The image features three vertical panels of contour plots on the left side. The top panel is orange and shows a series of concentric, wavy contours. The middle panel is blue and shows a more complex, irregular contour pattern. The bottom panel is green and shows a highly detailed, symmetrical contour pattern with many small, nested loops.

Convergent Algorithms for Infinite-Dimensional Nonlinear Spectral Problems

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University of Cambridge

28/11/2025

For papers and talk slides/videos, visit:
<http://www.damtp.cam.ac.uk/user/mjc249/home.html>

Motivating example throughout talk

- Klein–Gordon equation describes relativistic spinless particle:

$$\left(-\nabla^2 + m^2 + \left[\frac{\partial}{\partial t} - ieq \right]^2 \right) \psi = 0$$

- Separable solutions $\psi(x, t) = e^{izt} f(x)$ lead to spectral problem:

$$(-\nabla^2 + m^2 - [eq - z]^2) f = 0.$$

Nonlinear in spectral parameter z .

- Replace $-\nabla^2 + m^2$ by positive operator H_0 and eq by V :

$$T(z) = H_0 - (V - zI)^2.$$

World's 1st eigenvector was from a nonlinear spectral problem!

Modes of oscillations are roots of Bessel functions.

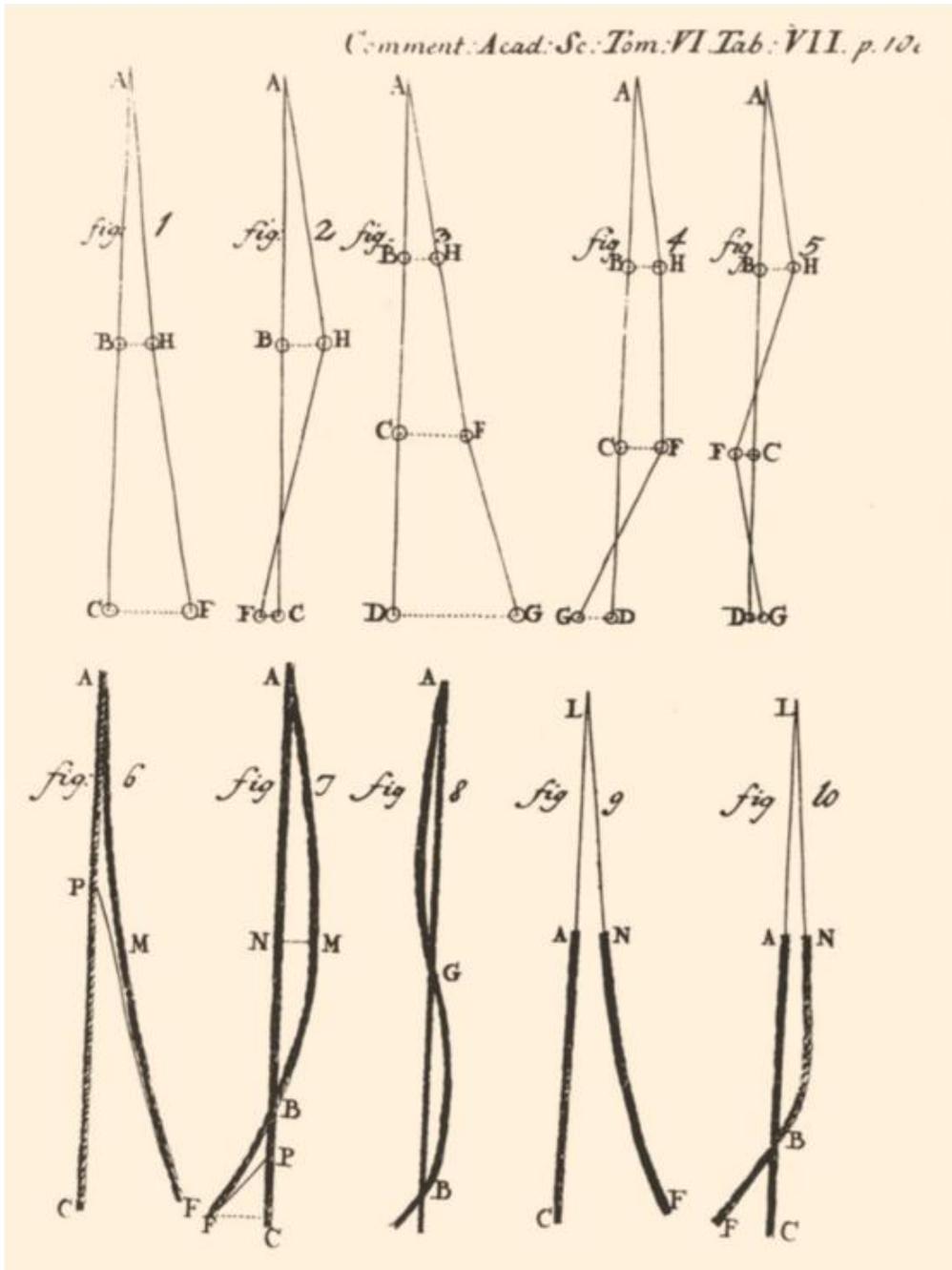


Figure A One of the earliest published images of an eigenvector from Daniel Bernoulli's 1733 masterpiece "Theoremata de oscillationibus corporum filo flexili connexorum et catenae verticaliter suspensae". (If a reader finds an earlier image of an eigenvector, please email the author!) A translation of this paper is available in [82, Appendix], and an extended discussion and interpretation can be found in [139, Section 23]. This remarkable paper establishes Bernoulli as the first to identify the simple modes and natural frequencies of an oscillating system. The development of vibration theory in the early 18th century serves as a prelude to linear analysis.

Figs. 1 and 2 in the image show the two simple modes of vibration for a string loaded by two weights. Figs. 3, 4 and 5 show the three simple modes of vibration for a string loaded by three weights. Figs. 6, 7 and 8 show the first three simple modes of vibration for the continuous heavy chord that is the limit of infinitely many weights.

The paper asserts that for a uniformly heavy hanging chord of length ℓ , the displacement y at a distance x from the bottom is given by $y = |CF|J_0(2\sqrt{x}/\alpha)$ where $|CF|$ is the displayed distance and α is such that $J_0(2\sqrt{\ell}/\alpha) = 0$. This is the first appearance of Bessel functions (given as a series expansion), and Bernoulli says there are infinitely many roots α , which we would now consider an eigenvalue parameter.

Many applications of “nonlinear” spectra

- Mechanical vibrations
- Fluid stability
- Delay systems
- Photonics
- Resonances
- ...

*Well-studied in finite dimensions,
but many applications arise from
infinite dimensions.*

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The nonlinear eigenvalue problem

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Nonlinear eigenvalue problems arise in a variety of science and engineering applications, and in the past ten years there have been numerous breakthroughs in the development of numerical methods. This article surveys nonlinear eigenvalue problems associated with matrix-valued functions which depend nonlinearly on a single scalar parameter, with a particular emphasis on their mathematical properties and available numerical solution techniques. Solvers based on Newton’s method, contour integration and sampling via rational interpolation are reviewed. Problems of selecting the appropriate parameters for each of the solver classes are discussed and illustrated with numerical examples. This survey also contains numerous MATLAB code snippets that can be used for interactive exploration of the discussed methods.

Example 1: Klein–Gordon equation

$$T(z) = H_0 - (V - zI)^2$$

$$H_0 = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & \frac{3}{2} & 2 & \frac{1}{2} & & & \\ & & \frac{1}{2} & |2| & \frac{3}{2} & & \\ & & \frac{3}{2} & 2 & \frac{1}{2} & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & & & \ddots \end{pmatrix}, \quad [Vx]_n = -5 \exp(-|n|)x_n, \quad x \in l^2(\mathbb{Z}), n \in \mathbb{Z}.$$

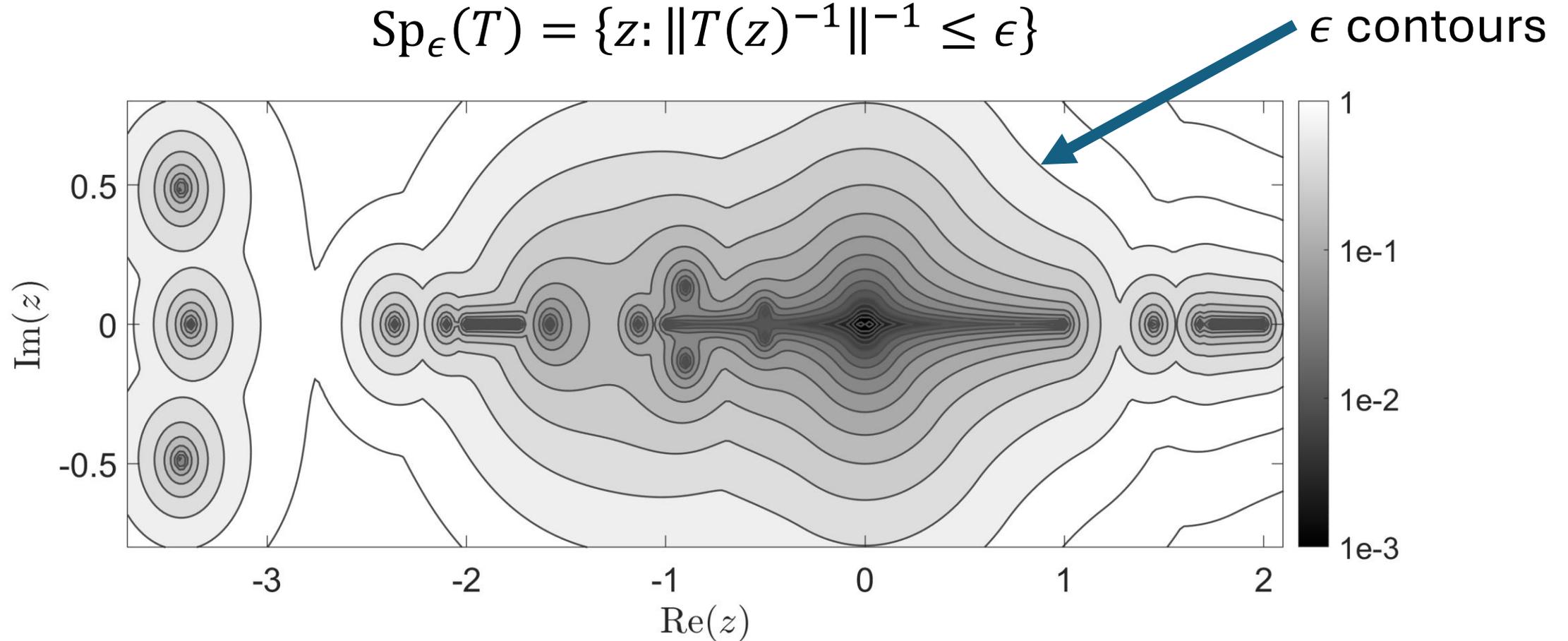
A nice bounded quadratic pencil.

