth of literature on this basic problem and, so far as we have been able to tell, ***there are no proven [general] techniques****."* W. Arveson, Berkeley (1994)

Arveson's work on C^*-algebras and finite sections of bounded operators

n

$$A \text{ "=" } \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad A\left(\sum_{k=1}^{\infty} x_k e_k\right) = \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{jk} x_k\right) e_j$$

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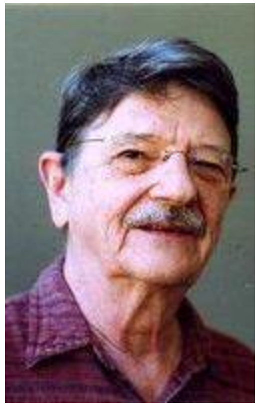

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When does $\mathrm{Sp}(A_n)$ converge? In what sense?
Can we compute $\mathrm{Sp}(A)$ from matrix entries?

Goal of talk: Explore mathematical foundations of computing $\mathrm{Sp}(A)$.

What can go wrong…

Spectral pollution

Definition: Let $\{S_n\} \subset \mathbb{C}$ be a sequence of closed sets (approximations of $\mathrm{Sp}(A)$). We say the sequence suffers from ***spectral pollution*** if there exists $\lambda \in \mathbb{C} \backslash \mathrm{Sp}(A)$ with

$$\liminf_{n\to\infty} \mathrm{dist}(\lambda, S_n) = 0.$$

Examples of $\{S_n\}$:

- **Matrix case ($l^2(\mathbb{N})$):** truncate to $\mathcal{P}_n A \mathcal{P}_n \in \mathbb{C}^{n\times n}, S_n = \mathrm{Sp}(\mathcal{P}_n A \mathcal{P}_n)$
- **PDE on unbounded domain:** truncate domain then discretise.

<u>Pervasive:</u> Dirac and Schrödinger operators, magnetohydrodynamics, matter physics, photonic waveguides, …

Spectral pollution

eigenvalues of
infinite multiplicity:
H is unitarily
equivalent to a
countable sum of
harmonic oscillators

Magnetic Schrödinger on $L^2(\mathbb{R}^2)$:

$$H = (i\partial_x + y/2)^2 + \left(i\partial_y - x/2\right)^2, \qquad \mathrm{Sp}(H) = \{1,3,5,\dots\}$$

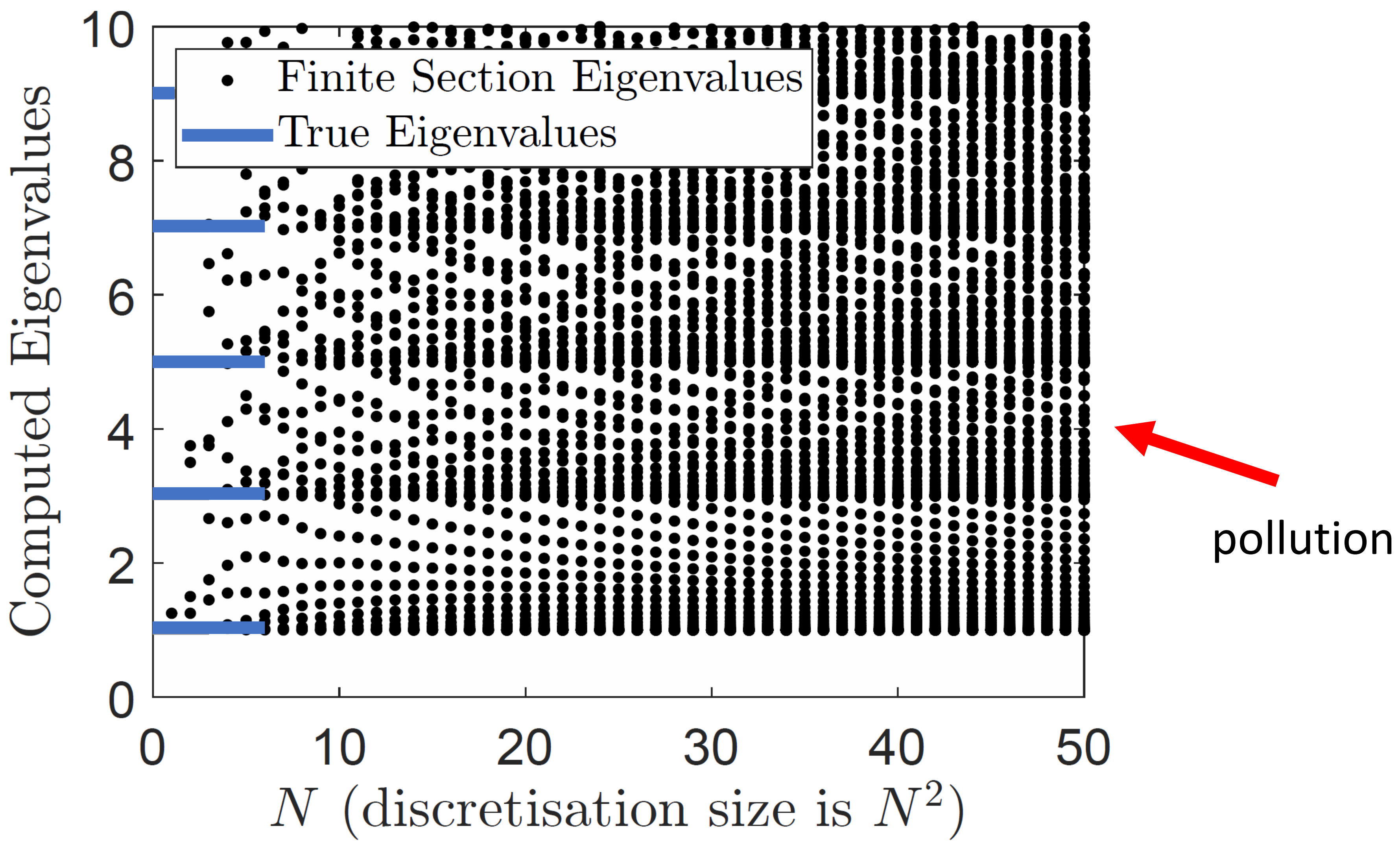
Use orthonormal basis $\psi_j(x) \otimes \psi_k(y)$,

$$\psi_k(x) = \frac{(-1)^k}{\sqrt{2^k k! \sqrt{\pi}}} e^{x^2/2} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-x^2}$$

Hermite functions

⟶ Sparse and self-adjoint “matrix”. BUT…

Spectral pollution



Where does spectral pollution occur?

Numerical range: $W(A)=\{\langle Ax,x\rangle : x\in\mathcal{D}(A), \|x\|=1\}$

Essential numerical range: $W_e(A)=\bigcap_{B\text{ compact}}\mathrm{Cl}(W(A+B))$

Theorem (Pokrzywa): Let A be a bounded operator on a separable Hilbert space $\mathcal{H}$, $\{\mathcal{P}_n\}$ finite-rank orthogonal projections that converge strongly to I.

- If $z\notin W_e(A)$, $z\in\mathrm{Sp}(A)$ if and only if $\lim_{n\to\infty}\mathrm{dist}(z,\mathrm{Sp}(\mathcal{P}_nA\mathcal{P}_n))=0$.
- If $S\subset W_e(A)$ compact, $\exists$ finite-rank orth. project. $\{\mathcal{Q}_n\}$ with $\mathcal{P}_n\le\mathcal{Q}_n$ and

$$\lim_{n\to\infty}\sup_{x\in\mathrm{Sp}(\mathcal{P}_nA\mathcal{P}_n)\cup S}\mathrm{dist}(x,\mathrm{Sp}(\mathcal{Q}_nA\mathcal{Q}_n))+\sup_{x\in\mathrm{Sp}(\mathcal{Q}_nA\mathcal{Q}_n)}\mathrm{dist}(x,\mathrm{Sp}(\mathcal{P}_nA\mathcal{P}_n)\cup S)=0$$

Spectral pollution occurs precisely on $W_e(A)\backslash\mathrm{Sp}(A)$.

Extensions to unbounded A, domain truncation (Bögli, Marletta, Tretter, 2020)

- Pokrzywa, *"Method of orthogonal projections and approximation of the spectrum of a bounded operator,"* **Studia Mathematica**, 1979.
- Bögli, Marletta, Tretter, *"The essential numerical range for unbounded linear operators,"* **Journal of Functional Analysis**, 2020

Spectral invisibility

Definition: Let $\{S_n\} \subset \mathbb{C}$ be a sequence of closed sets (approximations of $\mathrm{Sp}(A)$). We say the sequence suffers from ***spectral invisibility*** if there exists $\lambda \in \mathrm{Sp}(A)$ with

$$\limsup_{n\to\infty} \mathrm{dist}(\lambda, S_n) > 0.$$

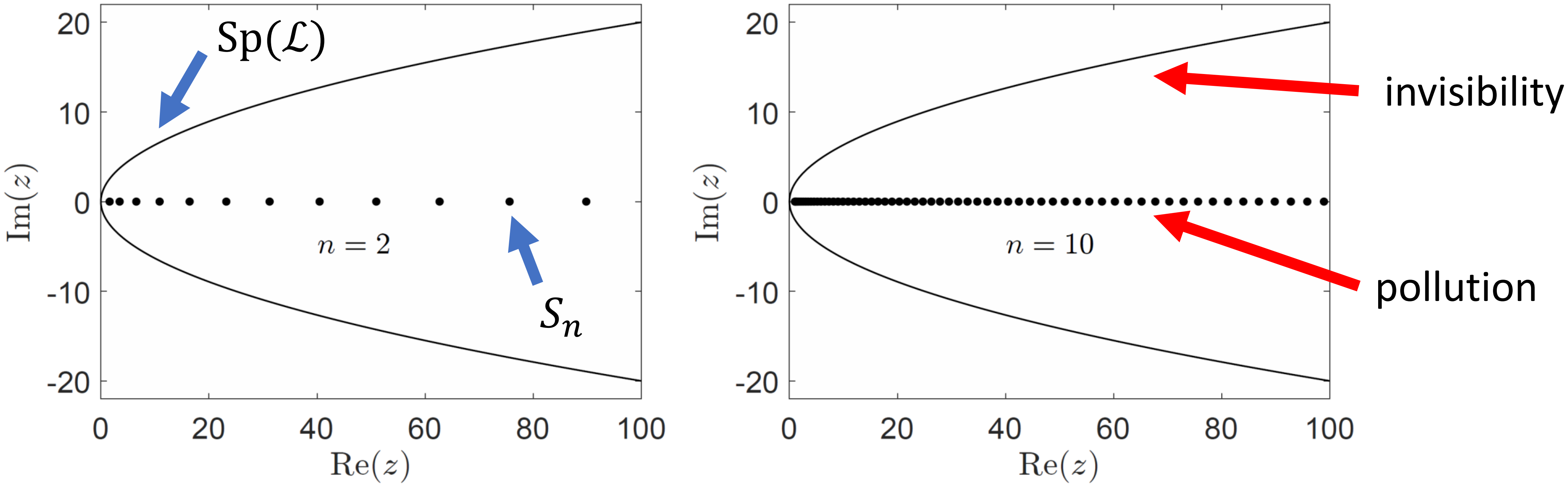
Currently no known characterization of invisibility
(i.e., no analog to $W_e(A)$ for spectral pollution).

Spectral invisibility

- Convection-diffusion operator (normal) on $L^2(\mathbb{R})$:

$$\mathcal{L}u = -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - 2\frac{\mathrm{d}u}{\mathrm{d}x}, \qquad \mathrm{Sp}(\mathcal{L}) = \{k^2 + 2ki : k \in \mathbb{R}\}$$

- Truncate to $[-n, n]$ + Dirichlet BCs, $S_n = \left\{1 + \frac{m^2\pi^2}{4n^2} : m \in \mathbb{N}\right\}$



• Davies, *"Non-self-adjoint differential operators,"* **Bulletin of the London Mathematical Society**, 2002

A method that <u>always</u> works…

Sketch of method

Spectra through injection moduli (smallest singular value)

Lipschitz-1 in z and A

$$\sigma_{\inf}(A) = \inf\{\|Av\|: v \in \mathfrak{D}(A), \|v\| = 1\}$$

$$\gamma(z, A) = \|(A - zI)^{-1}\|^{-1} = \min\{\sigma_{\inf}(A - zI), \sigma_{\inf}(A^* - \bar{z}I)\}$$

$$\mathrm{Sp}(A) = \{z \in \mathbb{C}: \gamma(z, A) = 0\}$$

Idea: $\mathcal{P}_n =$ orthog-projection onto span$\{e_1, \ldots, e_n\}$.

$$\gamma_{n,m}(z, A) = \min\{\sigma_{\inf}(\mathcal{P}_m(A - zI)\mathcal{P}_n), \sigma_{\inf}(\underbrace{\mathcal{P}_m(A^* - \bar{z})\mathcal{P}_n}_{\text{Rectangular finite section.}})\}$$

$$\gamma_n(z, A) = \min\{\sigma_{\inf}(\,(A - zI)\mathcal{P}_n), \sigma_{\inf}(\,(A^* - \bar{z}))\mathcal{P}_n\}$$

Dini's theorem: $\gamma_{n,m} \uparrow_{m\to\infty} \gamma_n \downarrow_{n\to\infty} \gamma$ uniformly on compacts

• Hansen, *"On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators,"* **J. Amer. Math. Soc.**, 2011.