What could possibly go wrong?



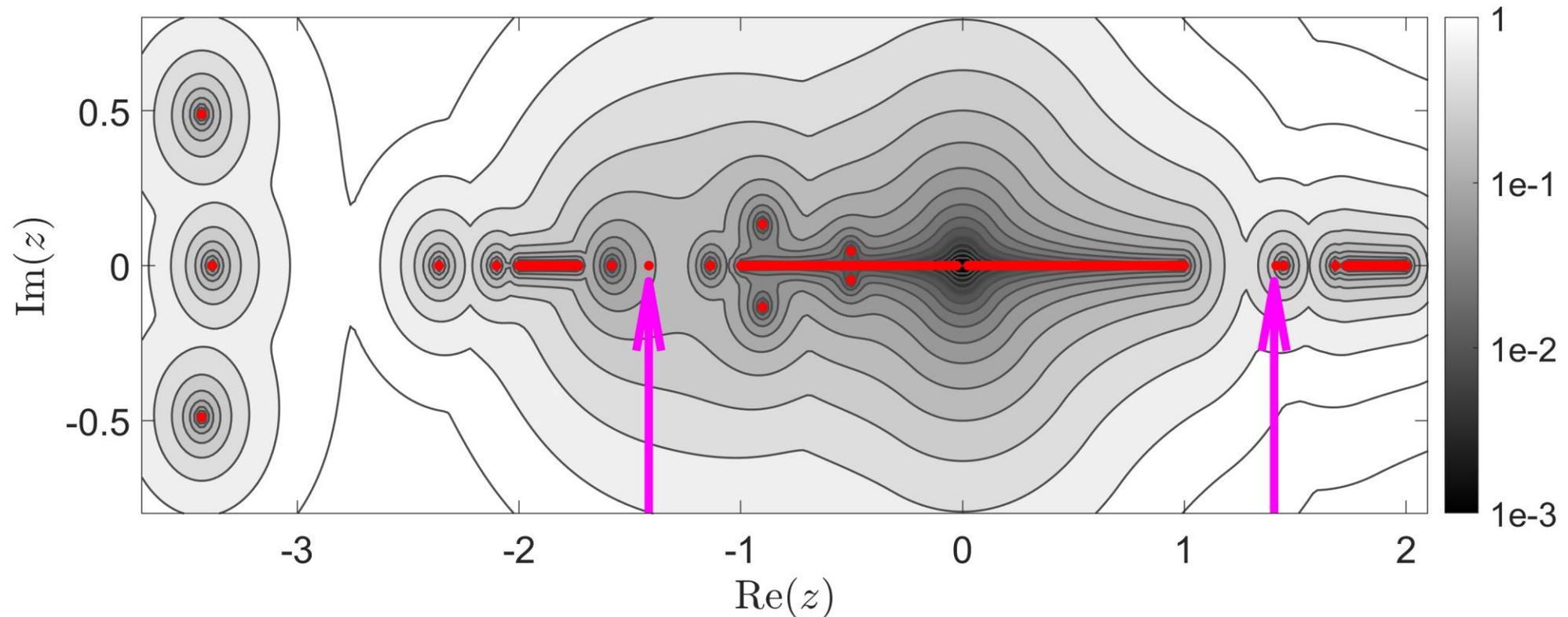
Example 1: Klein–Gordon equation

$$\text{Sp}_\epsilon(T) = \{z: \|T(z)^{-1}\|^{-1} \leq \epsilon\}$$



Example 1: Klein–Gordon equation

Truncate to $\text{span}\{e_{-N}, \dots, e_N\}$, $N = 100$ – spurious eigenvalues!



Persistent as we vary N !

Can we develop a systematic computational theory in infinite dimensions?

Want:

- Convergence to underlying applicational problem.
i.e., no truncation or discretization artifacts
- Bullet proof.
- Practical.

Part 1: Spectra and pseudospectra when $z \mapsto T(z)$ is continuous.

Part 2: Discrete spectrum when $z \mapsto T(z)$ is holomorphic.

An operator is a subspace

- **Fact of life:** Most applications involve differential operators.

An operator is a subspace

- **Fact of life:** Differential operators are unbounded.

An operator is a subspace

- **Fact of life:** Differential operators are unbounded.
- $\mathcal{H}_1, \mathcal{H}_2$ separable Hilbert spaces (countable o.n. basis)
- Operator $A: \mathcal{D}(A) \rightarrow \mathcal{H}_2$ with domain $\mathcal{D}(A) \subset \mathcal{H}_1$

$$\text{graph}(A) = \{(x, Ax): x \in \mathcal{D}(A)\} \subset \mathcal{H}_1 \times \mathcal{H}_2$$

A is closed when $\text{graph}(A)$ is a closed subspace.

John von Neumann



An operator is a subspace

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A is closed when $\text{graph}(A)$ is a closed subspace.

Not closed: $A = \frac{d^2}{dx^2}$ on $[-1, 1]$ with no BCs ❌

Closed: $A = \frac{d^2}{dx^2}$ on $[-1, 1]$, $\mathcal{D}(A) = \{u \in H^2(-1, 1), u(-1) = u(1) = 0\}$ ✅

John von Neumann



An operator is a subspace

- **Fact of life:** Differential operators are unbounded.
- $\mathcal{H}_1, \mathcal{H}_2$ separable Hilbert spaces (countable o.n. basis)
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A is closed when $\text{graph}(A)$ is a closed subspace.

Why am I telling you this?

1. Correct setting (spectra don't make sense otherwise).
2. Compare operators with different $\mathcal{D}(A)$ (e.g, z -dependent BCs).
3. We can use subspace theory to compute pseudospectra...

John von Neumann



The setting

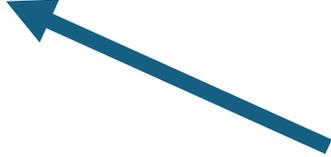
- $U \subset \mathbb{C}$ a bounded domain (non-empty, connected open set)
- $T: U \rightarrow \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$, i.e., each $T(z)$ closed linear operator.

Spectrum: $\text{Sp}(T) = \{z \in U: T(z)^{-1} \text{ doesn't exist}\}$

Pseudospectrum: $\text{Sp}_\epsilon(T) = \{z \in U: \|T(z)^{-1}\|^{-1} \leq \epsilon\}$

Assume: $\text{graph}(T(z))$ varies continuously in z .

Implies: $z \mapsto \|T(z)^{-1}\|^{-1}$ continuous on U .



Every problem
you meet in
the “wild”
satisfies this!

What type of convergence do we want?