Sketch of method

Hausdorff metric captures avoidance of pollution/invisibility:

$$d_{\mathrm{H}}(X,Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y}|x-y|, \sup_{y\in Y}\inf_{x\in X}|x-y|\right\}$$

Algorithm that converges in **3 limits**:

$$\Gamma_{n_3,n_2,n_1}(A) = \left\{z \in \frac{1}{n_2}(\mathbb{Z}+i\mathbb{Z}) : |z| \le n_2, \gamma_{n_2,n_1}(z,A) + \frac{1}{n_2} \le \frac{1}{n_3}\right\}$$

$$\lim_{n_2\to\infty}\lim_{n_1\to\infty}\Gamma_{n_3,n_2,n_1}(A) = \mathrm{Sp}_{\frac{1}{n_3}}(A), \qquad \lim_{n_3\to\infty}\mathrm{Sp}_{\frac{1}{n_3}}(A) = \mathrm{Sp}(A)$$

$$\boxed{\mathrm{Sp}_\epsilon(A) = \{z \in \mathbb{C} : \gamma(z,A) \le \epsilon\}}$$

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$$\lim_{n_2\to\infty}\lim_{n_1\to\infty}\Gamma_{n_3,n_2,n_1}(A)=\mathrm{Sp}_{\frac{1}{n_3}}(A),\qquad\lim_{n_3\to\infty}\mathrm{Sp}_{\frac{1}{n_3}}(A)=\mathrm{Sp}(A)$$

Can we do better (than 3 limits)?

$$\boxed{\mathrm{Sp}_{\epsilon}(A)=\{z\in\mathbb{C}:\gamma(z,A)\leq\epsilon\}}$$

- Hansen, *"On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators,"* **J. Amer. Math. Soc.**, 2011.

A mathematical structure…

Computational problem

Definition: A ***computational problem*** is a collection $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ consisting of:

- Input class Ω;
- Metric space $(\mathcal{M}, d)$;
- Problem function $\Xi: \Omega \to \mathcal{M}$;
- Evaluation set, Λ, of $\mathbb{C}$-valued functions on Ω;

such that for $A, B \in \Omega$:

$$f(A) = f(B) \;\; \forall f \in \Lambda \quad \Rightarrow \quad \Xi(A) = \Xi(B).$$

E.g., $\Omega = \mathcal{B}(l^2(\mathbb{N}))$

E.g., $\mathcal{M} = \mathcal{M}_H$ (Hausdorff metric)

Thing we want to compute E.g., $\Xi = \mathrm{Sp}$

Info available to algorithms
E.g. Matrix entries

$\Xi(A)$ determined by $\{f(A): f \in \Lambda\}$

General algorithm (consistency)

Info algorithm reads on input A.

Definition: Given $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$, a general algorithm is a map $\Gamma: \Omega \to \mathcal{M}$ such that for any $A \in \Omega$, there exists $\Lambda_\Gamma(A) \subset \Lambda$ finite and non-empty such that for $A, B \in \Omega$,

$$f(A) = f(B) \quad \forall f \in \Lambda_\Gamma(A) \quad \Rightarrow \quad \Lambda_\Gamma(A) = \Lambda_\Gamma(B), \Gamma(A) = \Gamma(B)$$

Can also consider restrictions (e.g., Turing or BSS machine)

Impossibility result for gen. alg. $\Longrightarrow$ impossibility result in any model

Solvability Complexity Index Hierarchy

- Δ_0: Solved in finite time (v. rare for cts problems).
- Δ_1: Solved in "one limit" with full error control:

$$d(\Gamma_n(A), \Xi(A)) \leq 2^{-n}$$

- Δ_2: Solved in "one limit":

$$\lim_{n\to\infty} \Gamma_n(A) = \Xi(A)$$

- Δ_3: Solved in "two successive limits":

$$\lim_{n\to\infty} \lim_{m\to\infty} \Gamma_{n,m}(A) = \Xi(A)$$

$\vdots$

Can work in *any* model. E.g., BSS machine, Turing machine, interval arithmetic, inexact input etc.

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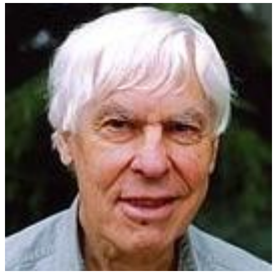
⋮


Steve Smale
(Berkeley,Fields Medal 1966)

Smale: *"Is there any purely rational iterative generally convergent algorithm for polynomial zero finding?"*


Curt McMullen
(Harvard, Fields Medal 1998)

McMullen: *"Yes, if the degree is three; no, if the degree is higher."*

Doyle & McMullen: *"The problem can be solved using successive limits for the quartic and quintic, but not the sextic."*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- McMullen, *"Families of rational maps and iterative root-finding algorithms,"* **Ann. of Math.**, 1987.
- Doyle, McMullen, *"Solving the quintic by iteration,"* **Acta Math.**, 1989.
- Smale, *"The fundamental theorem of algebra and complexity theory,"* **Bull. Amer. Math. Soc.**, 1981.

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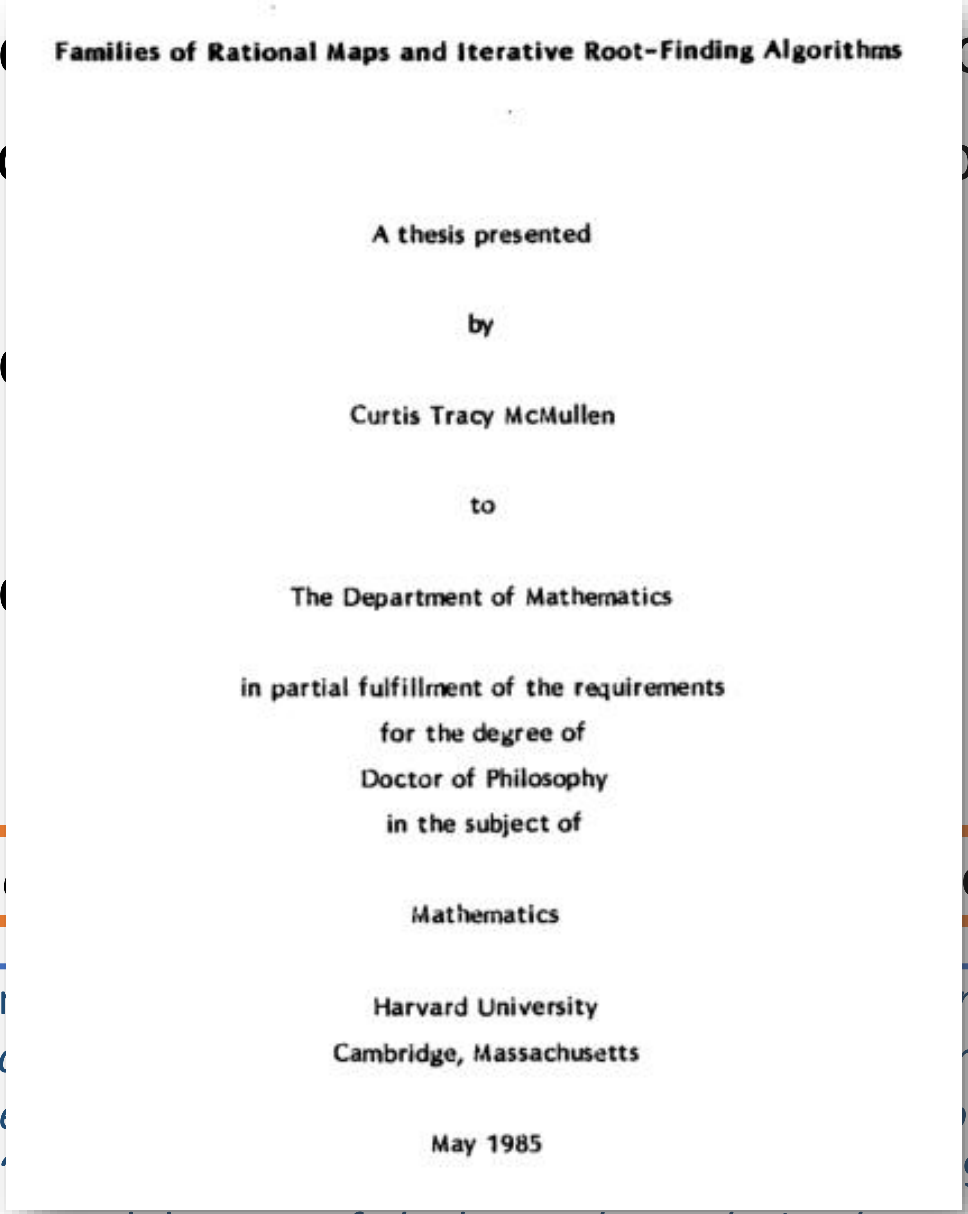


Families of Rational Maps and Iterative Root-Finding Algorithms

A thesis presented

by

Curtis Tracy McMullen

to

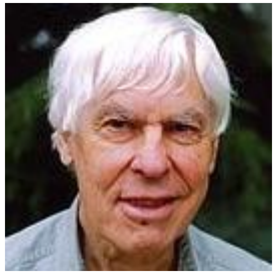
The Department of Mathematics

in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of

Mathematics

Harvard University
Cambridge, Massachusetts

May 1985



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- Doyle, McMullen, "[…]9.
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Why no proven techniques (Arveson)?

Ξ': $B \in \{0,1\}^{\mathbb{N}\times\mathbb{N}} = \Omega'$, does B have finitely many cols with finitely many 1s?

Descriptive set theory + SCI $\Longrightarrow \{\Xi', \Omega', \{0,1\}, \Lambda\} \notin \Delta_3$

- C., *"On the computation of geometric features of spectra of linear operators on Hilbert spaces,"* **Found. Comput. Math.**, 2022.

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For $\alpha \in \{0,1\}^{\mathbb{Z}}$ define

$$[C(\alpha)]_{k,l} = \begin{cases} 1, & k < l, \alpha_k = \alpha_l = 1, \alpha_n = 0 \text{ for } k < n < l \\ 0, & \text{otherwise.} \end{cases}$$

shift on $\mathrm{span}\{e_i : \alpha_i = 1\}$

Given $B \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ set

$$A(B) = \oplus_{j=1}^{\infty} C(\alpha_i^{(j)}), \qquad \alpha_i^{(j)} = \begin{cases} 1, & |i| \leq j \\ B_{|i|-j,j}, & \text{otherwise.} \end{cases}$$

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If $\Xi'(B) = 1$, $\mathrm{Sp}(B) = \{0\} \cup \mathbb{T}$. Otherwise $\mathrm{Sp}(B) = \{z : |z| \le 1\}$.

If classical spectral problem $\in \Delta_3$, so is $\{\Xi', \Omega', \{0,1\}, \Lambda\}$, contradiction!

• C., *"On the computation of geometric features of spectra of linear operators on Hilbert spaces,"* **Found. Comput. Math.**, 2022.

What about additional structure?
Computing spectra with error control…

Motivation: bounded diagonal operators

$$A = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \end{pmatrix}$$

$\boldsymbol{\Lambda}$: Matrix entries of A *(readable info)*

Algorithm: $\Gamma_n(A) = \{a_1, a_2, \dots, a_n\} \to \mathrm{Sp}(A) = \overline{\{a_1, a_2, \dots\}}$ in Haus. Metric.

One-sided error control: $\Gamma_n(A) \subset \mathrm{Sp}(A)$

$$d_{\mathrm{H}}(\Gamma_n(A), \mathrm{Sp}(A)) = \max\left\{ \sup_{x \in \Gamma_n(A)} d(x, \mathrm{Sp}(A)), \sup_{y \in \mathrm{Sp}(A)} d(y, \Gamma_n(A)) \right\}$$

Motivation: bounded diagonal operators

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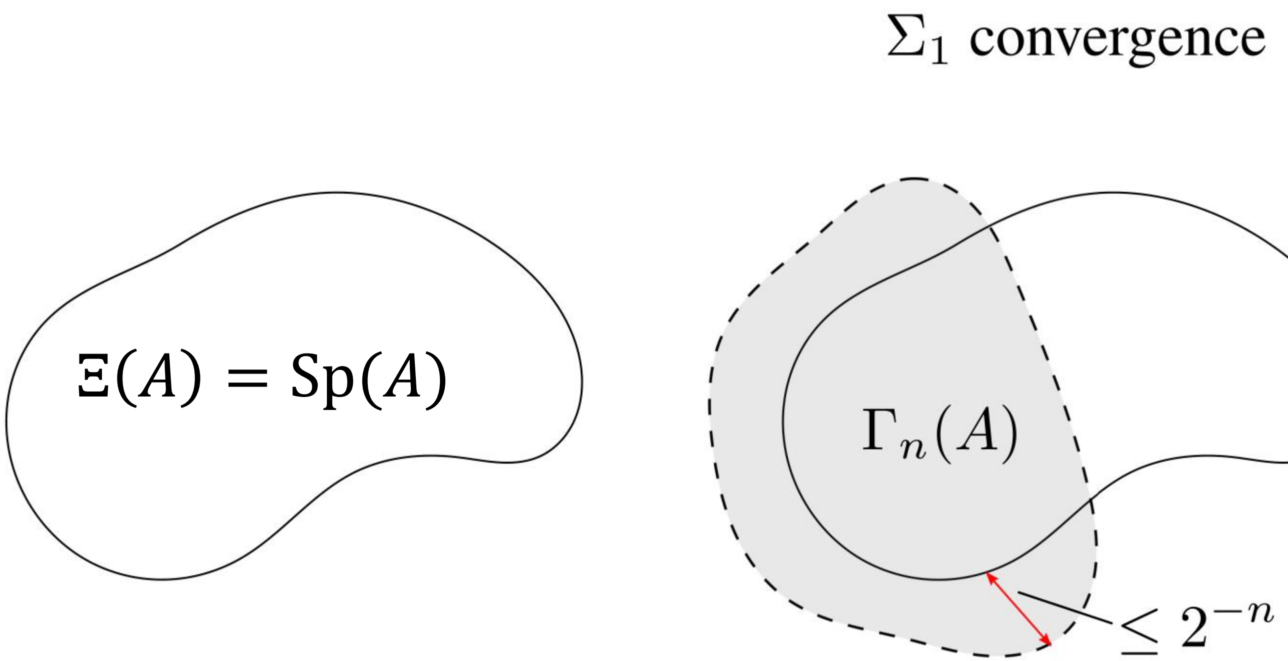
But: No algorithm with $\hat{\Gamma}_n(A) \to \mathrm{Sp}(A)$ with $\mathrm{Sp}(A) \subset \hat{\Gamma}_n(A)$.

Error control for spectral problems

$$d_{\mathrm{H}}(X,Y) = \max\left\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\right\}$$



- Σ_1: $\exists$ alg. $\{\Gamma_n\}$ s.t. $\lim_{n\to\infty} \Gamma_n(A) = \Xi(A)$, $\max_{z\in\Gamma_n(A)} \mathrm{dist}\big(z, \Xi(A)\big) \leq 2^{-n}$

- C., *"The foundations of infinite-dimensional spectral computations,"* **PhD diss., University of Cambridge**, 2020.