For $\Gamma_n(T) \subset \mathbb{C}$ to approximate $\text{Sp}(T) \subset \mathbb{C}$ as $n \rightarrow \infty$:

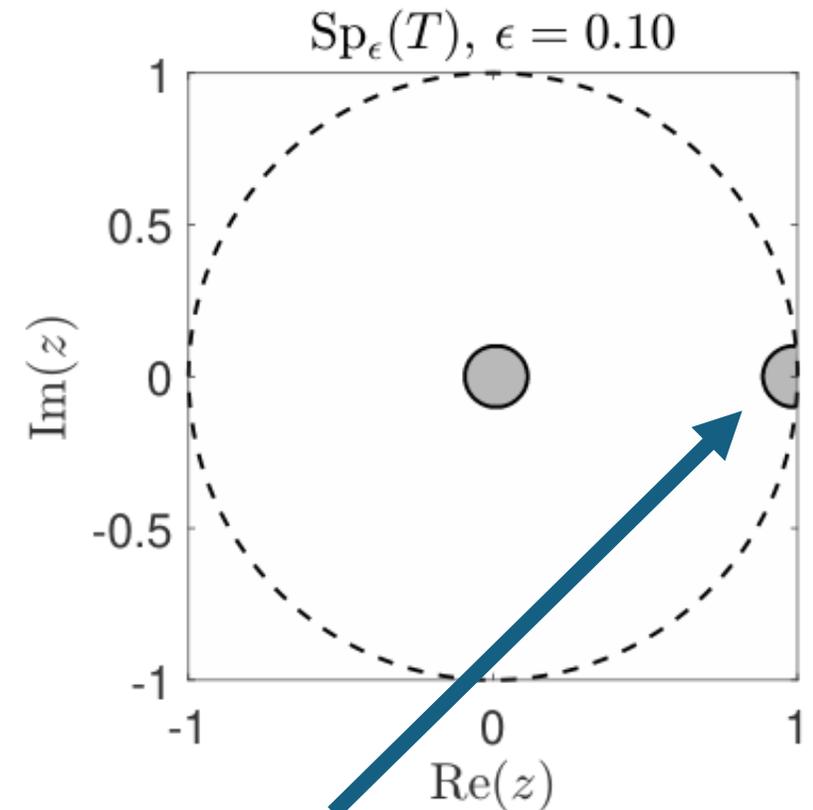
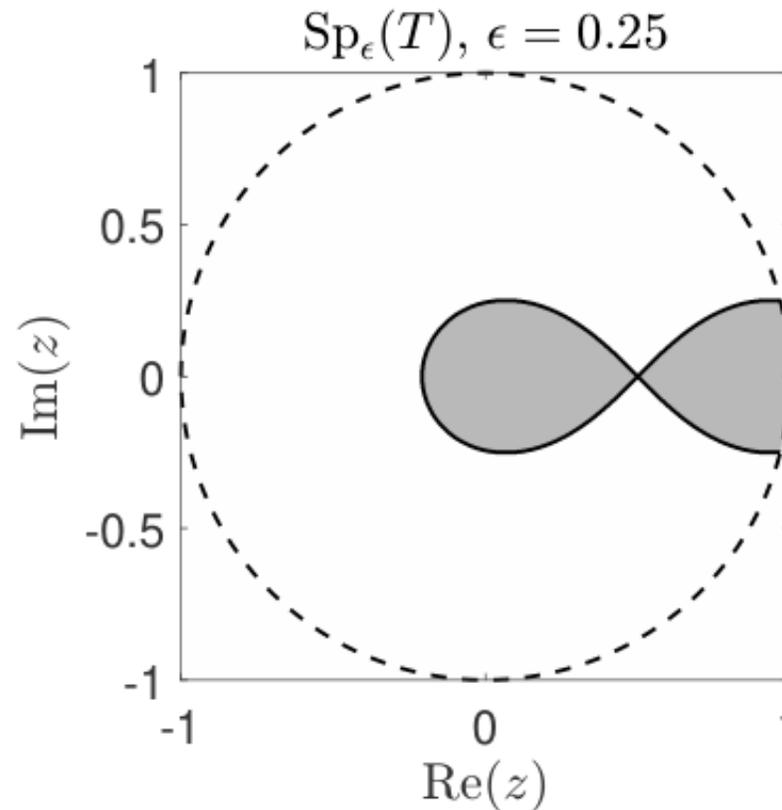
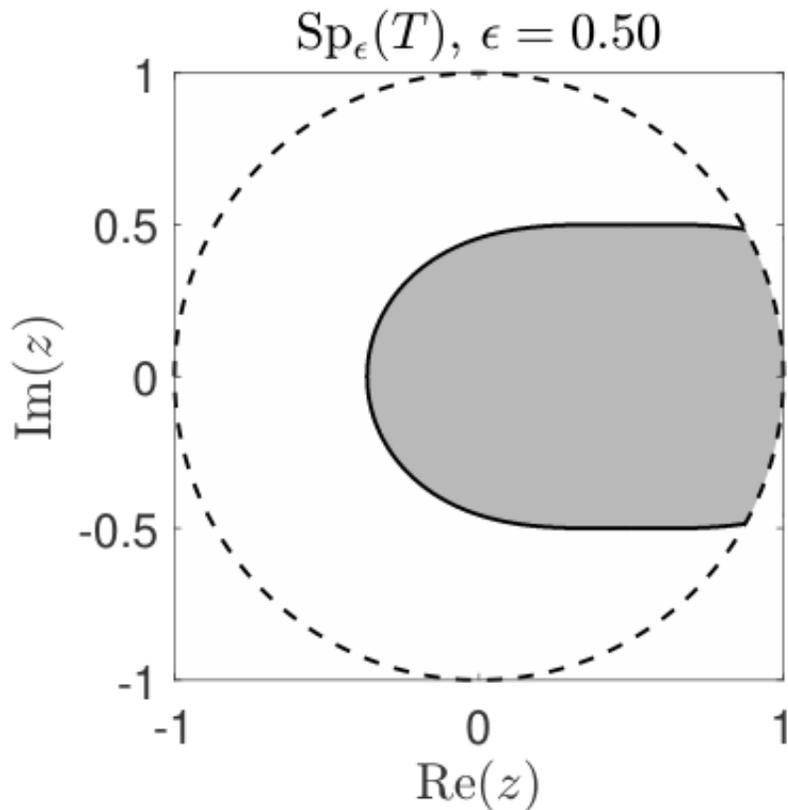
- **No spectral pollution:** Every $z \in U$ with $\liminf_{n \rightarrow \infty} \text{dist}(z, \Gamma_n(T)) = 0$ must have $z \in \text{Sp}(T)$. **NO BAD GUYS**
- **No spectral invisibility:** Every $z \in \text{Sp}(T)$ must have $\limsup_{n \rightarrow \infty} \text{dist}(z, \Gamma_n(T)) = 0$. **NO GOOD GUYS ARE MISSED**

Local uniform convergence on U .

Cannot expect anything stronger...

Example 2: Stefan's 2nd favorite NEP apparently (at least it was after a 🍺 on Weds)

$$U = \{z: |z| < 1\}, \quad T(z) = z(1 - z), \quad \text{Sp}(T) = \{0\}.$$



Convergence isn't globally uniform

Bullet proof algorithm: $\text{Sp}_\epsilon(T)$ to $\text{Sp}(T)$

- **Idea:** Approximate $\sigma_{\text{inf}}(T(z))$.

Let \mathcal{P}_N (possibly z -dependent) be orthogonal projection onto subspace V_N . Natural density and domain assumptions on V_N .

$$\sigma_{\text{inf}}(A) = \inf\{\|Ax\| : x \in \mathcal{D}(A), \|x\| = 1\}$$

$$\text{Fact: } \|T(z)^{-1}\|^{-1} = \min\{\sigma_{\text{inf}}(T(z)), \sigma_{\text{inf}}(T(z)^*)\}$$

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- **Solution:** One way truncation, $\sigma_{\text{inf}}(T(z) \mathcal{P}_N) \downarrow \sigma_{\text{inf}}(T(z))$ as $N \rightarrow \infty$.

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- **Ingredient:** Search for z s.t. $\min\{\sigma_{\inf}(T(z) \mathcal{P}_N), \sigma_{\inf}(T(z)^* \mathcal{P}_N)\} < \epsilon$.

Bullet proof algorithm: $\text{Sp}_\epsilon(T)$ to $\text{Sp}(T)$

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- **Ingredient:** Search for z s.t. $\min\{\sigma_{\inf}(T(z) \mathcal{P}_N), \sigma_{\inf}(T(z)^* \mathcal{P}_N)\} < \epsilon$.
- **Output:** $\Gamma_N^\epsilon(T)$ with $\Gamma_N^\epsilon(T) \subset \text{Sp}_\epsilon(T)$ and $\lim_{N \rightarrow \infty} \Gamma_N^\epsilon(T) = \text{Sp}_\epsilon(T)$.

$$\lim_{\epsilon \downarrow 0} \lim_{N \rightarrow \infty} \Gamma_N^\epsilon(T) = \text{Sp}(T)$$

Example 3: Wave equation with acoustic BCs

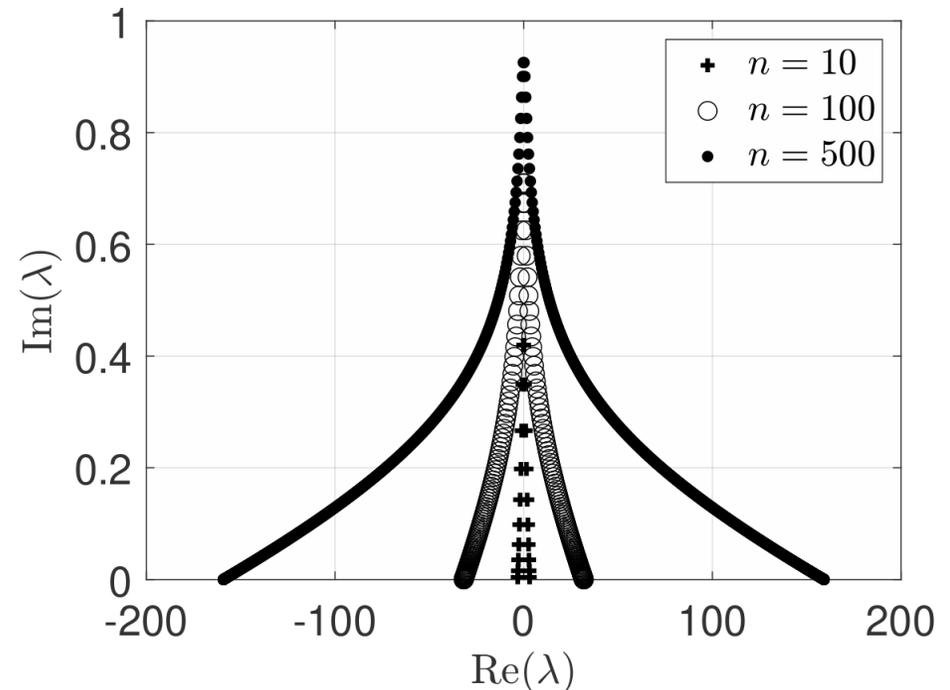
$$T(z)u = u'' + z^2 u, \quad \mathcal{D}(T(z)) = \{u \in \mathcal{W}^{2,2}(\mathbb{R}_{>0}) : -u'(0) + izu(0) = 0\}$$