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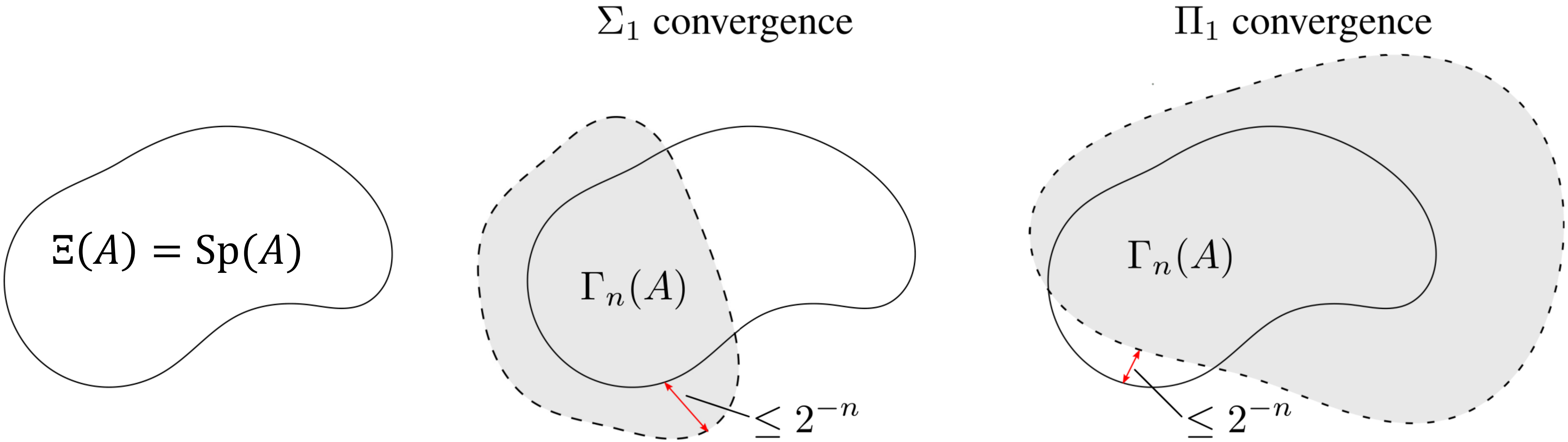

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Such problems can be used in a proof!

- C., *"The foundations of infinite-dimensional spectral computations,"* **PhD diss., University of Cambridge**, 2020.

Reasons it's hard I

$$A = \bigoplus_{r=1}^{\infty} J_{l_r}, \qquad J_{l_r} = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{C}^{l_r \times l_r}$$

Instability

$$\mathrm{Sp}(A) = \begin{cases} \{0\}, & \sup l_r < \infty \\ \{z : |z| \leq 1\}, & \text{otherwise} \end{cases}$$

No $\{\Gamma_n\}$ when given $\{l_r\}_{r=1}^{\infty}$ can determine if it is bounded.

$\Longrightarrow$ No $\{\Gamma_n\}$ computes spectra of gen. tridiagonal operators.

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Always have: $\|(A - zI)^{-1}\|^{-1} \leq \mathrm{dist}(z, \mathrm{Sp}(A))$

Extra assumption: $g(\mathrm{dist}(z, \mathrm{Sp}(A))) \leq \|(A - zI)^{-1}\|^{-1}$

known cts. bijection
$g: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$

Reasons it's hard II

$$A = \bigoplus_{r=1}^{\infty} A_{l_r}, \qquad A_{l_r} = \begin{pmatrix} 1 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 1 & & & & 1 \end{pmatrix} \in \mathbb{C}^{l_r \times l_r}$$

Info at ∞

$$\mathrm{Sp}(A) = \{0,2\}, \qquad \mathrm{Sp}\big(\mathrm{diag}(1,0,\ldots)\big) = \{0,1\}$$

More involved: Suppose for a contradiction $\{\Gamma_n\}$ converges, choose $\{l_r\}_{r=1}^{\infty}$ so $\Gamma_n(A)$ does not converge (try it!)

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Assume access to $\langle Ae_j, e_i\rangle,\ \langle Ae_j, Ae_i\rangle,\ \langle A^*e_j, A^*e_i\rangle$

Sketch of method with error control ($\in \Sigma_1$)

$$\sigma_{\inf}(A) = \inf\{\|Av\| : v \in \mathfrak{D}(A), \|v\| = 1\}$$

$$\gamma(z, A) = \|(A - zI)^{-1}\|^{-1} = \min\{\sigma_{\inf}(A - zI), \sigma_{\inf}(A^* - \bar{z}I)\}$$

$\mathcal{P}_n$ = orthog-projection onto span$\{e_1, \ldots, e_n\}$

Idea: $\sqrt{\sigma_{\inf}(\mathcal{P}_n(A - zI)^*(A - zI)\mathcal{P}_n)} = \sigma_{\inf}([A - zI]\mathcal{P}_n) \downarrow \sigma_{\inf}(A - zI)$

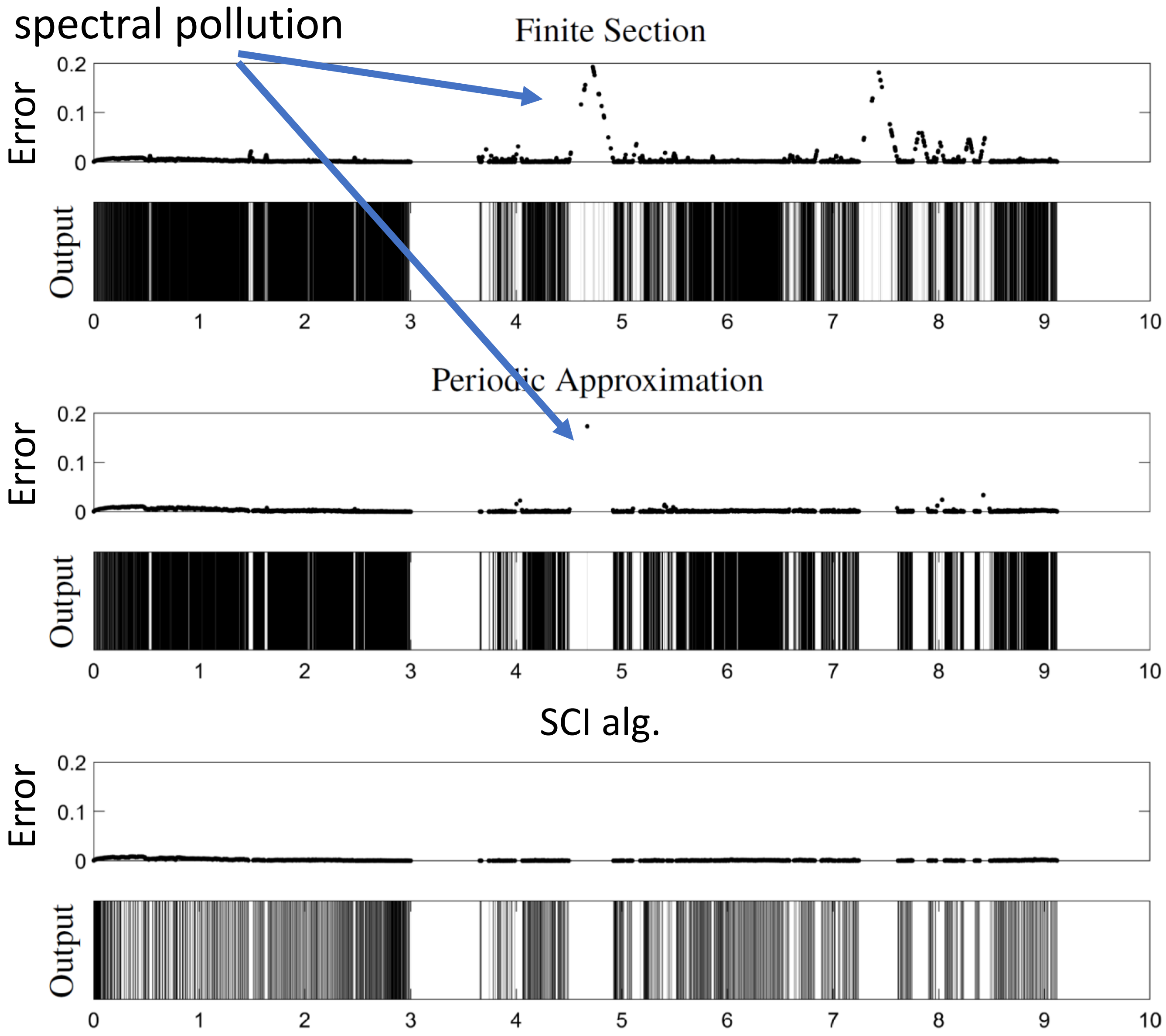
$$g^{-1}(\min\{\sigma_{\inf}([A - zI]\mathcal{P}_n), \sigma_{\inf}([A^* - \bar{z}I]\mathcal{P}_n)\}) \downarrow g^{-1}(\|(A - zI)^{-1}\|^{-1}) \geq \mathrm{dist}(z, \mathrm{Sp}(A))$$

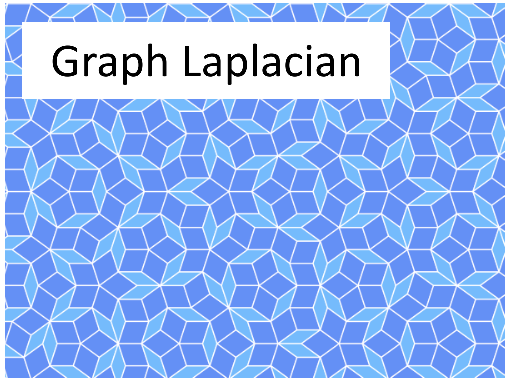
$$\boxed{\|(A - z)^{-1}\|^{-1} \geq g(\mathrm{dist}(z, \mathrm{Sp}(A)))}$$

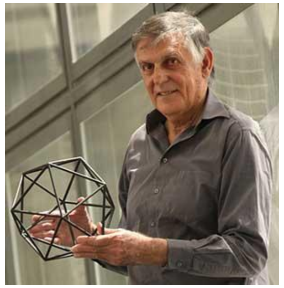
Final ingredient: adaptive search for local minimisers.

• C., Roman, Hansen, "How to compute spectra with error control," **Phys. Rev. Lett.**,2019.

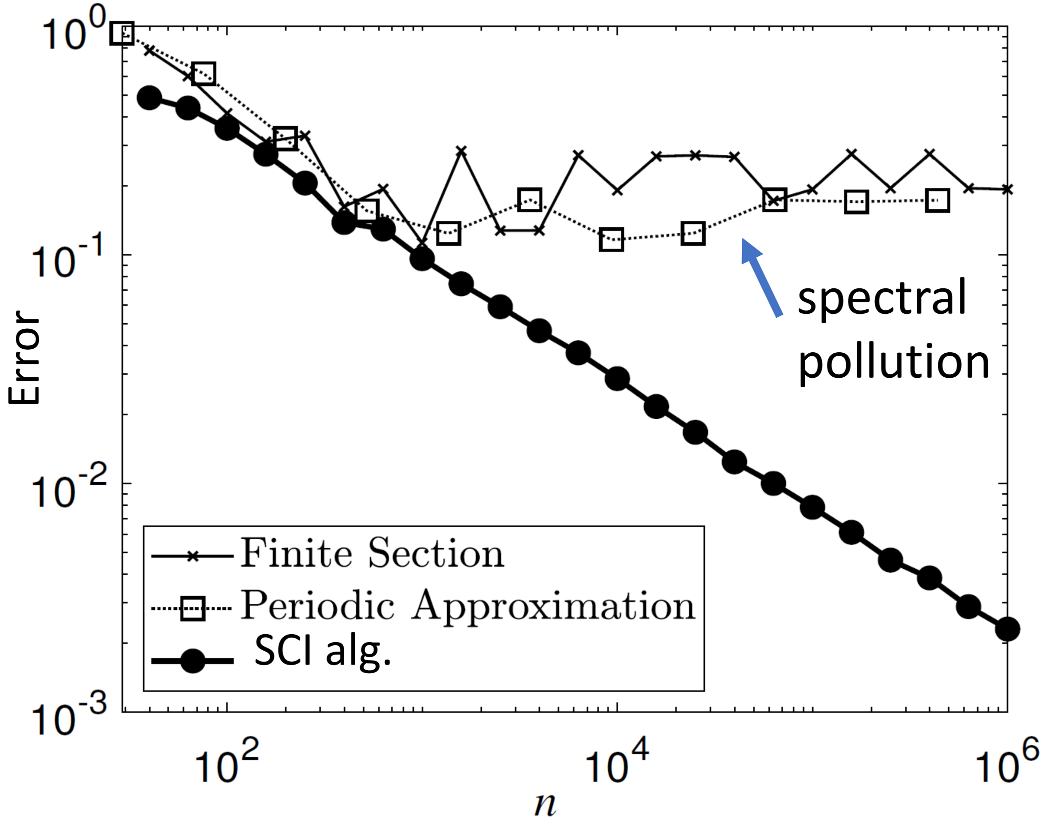
Example: Quasicrystal







Dan Shechtman
(Iowa State, Nobel Prize in Chemistry 2011)