Truncate to $[0, L]$ and rescale to $[0, 1]$:

$$S(\lambda)u = u'' + (2\pi\lambda)^2 u, \quad \mathcal{D}(S(\lambda)) = \{u \in \mathcal{W}^{2,2}((0, 1)) : -u'(0) + i2\pi\lambda u(0) = u(1) = 0\}$$

$$\text{Sp}(S) = \emptyset \quad \text{😲}$$

FEM-discretised version
in NLEVP collection



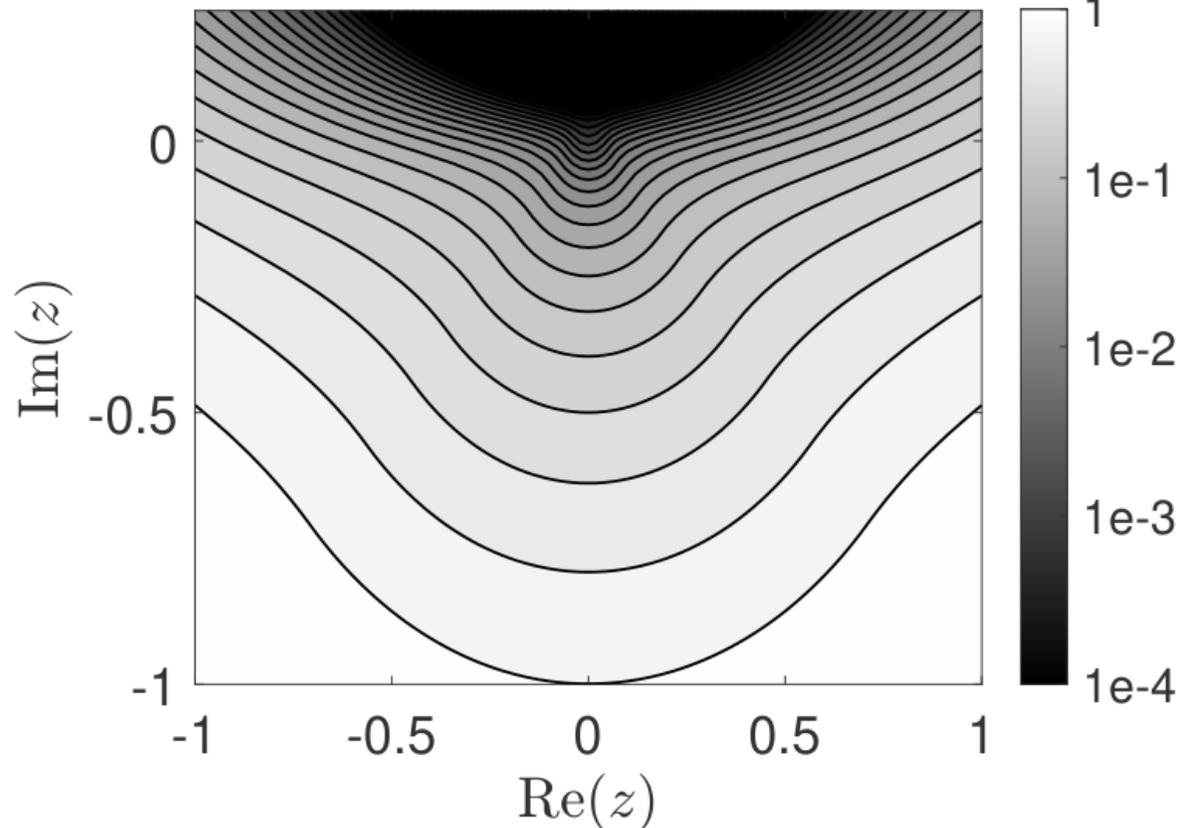
Example 3: Wave equation with acoustic BCs

$$\phi_n(x) = L_n(x)e^{-x/2}, \quad L_n(x) = \frac{1}{n!} \left(\frac{d}{dx} - 1 \right)^n x^n, \quad n = 0, 1, 2, \dots$$

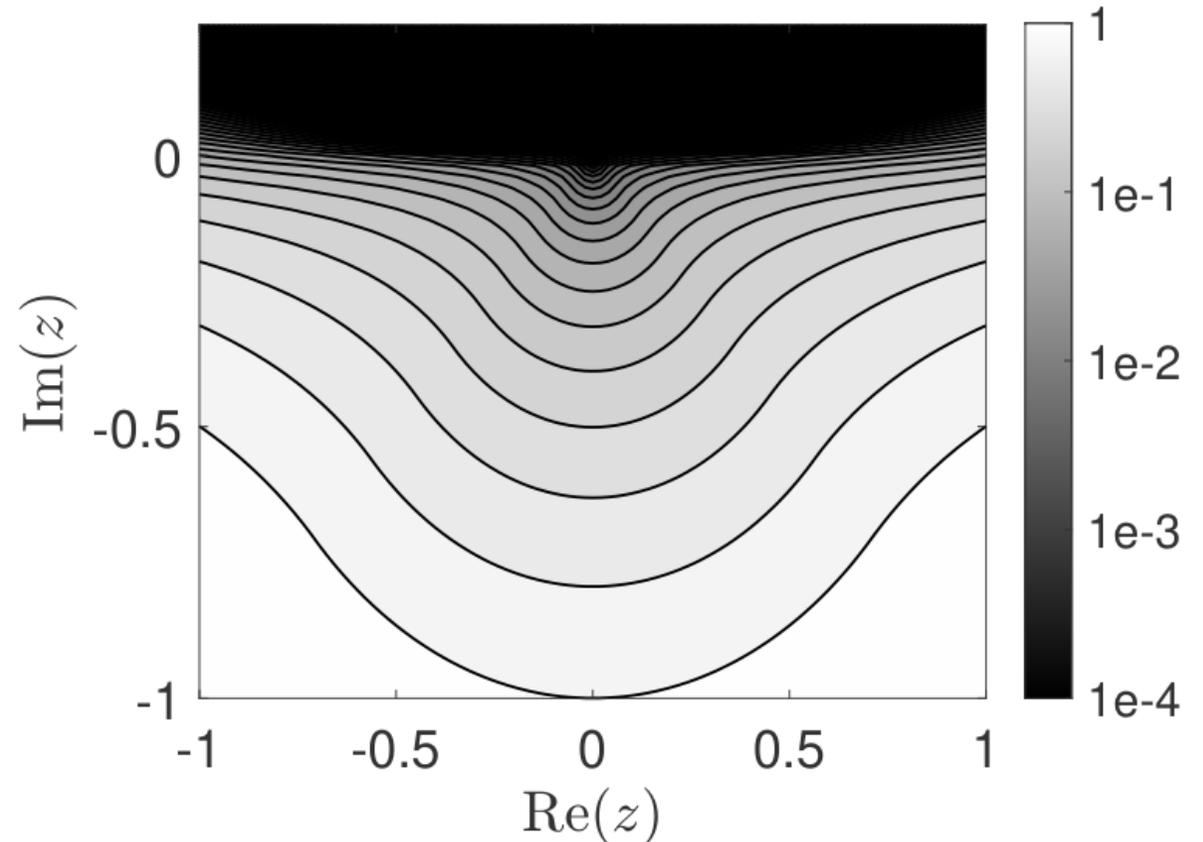
+ QR to compute $\sigma_{\text{inf}}(T(z)\mathcal{P}_N)$

$$\hat{\phi}_n(x) = \phi_n(x) + \alpha_n \phi_0(x), \quad \alpha_n = -(2iz + 2n + 1)/(2iz + 1), \quad n \in \mathbb{N}.$$