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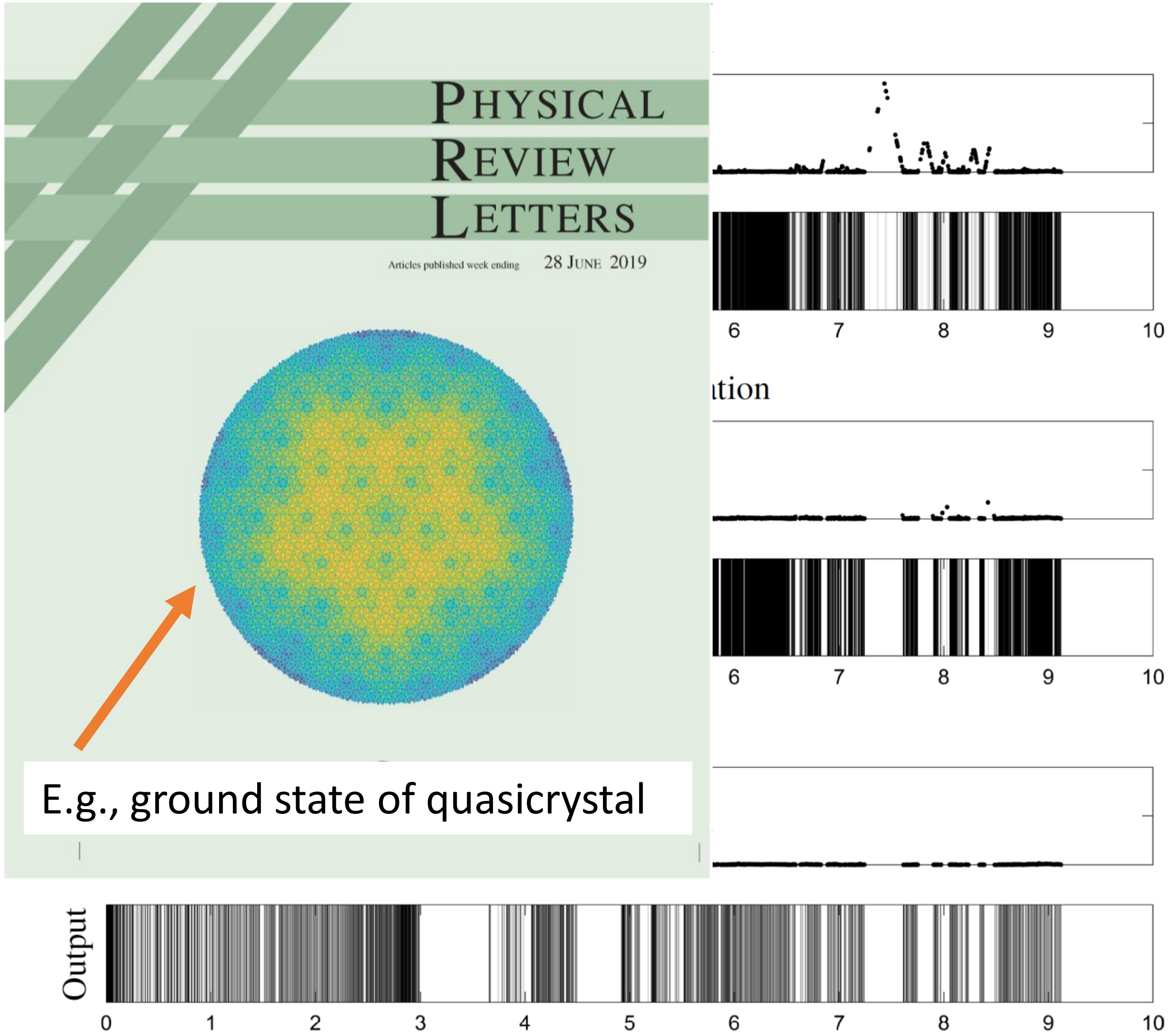

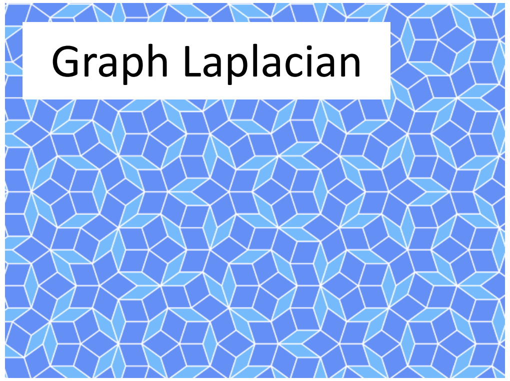

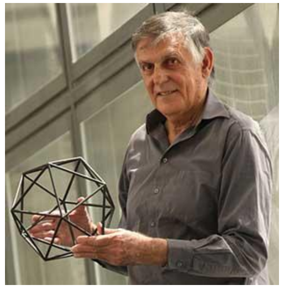

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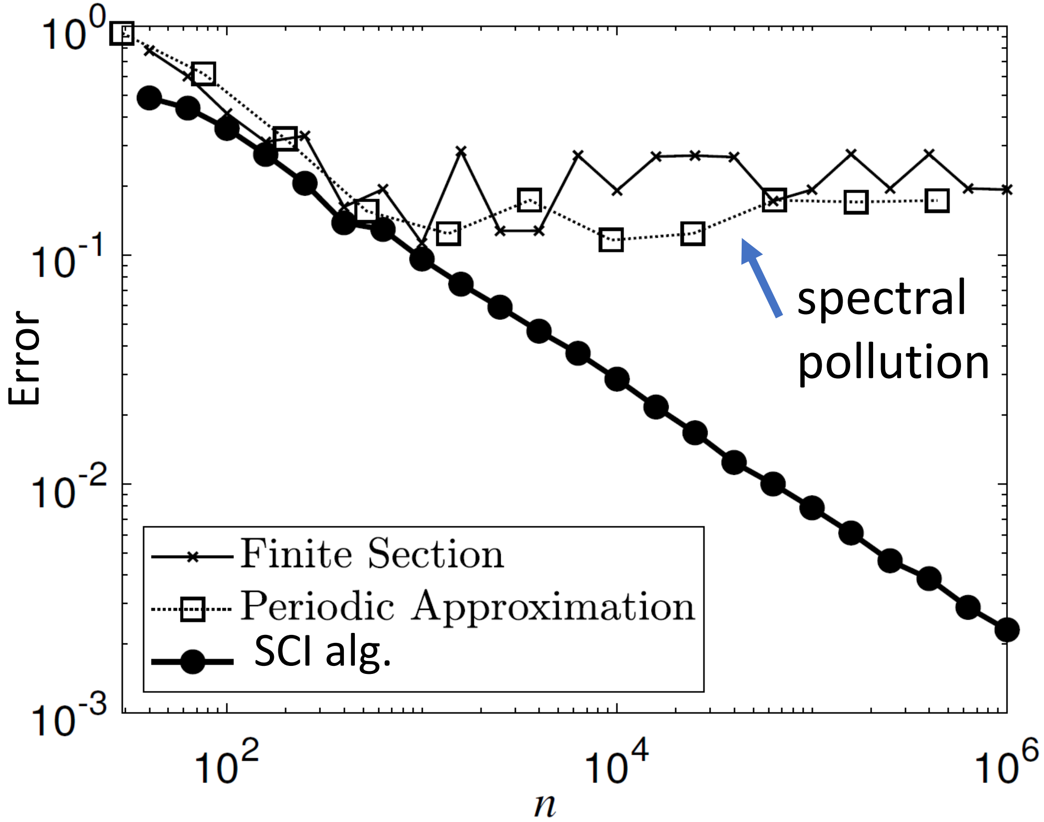

Schwinger's problem

Theorem: Ω: class of self-adjoint diff. operators on $L^2(\mathbb{R}^d)$

$$T = \sum_{k\in\mathbb{Z}^d_{\geq 0},|k|\leq N} c_k(x)\,\partial^k$$

- $C_0^\infty(\mathbb{R}^d)$ a core of T.
- $\{c_k\}$ poly bounded, locally bounded total variation.

Can access:

- $\{c_k(q)\}$ for $q \in \mathbb{Q}^d$.
- Polynomial that bounds $\{c_k\}$ on $\mathbb{R}^d$.

(a) Know $\|c_k\|_{\mathrm{TV}([-n,n]^d)} \leq b_n \Rightarrow \{\mathrm{Sp},\Omega\} \in \Sigma_1$.

(b) Know $\|c_k\|_{\mathrm{TV}([-n,n]^d)} = O(b_n) \Rightarrow \{\mathrm{Sp},\Omega\} \in \Delta_2 \backslash (\Sigma_1 \cup \Pi_1)$.

• C., Hansen, *"The foundations of spectral computations via the solvability complexity index hierarchy,"* **J. Eur. Math. Soc.**, 2022

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Sampling schemes to construct matrix.

Extends to other domains, singular coefficients etc.

Verifiable

Not verifiable

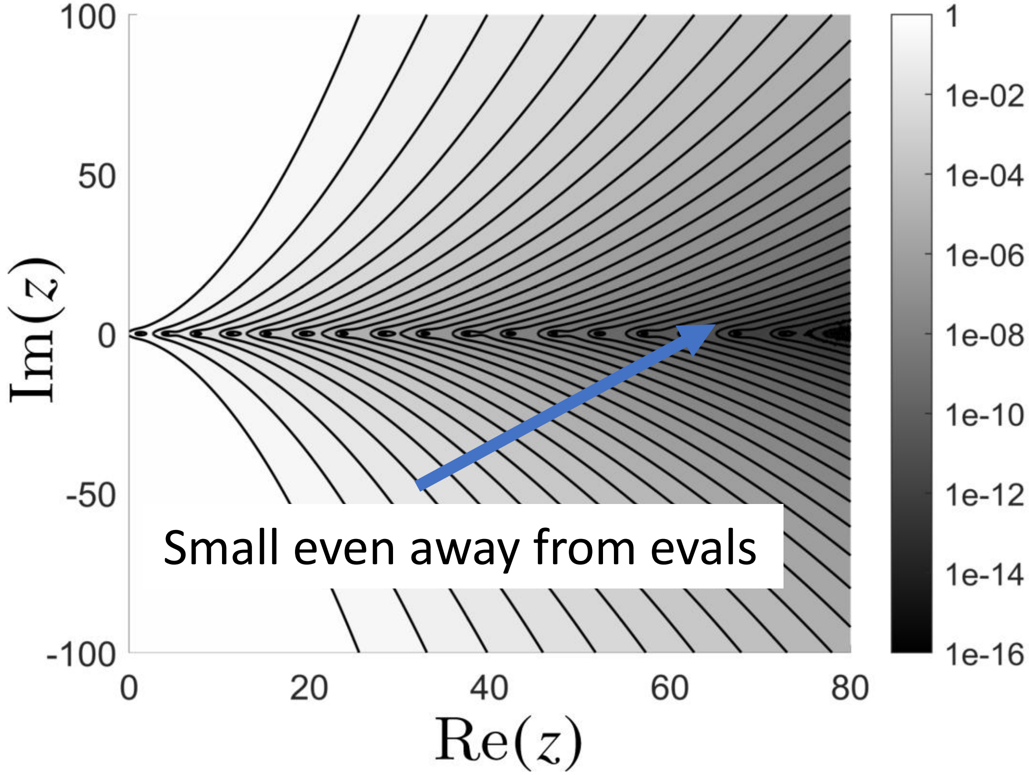
Non-self-adjoint example with non-trivial g

Carl Bender
(Washington, MIT, Heineman Prize 2017)

Michael Berry
(Bristol, Wolf Prize 1998)

$$T = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + ix^3 \text{ on } \mathbb{R}$$

$\|(T - zI)^{-1}\|^{-1}$



j	E_j to 30 digits with interval arithmetic
1	1.156 267 071 988 113 293 799 219 177 999 9
2	4.109 228 752 809 651 535 843 668 478 561 3
3	7.562 273 854 978 828 041 351 809 110 631 4
4	11.314 421 820 195 804 402 233 783 948 426 9
5	15.291 553 750 392 532 388 181 630 791 751 9
6	19.451 529 130 691 728 314 686 111 714 104 4
7	23.766 740 435 485 819 131 558 025 968 789 9
8	28.217 524 972 981 193 297 595 053 878 268 9
9	32.789 082 781 862 957 492 447 371 485 046 3
10	37.469 825 360 516 046 866 428 873 594 530 5
100	627.694 712 248 436 511 352 673 702 901 153 6

- C., *"INFINITE-DIMENSIONAL SPECTRAL COMPUTATIONS, Foundations, Algorithms, and Modern Applications.,"* **CUP**, to appear.

Sampler of results for bounded op. on $l^2(\mathbb{N})$

increasing difficulty

Error control **1 limit** **2 limits** **3 limits**

$$\begin{tikzcd}[column sep=small]
\Pi_0 \arrow[d, phantom, "\|"] & & \Pi_1 \arrow[dr, phantom, "\subsetneq" sloped] & & \Pi_2 \arrow[dr, phantom, "\subsetneq" sloped] & & \Pi_3 \arrow[dr, phantom, "\cdots" sloped] & \\
\Delta_0 \arrow[d, phantom, "\|"] \arrow[r, phantom, "\subsetneq"] & \Delta_1 \arrow[ur, phantom, "\subsetneq" sloped] \arrow[dr, phantom, "\subsetneq" sloped] \arrow[r, phantom, "\subsetneq"] & \Sigma_1 \cup \Pi_1 \arrow[r, phantom, "\subsetneq"] & \Delta_2 \arrow[ur, phantom, "\subsetneq" sloped] \arrow[dr, phantom, "\subsetneq" sloped] \arrow[r, phantom, "\subsetneq"] & \Sigma_2 \cup \Pi_2 \arrow[r, phantom, "\subsetneq"] & \Delta_3 \arrow[ur, phantom, "\subsetneq" sloped] \arrow[dr, phantom, "\subsetneq" sloped] \arrow[r, phantom, "\subsetneq"] & \Sigma_3 \cup \Pi_3 & \cdots \\
\Sigma_0 & & \Sigma_1 \arrow[ur, phantom, "\subsetneq" sloped] & & \Sigma_2 \arrow[ur, phantom, "\subsetneq" sloped] & & \Sigma_3 \arrow[ur, phantom, "\cdots" sloped] &
\end{tikzcd}$$

Sampler of results for bounded op. on $l^2(\mathbb{N})$

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Compact operators

$$\Pi_0 = \Delta_0 = \Sigma_0$$

$$\Delta_0 \subsetneq \Delta_1 \subsetneq \Sigma_1 \cup \Pi_1 \subsetneq \Delta_2 \subsetneq \Sigma_2 \cup \Pi_2 \subsetneq \Delta_3 \subsetneq \Sigma_3 \cup \Pi_3 \cdots$$

$$\Delta_1 \subsetneq \Pi_1 \subsetneq \Delta_2 \subsetneq \Pi_2 \subsetneq \Delta_3 \subsetneq \Pi_3 \cdots$$

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$$\begin{array}{ccccccccccc} & & \Pi_1 & & & & \Pi_2 & & & & \Pi_3 \\ & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \ddots \\ \Delta_1 & & & & \Delta_2 & & & & \Delta_3 & & & \cdots \\ & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \iddots \\ & & \Sigma_1 & & & & \Sigma_2 & & & & \Sigma_3 \end{array}$$

Normal operators

Sampler of results for bounded op. on $l^2(\mathbb{N})$

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Compact operators

"Sparse" operators

$$
\begin{array}{ccccccccccccc}
\Pi_0 & & & & \Pi_1 & & & & \Pi_2 & & & & \Pi_3 \\
\| & & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \ddots \\
\Delta_0 & \subsetneq & \Delta_1 & \subsetneq & \Sigma_1 \cup \Pi_1 & \subsetneq & \Delta_2 & \subsetneq & \Sigma_2 \cup \Pi_2 & \subsetneq & \Delta_3 & \subsetneq & \Sigma_3 \cup \Pi_3 \cdots \\
\| & & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \iddots \\
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\end{array}
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Compact operators: Δ_2

"Sparse" operators: Π_2

Approx. sparse normal op: Σ_1

Normal operators: Σ_2

General operators: Π_3

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\Delta_0 & \subsetneq & \Delta_1 & \subsetneq & \Sigma_1 \cup \Pi_1 & \subsetneq & \Delta_2 & \subsetneq & \Sigma_2 \cup \Pi_2 & \subsetneq & \Delta_3 & \subsetneq & \Sigma_3 \cup \Pi_3 \cdots \\
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\end{array}$$

Approx. sparse normal op

Normal operators

General operators

Zoo of problems: spectral type (pure point, absolutely continuous, singularly continuous), Lebesgue measure and fractal dimensions of spectra, discrete spectra, essential spectra, eigenspaces + multiplicity, spectral radii, essential numerical ranges, geometric features of spectrum (e.g., capacity), spectral gap problem, resonances ...