$N = 20$



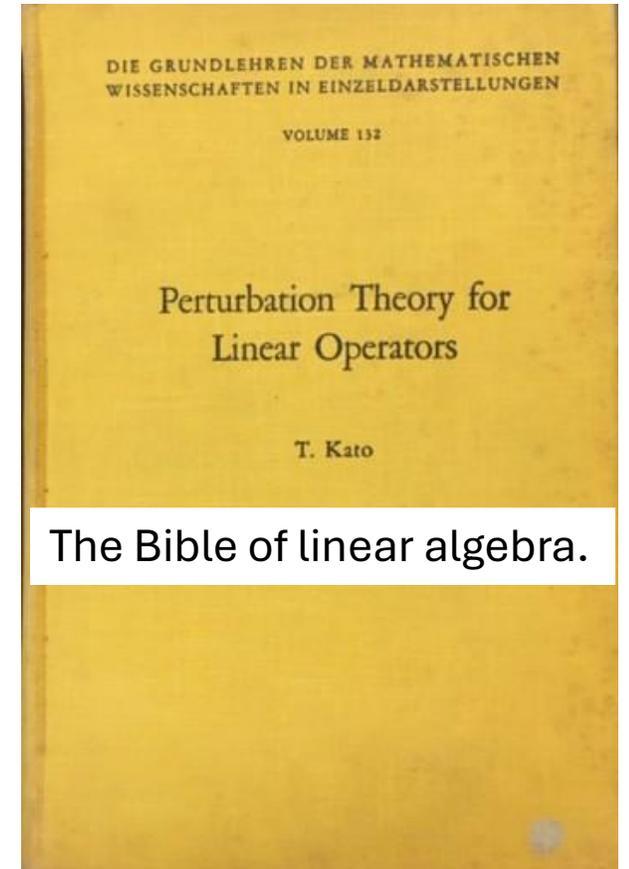
$N = 100$



Holomorphic families

- If $T(z)$ bounded, means local power series:

$$T(z) = \sum_{k=0}^{\infty} (z - z_0)^k T_k .$$

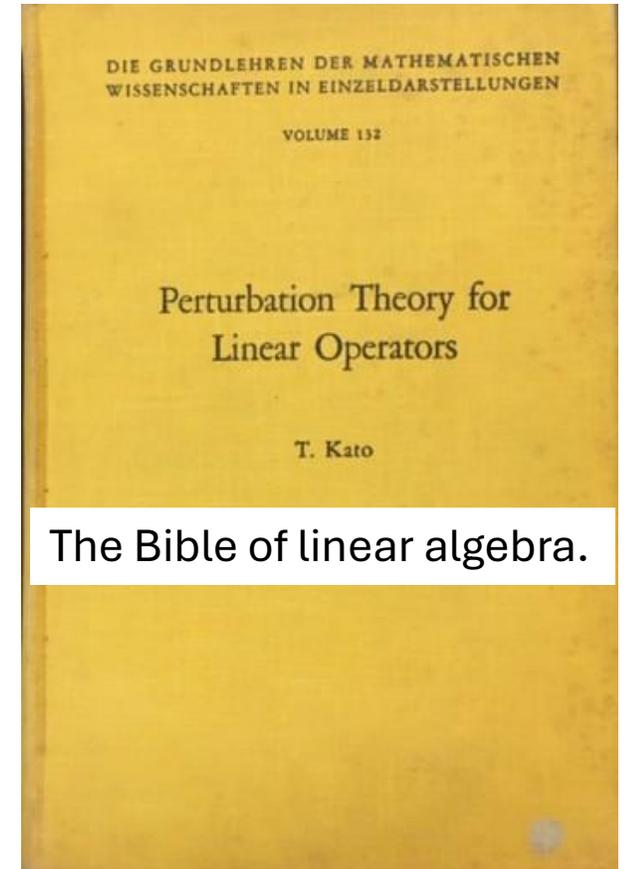


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- How to make sense of holomorphic BCs?!?!?



Holomorphic families

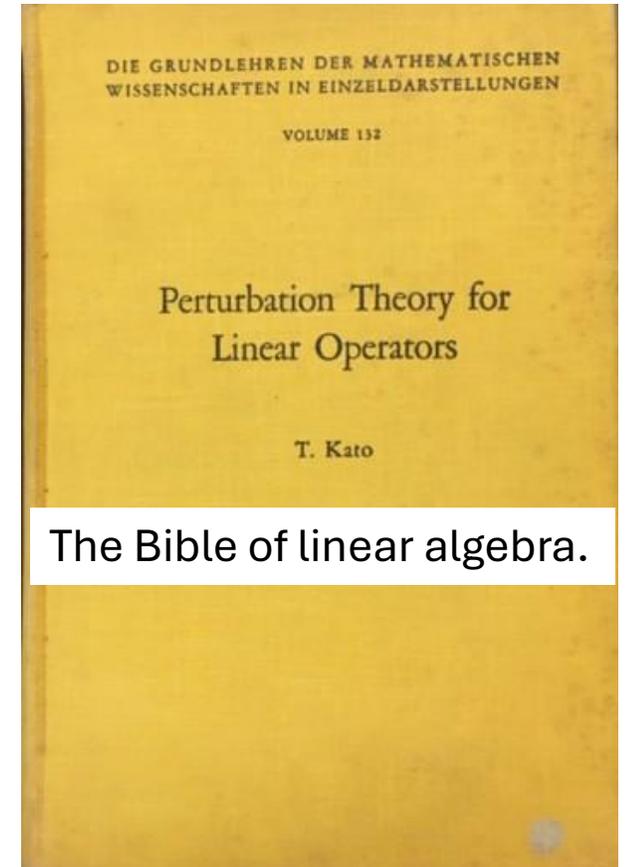
- If $T(z)$ bounded, means local power series:

$$T(z) = \sum_{k=0}^{\infty} (z - z_0)^k T_k .$$

- How to make sense of holomorphic BCs?!?!?
- For unbounded T , use a trick of Rellich:

$$A: U \rightarrow \mathcal{B}(\mathcal{H}_3, \mathcal{H}_1), \quad B: U \rightarrow \mathcal{B}(\mathcal{H}_3, \mathcal{H}_2)$$

bounded holomorphic with $T(z)A(z) = B(z)$.



Holomorphic families

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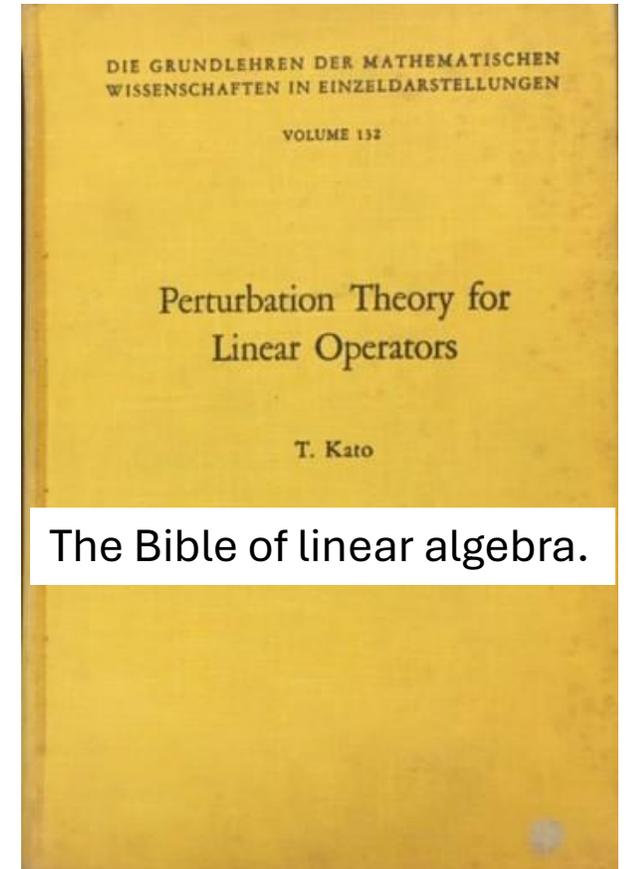
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bounded holomorphic with $T(z)A(z) = B(z)$.

- Lift results from the bounded case...



Keldysh's theorem (Laurent expansion)

Assume: T holomorphic, $\mathcal{D}(T(z))$ dense for all $z \in U$, $\text{Sp}(T) \neq U$, and $T(z)$ is Fredholm for all $z \in U$.

Consequence: Every $\lambda \in \text{Sp}(T)$ is an isolated eigenvalue. Quasimatrix V_λ of m_λ generalised eigenvectors in \mathcal{H}_1 , quasimatrix W_λ of m_λ (left) generalised eigenvectors in \mathcal{H}_2 , $m_\lambda \times m_\lambda$ Jordan matrix J_λ , and bounded holomorphic family $R_\lambda(z)$ s.t. near λ :

$$T(z)^{-1} = R_\lambda(z) + V_\lambda(zI - J_\lambda)^{-1}W_\lambda^*.$$

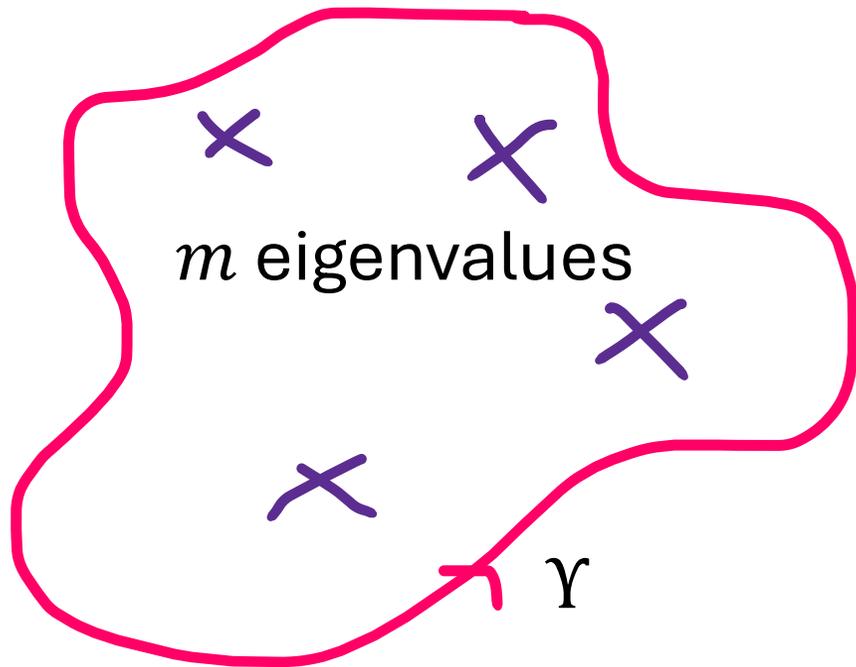
Combine: Assume there are finitely many eigenvalues in U . Stack the matrices and combine the holomorphic remainders to obtain:

$$T(z)^{-1} = R(z) + V(zI - J)^{-1}W^*.$$

Contour methods: \rightarrow finite linear problem

$$T(z)^{-1} = R(z) + V(zI - J)^{-1}W^*$$

Random quasimatrices: $F = (f_1 \cdots f_l)$, $G = (g_1 \cdots g_r)$ (see later!)