Example: Analysis helps in applications in dynamical systems…

Operator theory for dynamical systems

- Compact metric space $(\mathcal{X}, d)$ – the state space
- $x \in \mathcal{X}$ – the state

cts $F: \mathcal{X} \to \mathcal{X}$ – the dynamics: $x_{n+1} = F(x_n)$

Henri Poincaré
(Sorbonne)

- Poincaré, ''*Les méthodes nouvelles de la mécanique céleste,*'' **Vol. 2. Gauthier-Villars et fils, imprimeurs-libraires**, 1893.

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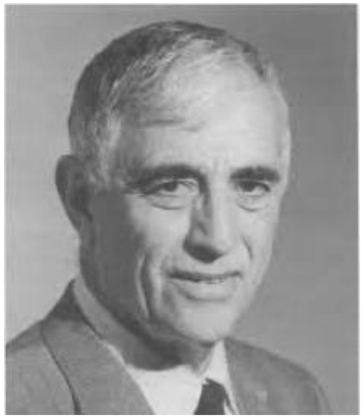
- Borel measure ω on $\mathcal{X}$
- Function space $L^2 = L^2(\mathcal{X}, \omega)$ (elements g called "observables")
- Koopman operator $\mathcal{K}_F: L^2 \to L^2$; $[\mathcal{K}_F g](x) = g(F(x))$

Bernard Koopman (Columbia)



John von Neumann (IAS)

NB: Pointwise definition of $\mathcal{K}_F$ needs $F\#\omega \ll \omega$ – this will hold throughout.
NB: $\mathcal{K}_F$ bounded equivalent to $\mathrm{d}F\#\omega/\mathrm{d}\omega \in L^\infty$ – this will hold throughout (can be dropped).

- Koopman, "*Hamiltonian systems and transformation in Hilbert space,*" **Proc. Natl. Acad. Sci. USA**, 1931.
- Koopman, v. Neumann, "*Dynamical systems of continuous spectra,*" **Proc. Natl. Acad. Sci. USA**, 1932.

Operator theory for dynamical systems

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- ***Unknown*** cts $F: \mathcal{X} \to \mathcal{X}$ – the dynamics: $x_{n+1} = F(x_n)$
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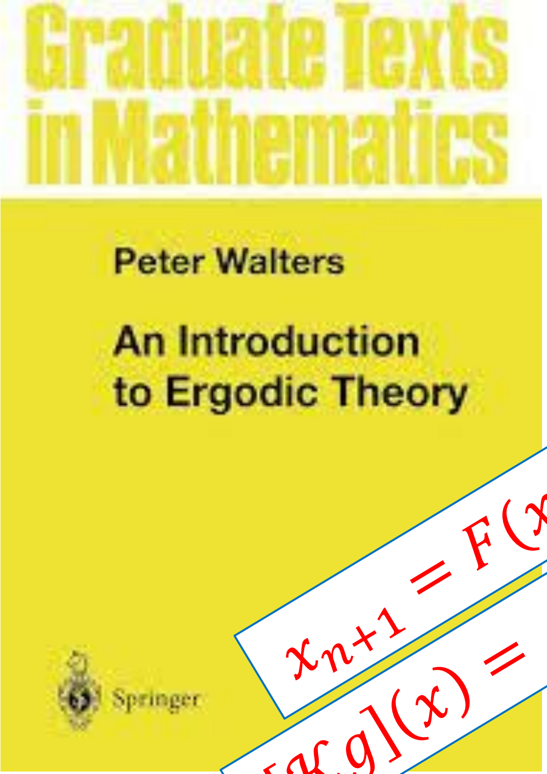
- Koopman operator $\mathcal{K}_F: L^2 \to L^2$; $[\mathcal{K}_F g](x) = g(F(x))$
- ***Available*** snapshot data: $\left\{\left(x^{(m)}, y^{(m)} = F\left(x^{(m)}\right)\right): m = 1, \ldots, M\right\}$

Can we compute spectral properties from sampling trajectories?

Fundamental object

Fundamental in ergodic theory



$$x_{n+1} = F(x_n)$$

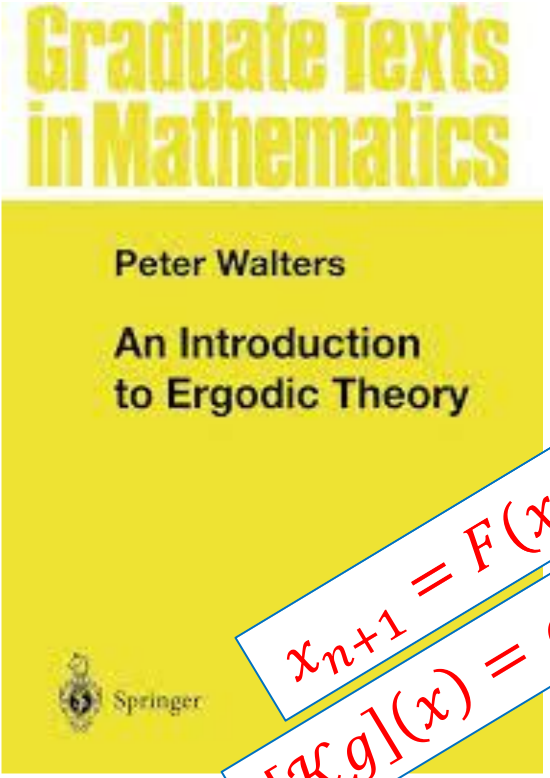
$$[\mathcal{K}g](x) = g(F(x))$$

E.g., key to ergodic theorems of Birkhoff and von Neumann.

Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

Fundamental object

Fundamental in ergodic theory



$x_{n+1} = F(x_n)$

$[\mathcal{K}g](x) = g(F(x))$

E.g., key to ergodic theorems of Birkhoff and von Neumann.

Can provide a *diagonalization* of a nonlinear system.

eigenfunction of $\mathcal{K}$

continuous spectrum

$$g(x) = \sum_{\text{eigenvalues } \lambda_j} c_{\lambda_j} \varphi_{\lambda_j}(x) + \int_{-\pi}^{\pi} \phi_{\theta,g}(x)\, \mathrm{d}\theta$$

$$g(x_n) = [\mathcal{K}^n g](x_0)$$

$$= \sum_{\text{eigenvalues } \lambda_j} c_{\lambda_j} \boxed{\lambda_j{}^n} \varphi_{\lambda_j}(x_0) + \int_{-\pi}^{\pi} \boxed{e^{in\theta}} \phi_{\theta,g}(x_0)\, \mathrm{d}\theta$$

Spectral properties encode: geometric features, invariant measures, transient behavior, long-time behavior, coherent structures, quasiperiodicity, etc.

Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

Fundamental object

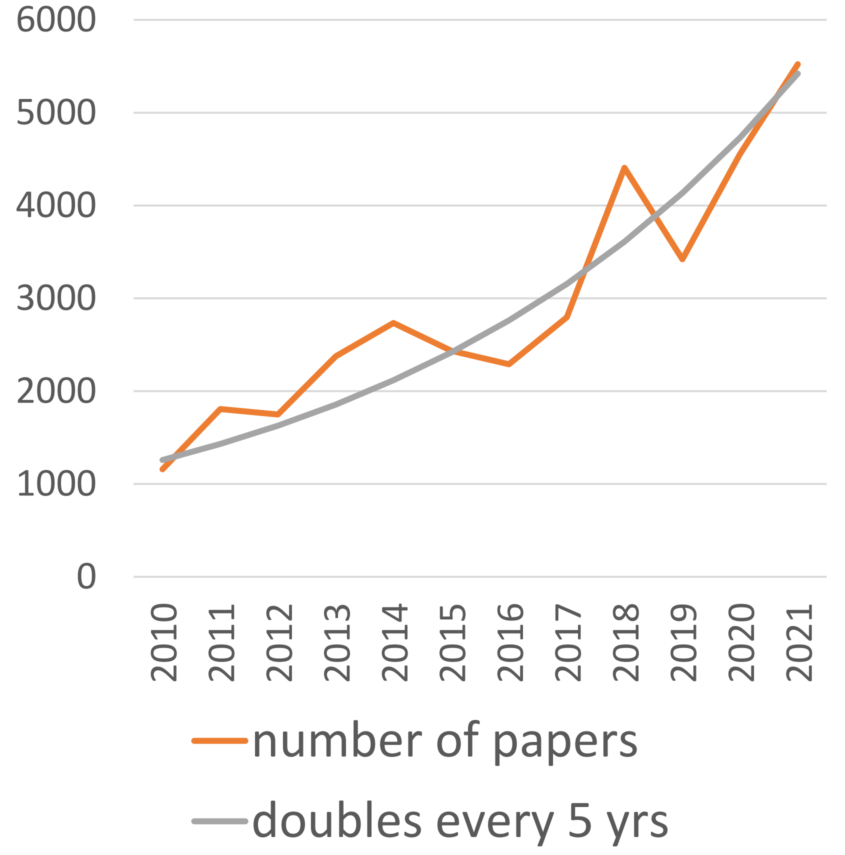
Fundamental in ergodic theory

Can provide a *diagon*

$g(x) = \sum$ eigenvalues

+ huge recent interest in applications of spectra from trajectories

New papers on spectra of Koopman operators



E.g., key to ergodic theorems of Birkhoff and von Neumann.

Spectral properties e
invariant measures, t
behavior, coherent st...uctures, quasiperiodicity, etc.

Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

Extended Dynamic Mode Decomposition (EDMD)

Functions $\psi_j : \mathcal{X} \to \mathbb{C},\ j = 1, \dots, N$

$$\boxed{\left\{x^{(m)}, y^{(m)} = F\left(x^{(m)}\right)\right\}_{m=1}^{M}}$$

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} \underbrace{w_m}_{\text{quadrature weights}} \overline{\psi_j\left(x^{(m)}\right)} \psi_k\left(\overbrace{x^{(m)}}^{\text{quadrature points}}\right) = \left[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}^{*}}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X} \right]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j\left(x^{(m)}\right)} \underbrace{\psi_k\left(y^{(m)}\right)}_{[\mathcal{K}\psi_k](x^{(m)})} = \left[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}^{*}}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \psi_1(y^{(1)}) & \cdots & \psi_N(y^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(y^{(M)}) & \cdots & \psi_N(y^{(M)}) \end{pmatrix}}_{\Psi_Y} \right]_{jk}$$

Finite Section Approximation

$$\mathcal{K} \longrightarrow \left(\Psi_X^{\ *} W \Psi_X\right)^{-1} \Psi_X^{\ *} W \Psi_Y \in \mathbb{C}^{N \times N}$$

- Schmid, *"Dynamic mode decomposition of numerical and experimental data,"* **J. Fluid Mech.**, 2010.
- Rowley, Mezić, Bagheri, Schlatter, Henningson, *"Spectral analysis of nonlinear flows,"* **J. Fluid Mech.**, 2009.
- Williams, Kevrekidis, Rowley *"A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition,"* **J. Nonlinear Sci.**, 2015.

Residual DMD (ResDMD)

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \psi_k(x^{(m)}) = [\Psi_X{}^* W \Psi_X]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \underbrace{\psi_k(y^{(m)})}_{[\mathcal{K}\psi_k](x^{(m)})} = [\Psi_X{}^* W \Psi_Y]_{jk}$$

New matrix: $$\langle \mathcal{K}\psi_k, \mathcal{K}\psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(y^{(m)})}\, \psi_k(y^{(m)}) = \underbrace{[\Psi_Y{}^* W \Psi_Y]_{jk}}_{\text{Uses same trajectory data}}$$

- C., Townsend, "*Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems,*" **Commun. Pure Appl. Math.**, 2023.
- C., Ayton, Szőke, "*Residual Dynamic Mode Decomposition,*" **J. Fluid Mech.**, 2023.
- Code: https://github.com/MColbrook/Residual-Dynamic-Mode-Decomposition

Upper bound on SCI: $\in \Sigma_2$

Implies $\mathcal{K}$ is unitary

Class of systems: $\Omega_{\mathcal{X}} = \{F: \mathcal{X} \to \mathcal{X} \mid F \text{ cts, measure preserving, invertible}\}$.

Data an algorithm can use: $\mathcal{T}_F = \{(x, y_m) | x \in \mathcal{X}, d(y_m, F(x)) \leq 2^{-m}\}$.