$h(z)$ holomorphic

$$\begin{aligned} M_h &= \frac{1}{2\pi i} \int_{\gamma} h(z) F^* T(z)^{-1} G \, dz \\ &= F^* V h(J) W^* G \in \mathbb{C}^{l \times r} \end{aligned}$$

Contour methods: → finite linear problem

$$h_{i,j}(z) = h_{\text{left},i}(z)h_{\text{right},j}(z), \quad i = 1, \dots, p, \quad j = 1, \dots, q$$

$$B_0 = \begin{pmatrix} M_{h_{1,1}} & \cdots & M_{h_{1,q}} \\ \vdots & & \vdots \\ M_{h_{p,1}} & \cdots & M_{h_{p,q}} \end{pmatrix} \in \mathbb{C}^{pl \times qr}, \quad B_1 = \begin{pmatrix} M_{h_{1,1} \cdot (z-z_0)} & \cdots & M_{h_{1,q} \cdot (z-z_0)} \\ \vdots & & \vdots \\ M_{h_{p,1} \cdot (z-z_0)} & \cdots & M_{h_{p,q} \cdot (z-z_0)} \end{pmatrix} \in \mathbb{C}^{pl \times qr}$$

$$V_p = \begin{pmatrix} F_{\text{left}}^* V h_{\text{left},1}(J) \\ \vdots \\ F_{\text{left}}^* V h_{\text{left},p}(J) \end{pmatrix} \in \mathbb{C}^{pl \times m}, \quad W_q^* = (h_{\text{right},1}(J)W^*G_{\text{right}} \quad \cdots \quad h_{\text{right},q}(J)W^*G_{\text{right}}) \in \mathbb{C}^{m \times qr}$$

Factorised form: $B_0 = V_p W_q^*, \quad B_1 = V_p (J - z_0 I) W_q^*$

Contour methods: → finite linear problem

Lemma: Suppose that V_p and W_q have rank m . Let $B_0 = U_0 \Sigma_0 V_0^*$ be an economy SVD. Then $X = U_0^* V_p$ and $Y = V_0^* W_p$ are invertible and

$$S(z) = U_0^* [B_1 - (z - z_0) B_0] V_0 = X (J - zI)^{-1} X^{-1} \Sigma_0, \quad (Y^* = X^{-1} \Sigma_0)$$

- Reduces to finite-dimensional linear problem S !
- Similar result recovers eigenvectors V and W .
- **In practice:** M_h approximated using quadrature, system solves $F^* T(z)^{-1} G$ (embarrassingly parallelisable).

$$M_h = \frac{1}{2\pi i} \int_{\gamma} h(z) F^* T(z)^{-1} G \, dz$$

Unifies methods in the literature

Paper	Corresponding parameters
Sakurai and Sugiura 2003	Polynomial $h, p = q, l = r = 1$
Asakura, Sakurai, Tadano, Ikegami and Kimura 2009	Polynomial $h, p = q, l = r$
Beyn 2012	Polynomial $h, p = q$, no sketching on left
Yokota and Sakurai 2013	Polynomial $h, p = q, l = r$
Brennan, Embree and Gugercin 2023	Rational h (connections with system realisation)

- Rank condition generically holds for sufficiently large p, q .
- Other methods use iteration (not in this talk).

Key Questions:

- How does error in approximating M_h affect results?
- How does use of random vectors affect results?

Perturbation analysis

$M_h = \frac{1}{2\pi i} \int_{\Upsilon} h(z) F^* T(z)^{-1} G \, dz$ approximated by \widetilde{M}_h

$\Rightarrow \widetilde{B}_0 \approx \widetilde{U}_0 \widetilde{\Sigma}_0 \widetilde{V}_0^*$ (m -truncated SVD), $\widetilde{B}_1, \widetilde{S}(z) = \widetilde{U}_0^* [\widetilde{B}_1 - (z - z_0) \widetilde{B}_0] \widetilde{V}_0$.

Assume: $\|B_0 - \widetilde{B}_0\| \leq \epsilon_B \|B_0\|, \|B_1 - \widetilde{B}_1\| \leq \epsilon_B \|B_1\|$.

Consequence: If λ an eigenvalue of \widetilde{S} then

$$\|(J - \lambda I)^{-1}\|^{-1} \leq \|V_p^\dagger\| \|W_q^\dagger\| \left(4 \|B_1\| \frac{\sigma_1(B_0)}{\sigma_m(B_0)} + 2 |\lambda - z_0| \|B_0\| + \|B_1\| \right) \epsilon_B.$$

Choosing random F and G

- Positive, self-adjoint, trace-class operator C on \mathcal{H} .
- Orthonormal basis of e-vectors ψ_β , e-values $c_\beta^2 \geq 0$, $\sum_\beta c_\beta^2 < \infty$.
- **Karhunen–Loève** expansion:

$$g = \sum_\beta c_\beta \zeta_\beta \psi_\beta, \quad \zeta_\beta = \frac{1}{\sqrt{2}} (a_\beta + i b_\beta) \sim \text{i.i.d. standard complex Gaussian}$$

Write $g \sim \text{GP}(0, C)$, C the covariance operator.

$$F_{\text{left}} = (f_1 \cdots f_l), f_j \sim \text{GP}(0, C_{\text{left}}), \quad G_{\text{right}} = (g_1 \cdots g_r), g_j \sim \text{GP}(0, C_{\text{right}})$$

Sketching doesn't degrade things

$$F_{\text{left}} = (f_1 \cdots f_l)$$

$$f_j \sim \text{GP}(0, C_{\text{left}})$$

Lemma: Let $p \in \mathbb{N}$ and, for each $j = 1, \dots, p$, let $A_j \in \mathbb{C}^{m \times m}$. Let L be a quasimatrix with $m \geq \text{rank}(V) \geq 2$ columns in a separable (possibly finite-dimensional) Hilbert space \mathcal{H} . Define the sketched and unsketched matrices

$$A_{\text{sketch}} = \begin{pmatrix} F_{\text{left}}^* V A_1 \\ \vdots \\ F_{\text{left}}^* V A_p \end{pmatrix} L^*, \quad A = \begin{pmatrix} V A_1 \\ \vdots \\ V A_p \end{pmatrix} L^*.$$

Let \mathcal{P} denote the orthogonal projection onto the column space of V , and assume that

$$\text{rank}(\sqrt{C_{\text{left}}} \mathcal{P}^*) = \text{rank}(V) \leq l, \quad \text{rank}(A) = m.$$

Then, for every $\epsilon > 0$ satisfying

$$\epsilon < \frac{1}{2\pi} \left(\frac{6.298}{l - \text{rank}(V) + 1} \right)^{2(l - \text{rank}(V) + 1)},$$

we have

$$\mathbb{P} \left(\frac{\sigma_1(A_{\text{sketch}})}{\sigma_m(A_{\text{sketch}})} \geq \frac{\sigma_1(A)}{\sigma_m(A)} \frac{\sigma_1(\sqrt{C_{\text{left}}} \mathcal{P}^*)}{\sigma_{\text{rank}(V)}(\sqrt{C_{\text{left}}} \mathcal{P}^*)} \frac{6.298}{(2\pi\epsilon)^{\frac{1}{2(l - \text{rank}(V) + 1)}}} \frac{l}{l - \text{rank}(V) + 1} \right) \leq \epsilon.$$

Set $l = \text{rank}(V) + 5$,

$$\frac{\sigma_1(A_{\text{sketch}})}{\sigma_m(A_{\text{sketch}})} \leq 1.7l \cdot \frac{\sigma_1(A)}{\sigma_m(A)} \cdot \frac{\sigma_1(\sqrt{C_{\text{left}}} \mathcal{P}^*)}{\sigma_{\text{rank}(V)}(\sqrt{C_{\text{left}}} \mathcal{P}^*)}$$

with probability > 0.999 .