Theorem: There **exists** *deterministic* algorithms $\{\Gamma_{N,M}\}$ using $\mathcal{T}_F$ such that $\lim_{N\to\infty}\lim_{M\to\infty}\Gamma_{N,M}(F) = \mathrm{Sp}(\mathcal{K}_F)$ for all $F \in \Omega_{\mathcal{X}}$. (Moreover $\in \Sigma_2^A$)

Idea: Use the above matrices to compute

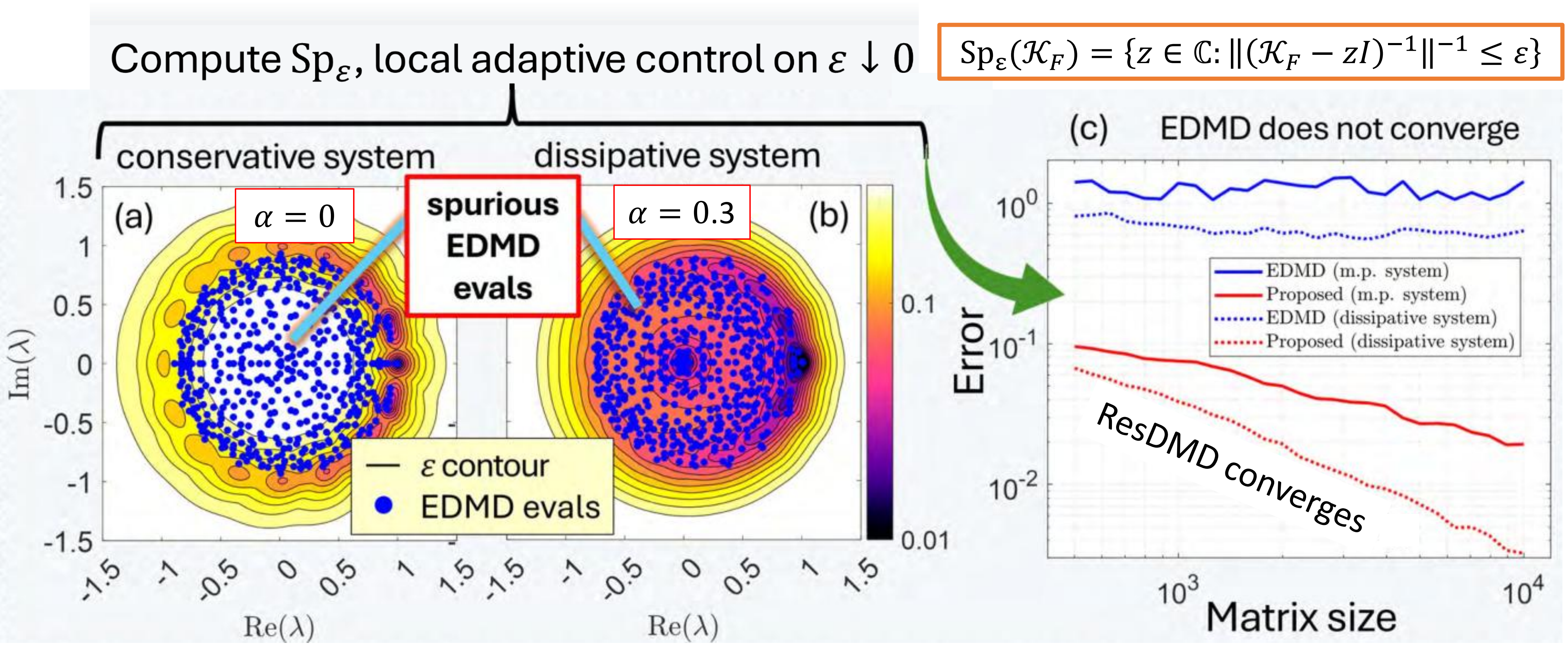
$$\lim_{N\to\infty}\lim_{M\to\infty}\gamma_{N,M}(z, F) = \|(\mathcal{K}_F - zI)^{-1}\|^{-1} = \mathrm{dist}(z, \mathrm{Sp}(\mathcal{K}_F))$$

N = size of basis, M = amount of data (quadrature)

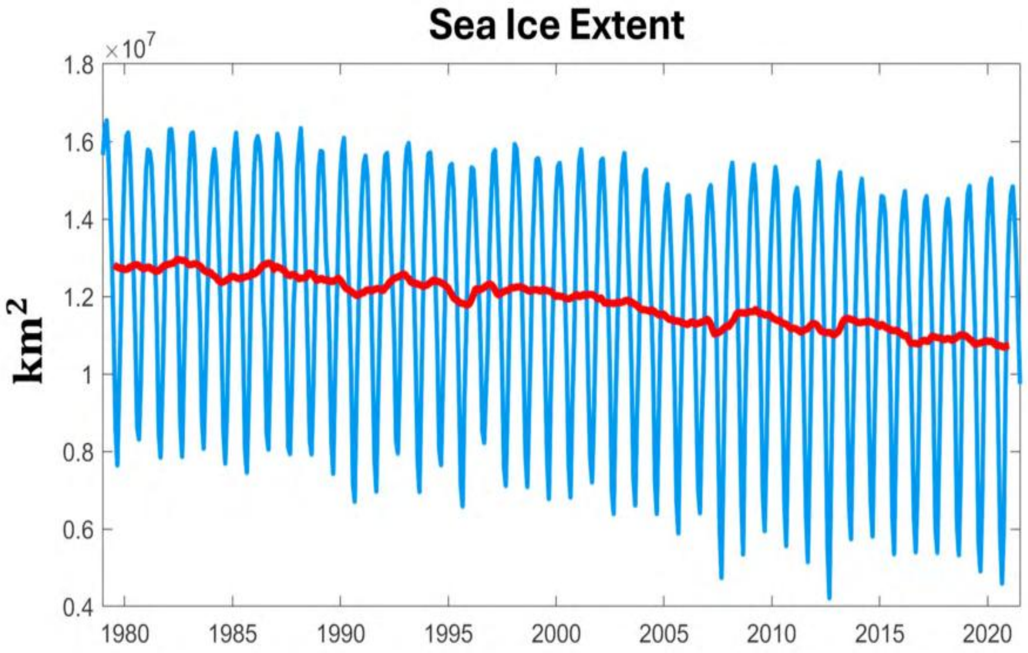
Example: Finite sections don't converge

- Duffing oscillator: $\dot{x} = y, \dot{y} = -\alpha y + x(1 - x^2)$, sampled $\Delta t = 0.3$.
- Gaussian radial basis functions, Monte Carlo integration ($M = 50000$)

$\mathrm{Sp}_\varepsilon(\mathcal{K}_F) = \{z \in \mathbb{C} : \|(\mathcal{K}_F - zI)^{-1}\|^{-1} \leq \varepsilon\}$



Practical Gains: Arctic Sea Ice Forecasting



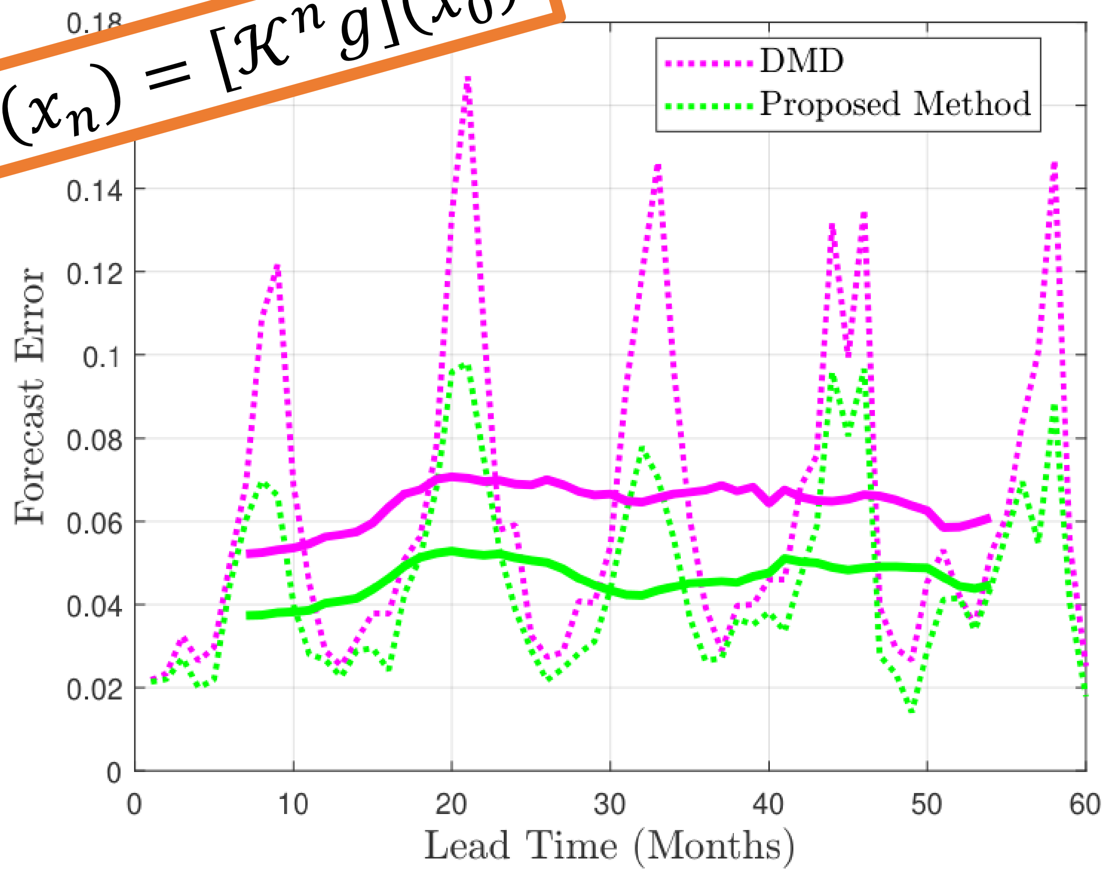
Monthly average from satellite passive microwave sensors.

Motivation: Arctic amplification, polar bears, local communities, effect on extreme weather in Northern hemisphere,…

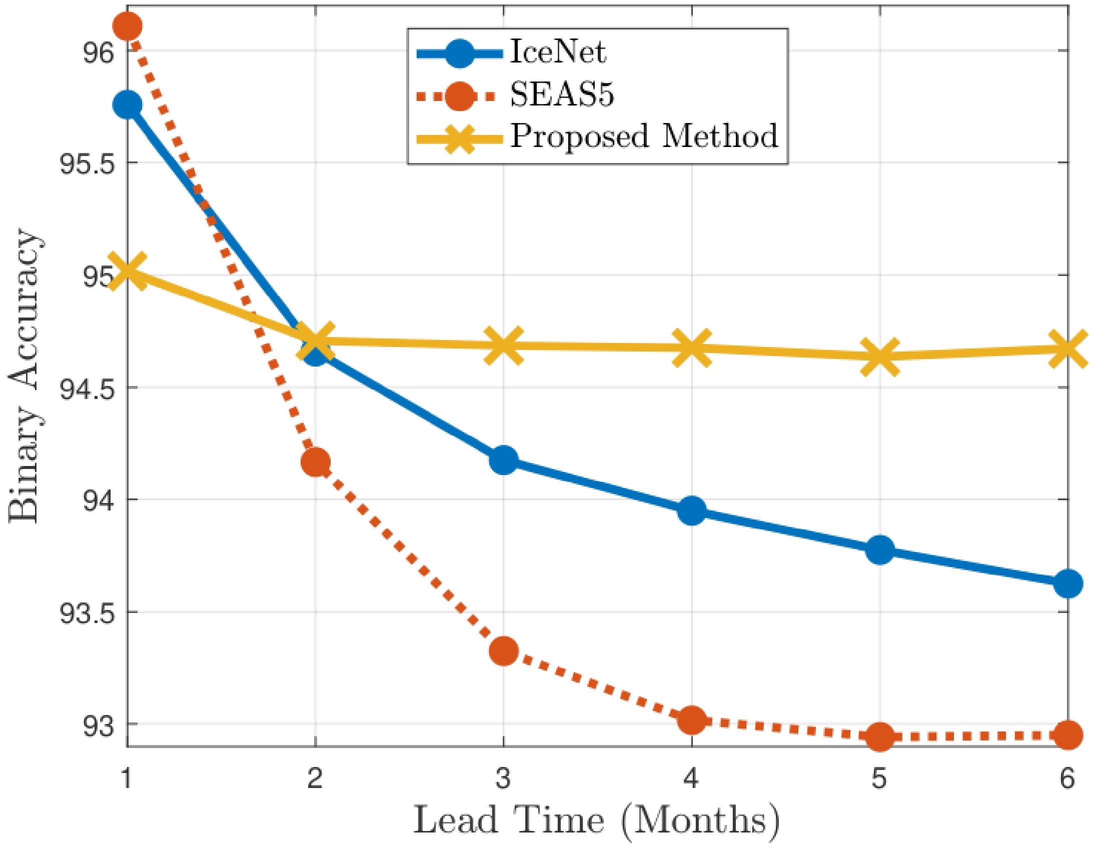
Problem: Very hard to predict more than two months in advance.

Arctic case: Avoiding spurious eigenvalues helps!

$$g(x_n) = [\mathcal{K}^n g](x_0)$$



Relative mean squared error over 2016-2020. Model built from 2005-2015 data. (Solid lines moving 12-month mean.)



Mean binary accuracy over test years 2012-2020. *(IceNet: Andersson et al, "Seasonal Arctic sea ice forecasting with probabilistic deep learning." Nature Communications, 2021.)*

Lower bound on SCI: $\notin \Delta_2$

Implies $\mathcal{K}$ is unitary

Class of systems: $\Omega_{\mathbb{D}} = \{F: \overline{\mathbb{D}} \to \overline{\mathbb{D}} \mid F \text{ cts, measure preserving, invertible}\}$.

Data an algorithm can use: $\mathcal{T}_F = \{(x, y_m) \mid x \in \overline{\mathbb{D}}, \|F(x) - y_m\| \leq 2^{-m}\}$.