Apply twice to control

$$\|V_p^\dagger\| \|W_q^\dagger\| \text{ and } \frac{\sigma_1(B_0)}{\sigma_m(B_0)}.$$

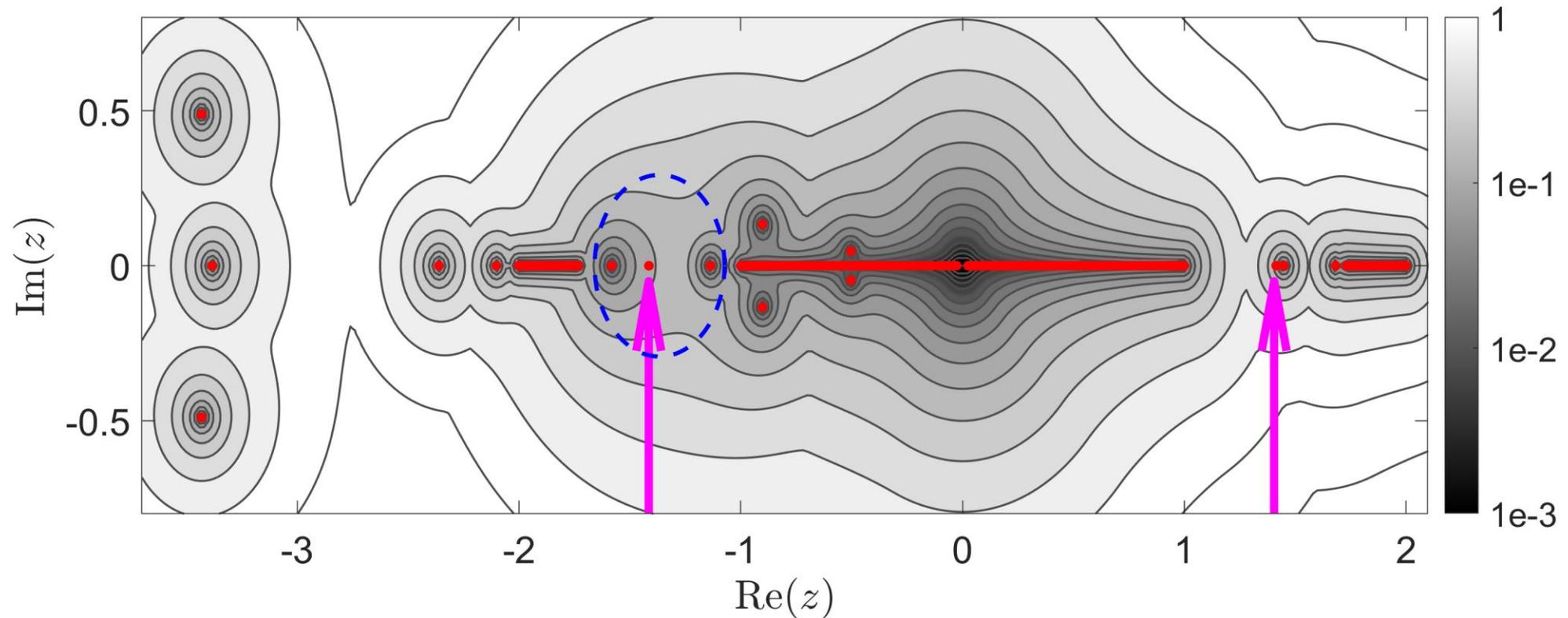
Stability no worse than

unsketched matrix

(depends on T & chosen h)

Example 4: Above matters in practice

Blue contour that contains 2 true eigenvalues...



Example 4: Above matters in practice

- Choose C with $\psi_\beta = e_\beta$ (canonical basis of $\ell^2(\mathbb{Z})$).
- $N = 300$ (601 basis functions), trapezoidal rule.

KL expansion:

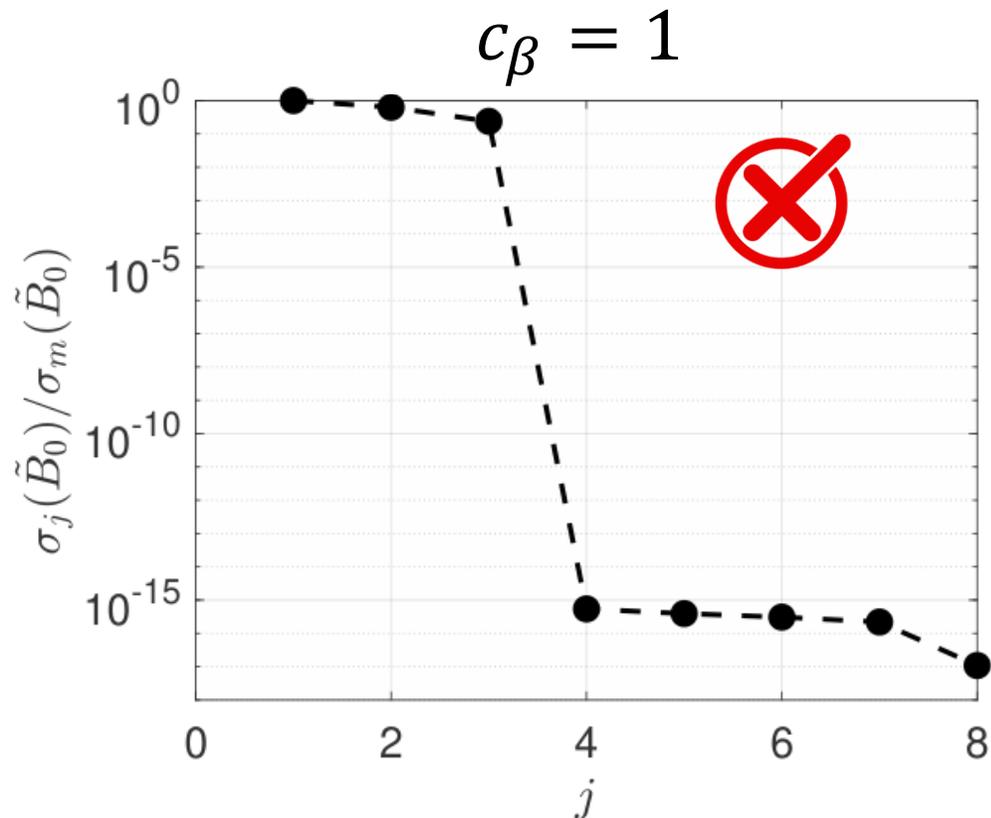
$$\sum_{\beta} c_{\beta} \zeta_{\beta} \psi_{\beta}$$

KL expansion:

$$\sum_{\beta} c_{\beta} \zeta_{\beta} \psi_{\beta}$$

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- Choose C with $\psi_{\beta} = e_{\beta}$ (canonical basis of $\ell^2(\mathbb{Z})$).
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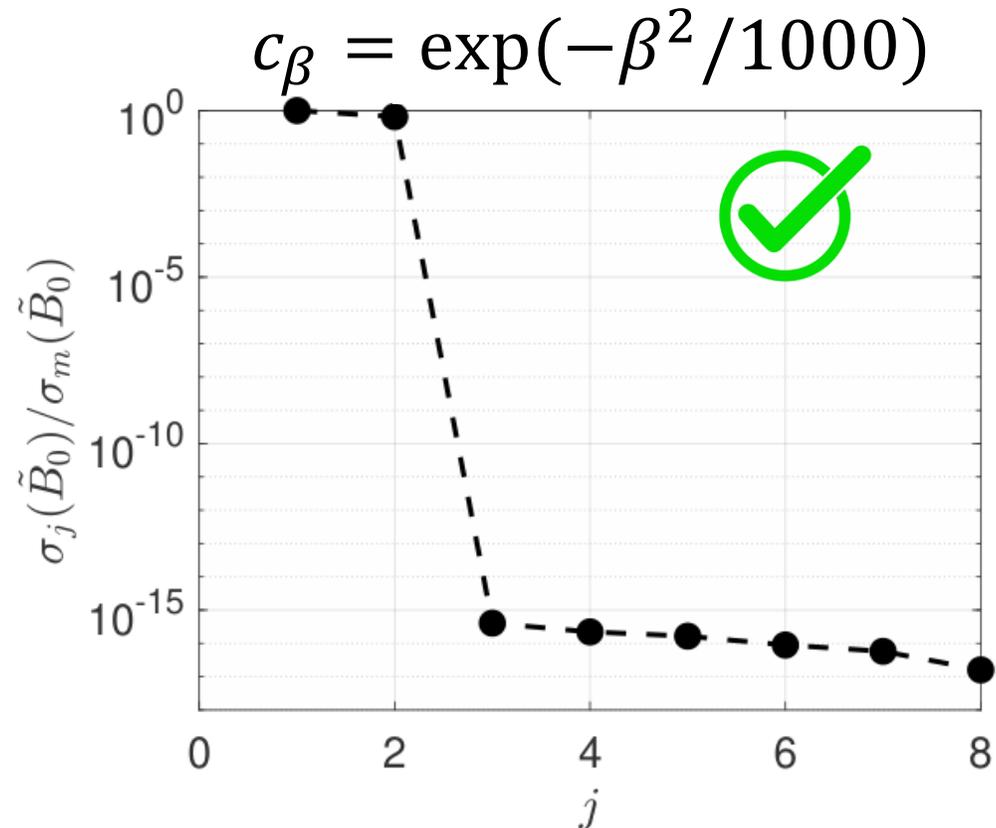
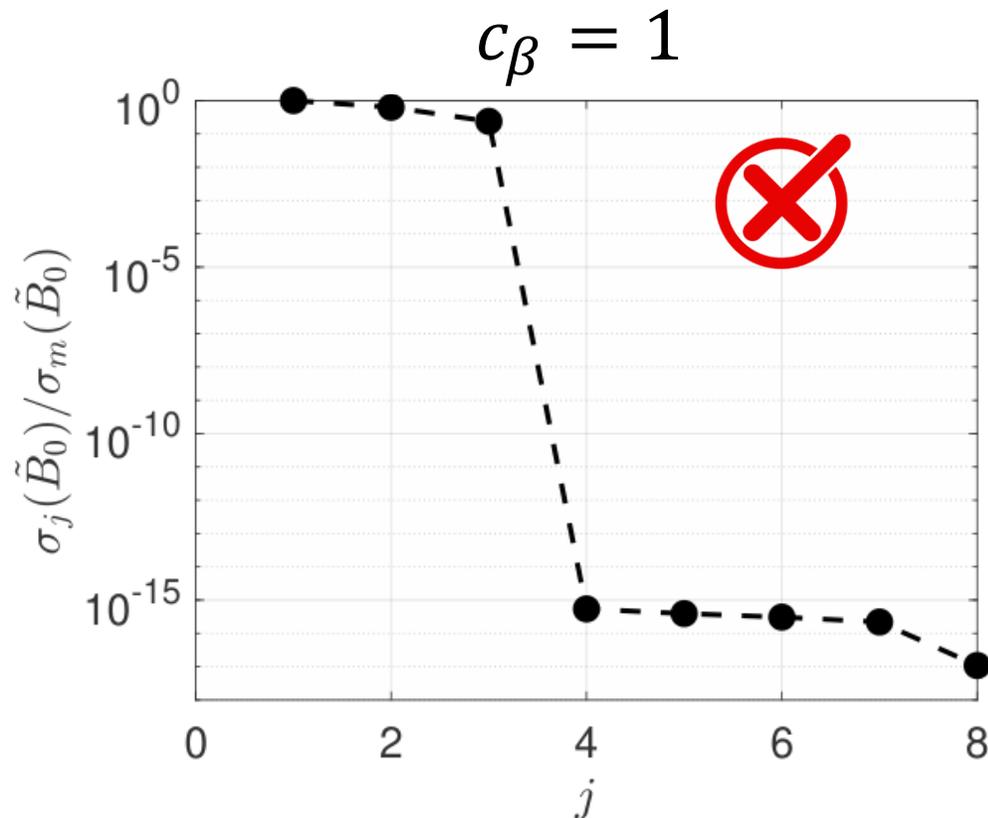