Theorem: There **does not exist** any sequence of deterministic algorithms $\{\Gamma_n\}$ using $\mathcal{T}_F$ such that $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

NB: Similarly, no random algorithms converging with probability $> 1/2$.

Double limit is necessary.

• C., Mezić, Stepanenko, "Adversarial Dynamical Systems Reveal Limits and Rules for Trustworthy Data-Driven Learning," **preprint**, 2025.

Proof idea: Constructing an adversary

F_0: rotation by π, $\mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

Phase transition lemma: Let $X = \{x_1, \ldots, x_N\}, Y = \{y_1, \ldots, y_N\}$ be distinct points in annulus $\mathcal{A} = \{x \in \mathbb{D} | 0 < R < \|x\| < r < 1\}$ with $X \cap Y = \emptyset$. There exists a measure-preserving homeomorphism H such that H acts as the identity on $\mathbb{D} \backslash \mathcal{A}$ and $H(y_j) = F_0(H(x_j)), j = 1, \ldots, N$.

Conjugacy of data ($x_j \to y_j$) with F_0

Idea: Use lemma to trick any algorithm into oscillating between spectra.

- Brown and Halperin. "*On certain area-preserving maps.*" **Annals of Mathematics**, 1935.

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

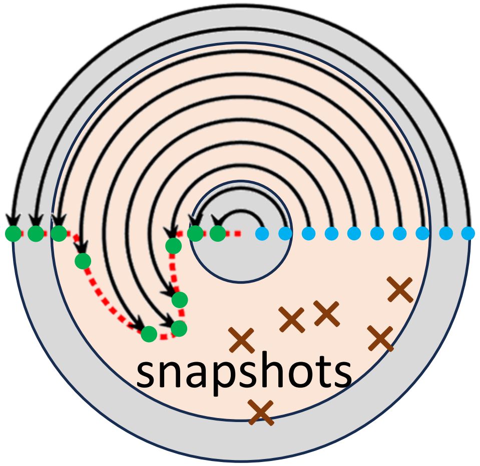
Build an adversarial F…

$\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

Build an adversarial F…



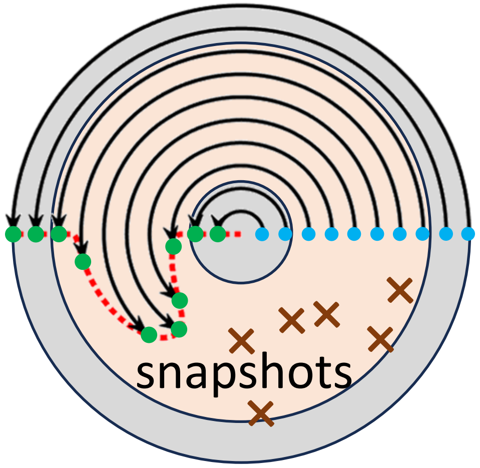
$\widetilde{F_1}(r,\theta) = \big(r, \theta + \pi + \phi(r)\big)$, $\mathrm{supp}(\phi) \subset [1/4, 3/4]$
$\mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) = \mathbb{T}$ (unit circle).

$\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \forall F \in \Omega_{\mathbb{D}}$.

Build an adversarial F...



$\widetilde{F_1}(r,\theta) = \big(r, \theta + \pi + \phi(r)\big)$, $\mathrm{supp}(\phi) \subset [1/4, 3/4]$

$\mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) = \mathbb{T}$ (unit circle).

$\lim_{n\to\infty} \Gamma_n\big(\widetilde{F_1}\big) = \mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) \Rightarrow \exists n_1$ s.t. $\mathrm{dist}(i, \Gamma_{n_1}\big(\widetilde{F_1}\big)) \le 1$.

BUT Γ_{n_1} uses finite amount of info to output $\Gamma_{n_1}\big(\widetilde{F_1}\big)$.

Let X, Y correspond to these snapshots.

$\boxed{\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \le 2^{-m}\}}$

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty}\Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

Build an adversarial F…

$\widetilde{F_1}(r,\theta) = \big(r, \theta + \pi + \phi(r)\big)$, $\mathrm{supp}(\phi) \subset [1/4, 3/4]$

$\mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) = \mathbb{T}$ (unit circle).

$\lim_{n\to\infty}\Gamma_n\big(\widetilde{F_1}\big) = \mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) \Rightarrow \exists n_1$ s.t. $\mathrm{dist}\big(i, \Gamma_{n_1}\big(\widetilde{F_1}\big)\big) \leq 1$.

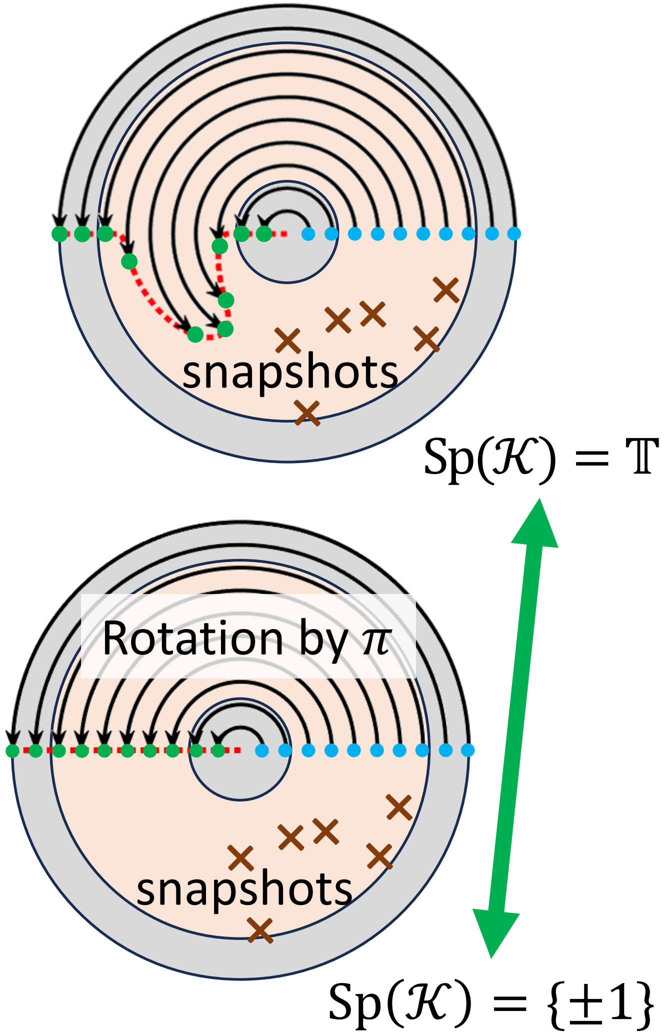
BUT Γ_{n_1} uses finite amount of info to output $\Gamma_{n_1}\big(\widetilde{F_1}\big)$.

Let X, Y correspond to these snapshots.

Lemma: $F_1 = H_1^{-1} \circ F_0 \circ H_1$ on annulus $\mathcal{A}_1$.

<u>Consistent data</u> $\Rightarrow \Gamma_{n_1}(F_1) = \Gamma_{n_1}\big(\widetilde{F_1}\big)$, $\mathrm{dist}(i, \Gamma_{n_1}(F_1)) \leq 1$

BUT $\mathrm{Sp}(\mathcal{K}_{F_1}) = \mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$



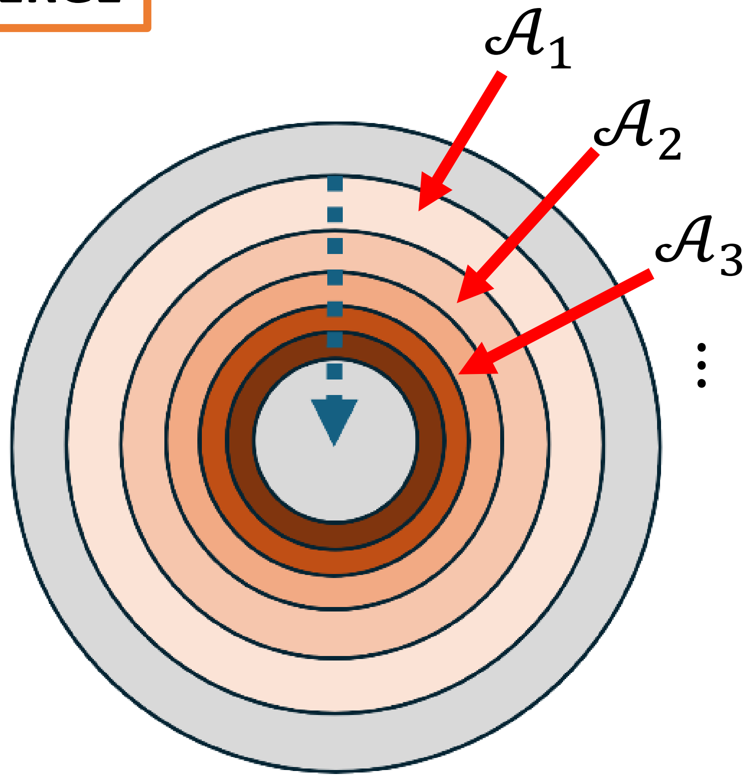
Proof idea: Constructing an adversary

Inductive step: Repeat on annuli, $F_k = H_k^{-1} \circ F_0 \circ H_k$ on $\mathcal{A}_k$. $F = \lim_{k\to\infty} F_k$

<u>Consistent data</u> $\Rightarrow \Gamma_{n_k}(F) = \Gamma_{n_k}(\widetilde{F_k})$, $\mathrm{dist}(i, \Gamma_{n_k}(F)) \leq 1$, $n_k \to \infty$

BUT $\mathrm{Sp}(\mathcal{K}_F) = \mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

CANNOT CONVERGE



Cascade of disks

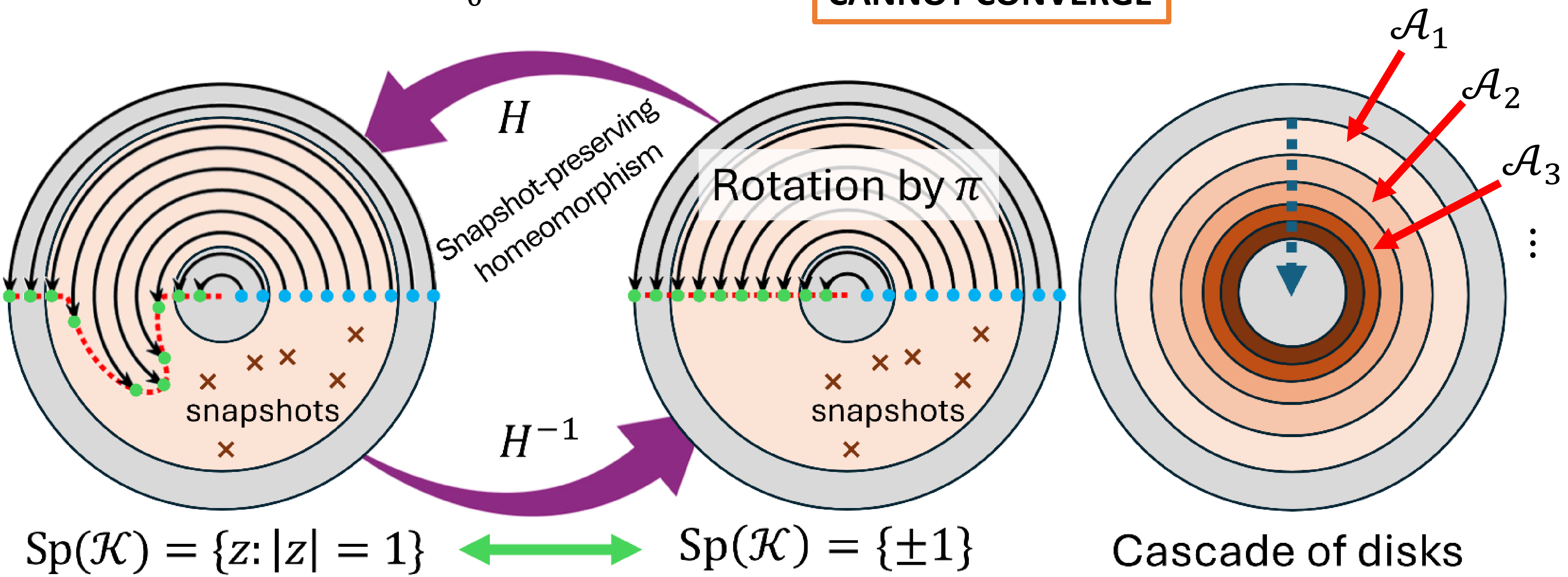
Proof idea: Constructing an adversary

Inductive step: Repeat on annuli, $F_k = H_k^{-1} \circ F_0 \circ H_k$ on $\mathcal{A}_k$. $F = \lim_{k\to\infty} F_k$

Consistent data $\Rightarrow \Gamma_{n_k}(F) = \Gamma_{n_k}(\widetilde{F_k})$, $\mathrm{dist}(i, \Gamma_{n_k}(F)) \leq 1$, $n_k \to \infty$

BUT $\mathrm{Sp}(\mathcal{K}_F) = \mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

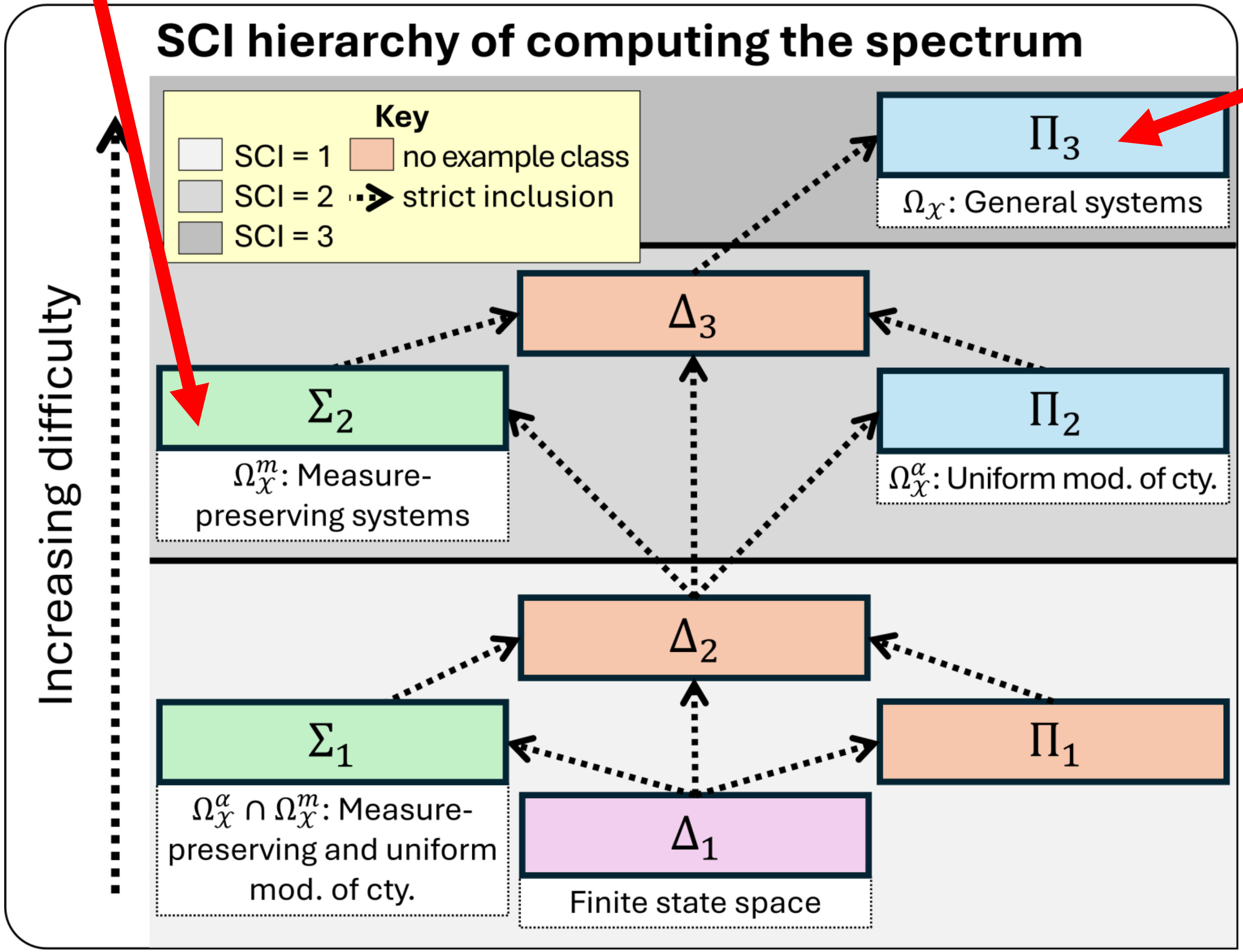
CANNOT CONVERGE



Classification for Koopman

Lower + upper bounds

3 limits needed in general!



Different classes:

$\Omega_{\mathcal{X}} = \{F: \mathcal{X} \to \mathcal{X} \mid F \text{ cts}\}$

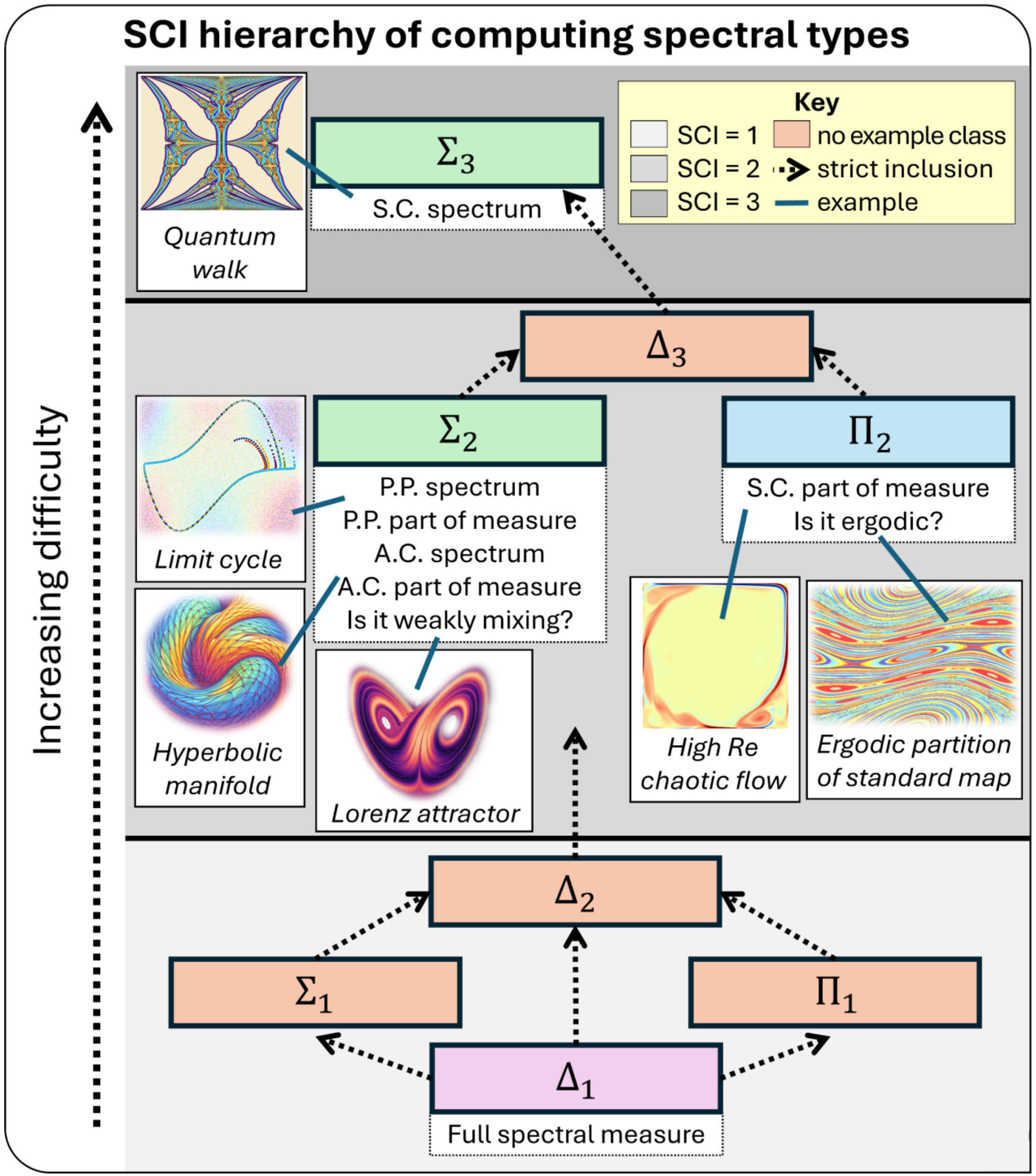
$\Omega_{\mathcal{X}}^m = \{F: \mathcal{X} \to \mathcal{X} \mid F \text{ cts, m. p.}\}$

$\Omega_{\mathcal{X}}^\alpha = \{F: \mathcal{X} \to \mathcal{X} \mid F \text{ mod. cty. } \alpha\}$

$[d_{\mathcal{X}}(F(x), F(y)) \leq \alpha(d_{\mathcal{X}}(x, y))]$

Optimal algorithms and classifications of dynamical systems.

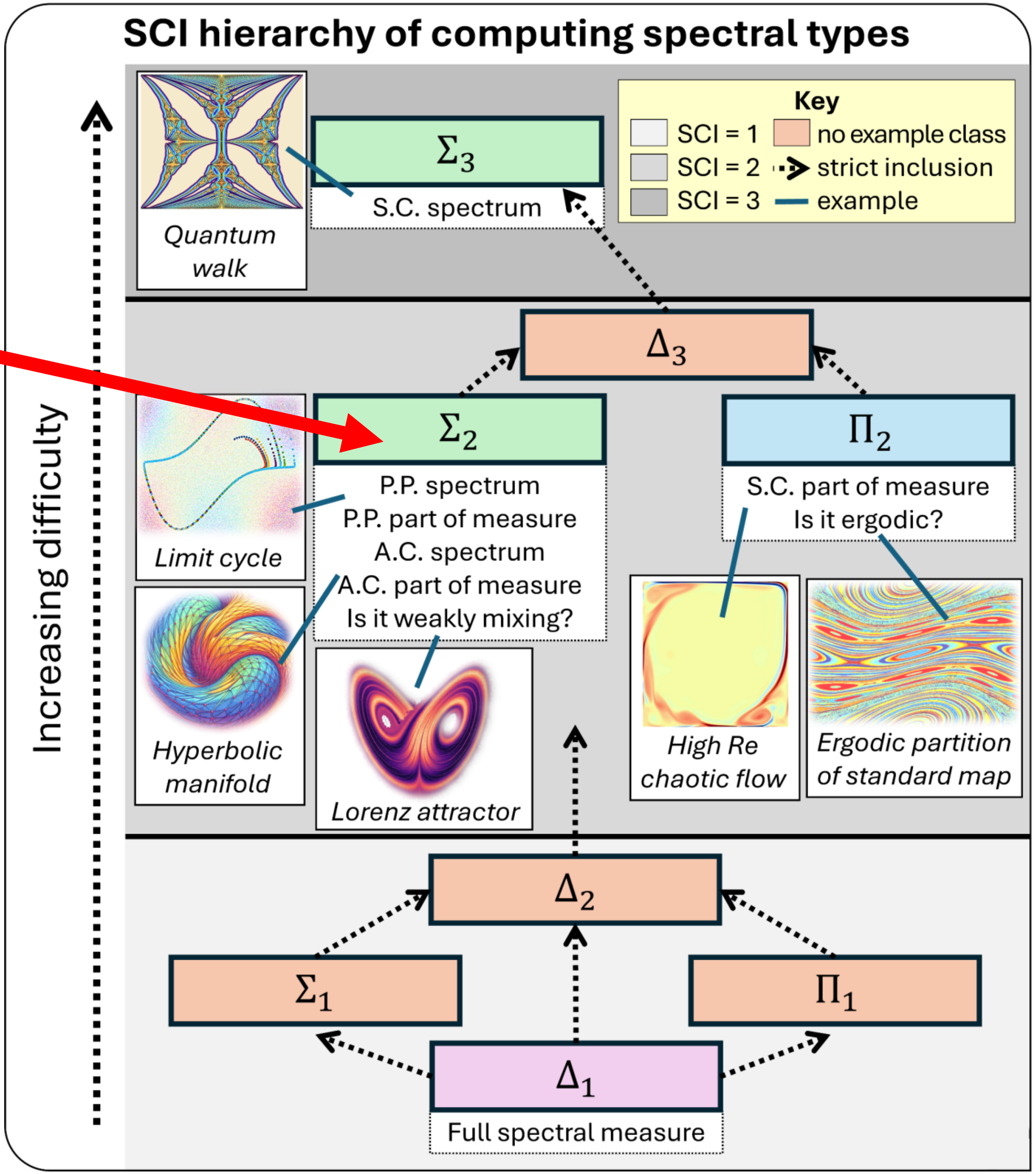
Classification for Koopman II



Classification for Koopman II

Example: **Theorem**
For smooth, measure-preserving systems on a torus, learning eigenfunctions or even determining if there are any has SCI $= 2$ (even if we can sample derivatives).

Finding finite-dimensional embeddings in which the dynamics are linear is very hard!



Conclusion: **FOUNDATIONS $\longleftrightarrow$ METHODS**

- $\exists$ interesting mathematical structure in inf.-dim. spectral computations.
- Many spectral problems in inf. dim. are impossible. *Some harder than others*
- ***SCI hierarchy*** is a tool for discovering the foundations of computation.

  Lower bounds $\Longrightarrow$ spot assumptions needed to lower SCI.

  Upper bounds $\Longrightarrow$ new "inf.-dim." algorithms. *Rigorous, optimal, practical.*
- $\Sigma_1 \cup \Pi_1 \Longrightarrow$ computer-assisted proofs (e.g., Dirac-Schwinger poof implicit Σ_1)

Further examples not covered in talk: foundations of AI, optimization, PDEs, resonances, computer-assisted proofs, spectral measures,…

Shameless final plug…

Upcoming book with CUP:

INFINITE-DIMENSIONAL SPECTRAL COMPUTATIONS

Foundations, Algorithms, and Modern Applications

100s of: classifications, algorithms, examples (including full code), figures, exercises (including full solutions).

Out this (2025) holiday season (hopefully!)…

Contents

If something interests you, please speak to me after.

Shameless final plug…

Upcoming book with CUP:

INFINITE-DIMENSIONAL SPECTRAL COMPUTATIONS

Foundations, Algor n
Applic

Connections with harmonic analysis. Main tool is computing $(A, z, u) \mapsto (A - zI)^{-1}u$
Lower bounds through things like Anderson localization.

100s of: classifications examples (including fu exercises (including full solutions).

*****Out this (2025) holiday season (hopefully!)…*****

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Shameless final plug…

Upcoming book with CUP:

INFINITE-DIMENSIONAL SPECTRAL COMPUTATIONS

Foundations, Algor … n
Applic…

Use $\gamma(z, A)$ to compute more complex objects like fractal dimensions. Applications include aperiodic operators.

100s of: classifications, algorithms, examples (including full code), figures, exercises (including full solutions).

Out this (2025) holiday season (hopefully!)…

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Shameless final plug…

Upcoming book with CUP:

INFINITE-DIMENSIONAL SPECTRAL COMPUTATIONS

Foundations, Algorithms, and Modern

Main tool is essential injection moduli (Edmunds & Evans 1987):

$$\tau_{\inf}(A)=\inf\left\{\liminf_{n\to\infty}\|Ax_n\|: x_n\in\mathcal{D}(A), \|x_n\|=1, x_n\to^w 0\right\}$$

Typically incurs and extra limit. $W_e(A)$ *is universally* Π_2

classifications, algorithms, examples (including full code), figures, exercises (including full solutions).

****Out this (2025) holiday season (hopefully!)…*****

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Foundations, Algorithms, and Modern Applications

100s of: classificat
examples (includin
exercises (including full solutions).

Injection moduli for $T(z)$.
Contour methods for discrete spectra of holomorphic familes.

****Out this (2025) holiday season (hopefully!)…***

Contents

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