KL expansion:

$$\sum_{\beta} c_{\beta} \zeta_{\beta} \psi_{\beta}$$

Example 4: Above matters in practice

- Choose C with $\psi_{\beta} = e_{\beta}$ (canonical basis of $\ell^2(\mathbb{Z})$).
- $N = 300$ (601 basis functions), trapezoidal rule.



But I thought we had convergence?!?!

- $c_\beta = 1$ does not lead to a convergent series $\sum_\beta c_\beta \zeta_\beta \psi_\beta$

Analysis only holds if F_{left} and G_{right} in the correct Hilbert space.

- Eigenvectors v_N corresponding to pollution converge weakly to 0:

$$\lim_{N \rightarrow \infty} \langle v_N, u \rangle = 0 \quad \forall u \in \ell^2(\mathbb{Z}).$$

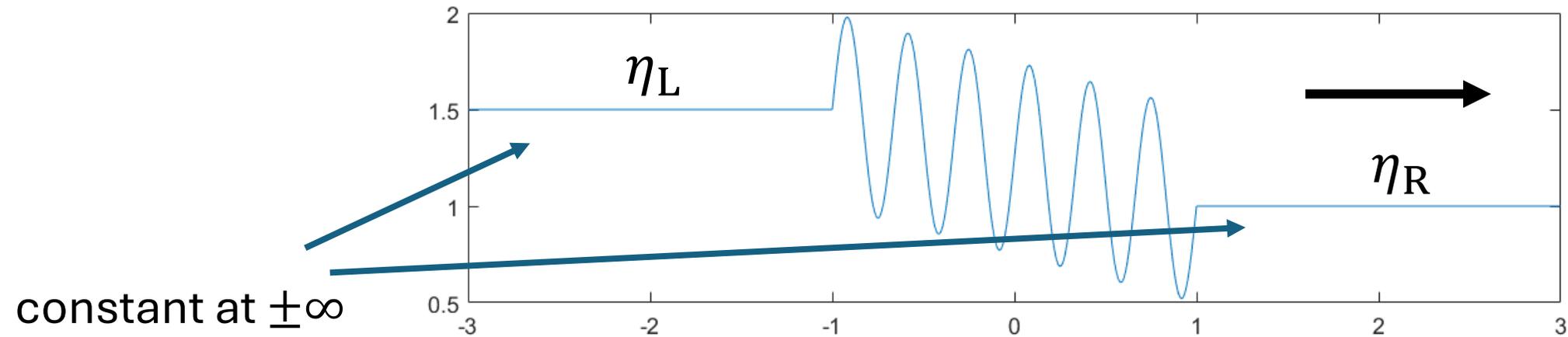
When we sketch and truncate the SVD of \widetilde{B}_0 we suppress them!

\Rightarrow With care, contour methods defend against pollution!



Example 5: Resonances

Planar waveguide with refractive index η :



Modes $\sim \phi(x) \exp(ik(nx - t))$ described by

$$\frac{d^2\phi}{dx^2} + k^2[\eta(x)^2 - z^2]\phi(x) = 0, \quad x \in \mathbb{R}.$$

Take $k = 11$, consider **resonances** – physical solutions $\notin L^2(\mathbb{R})$.
(very useful, very hard to compute!)

Example 5: Resonances

Truncate to $[-1,1]$, use Dirichlet to Neumann map:

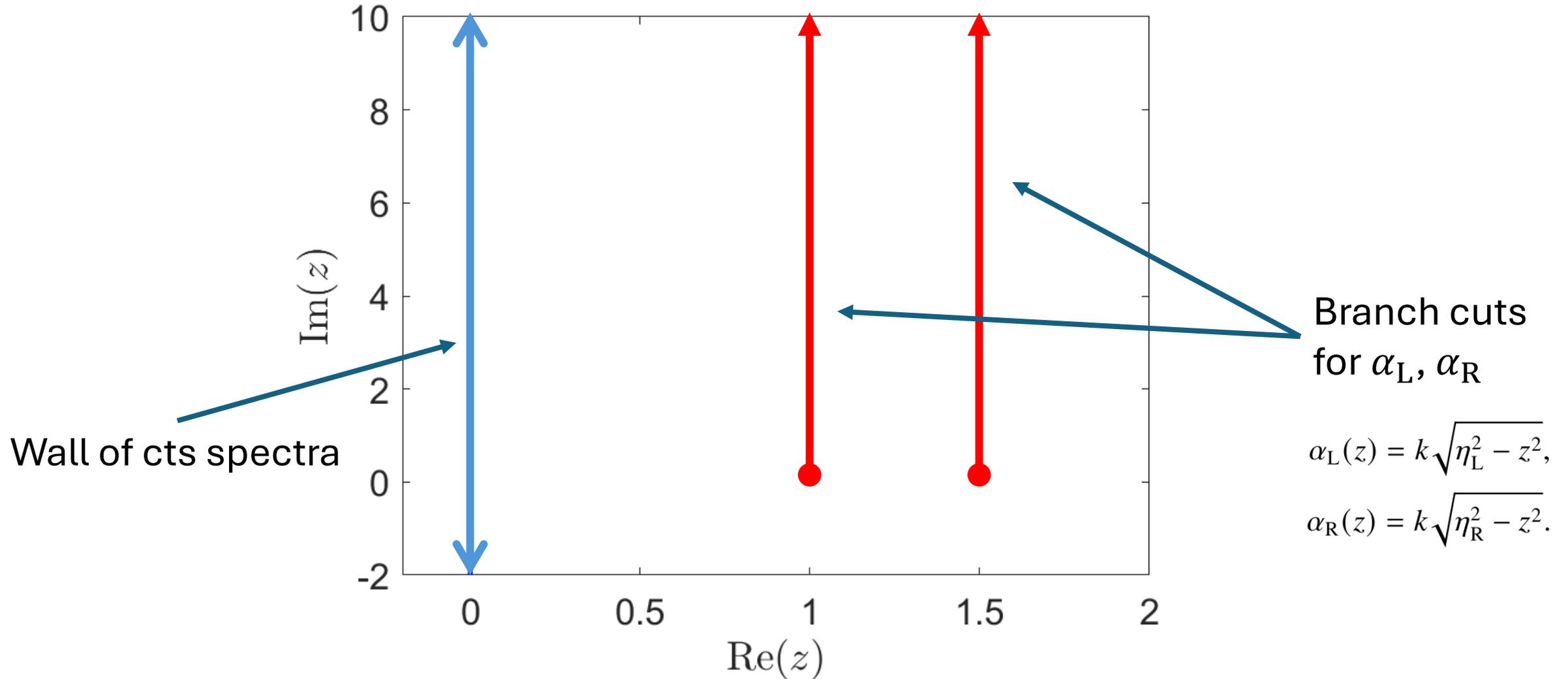
$$\phi(x) = \begin{cases} \exp(i\alpha_L(z)(x+1))\phi(-1), & x < -1 \\ \exp(-i\alpha_R(z)(x-1))\phi(1), & x > 1 \end{cases} \quad \begin{aligned} \alpha_L(z) &= k\sqrt{\eta_L^2 - z^2}, \\ \alpha_R(z) &= k\sqrt{\eta_R^2 - z^2}. \end{aligned}$$

$$\frac{d^2\phi}{dx^2}(x) + k^2([\eta(x)]^2 - z^2)\phi(x) = 0, \quad x \in (-1, 1),$$

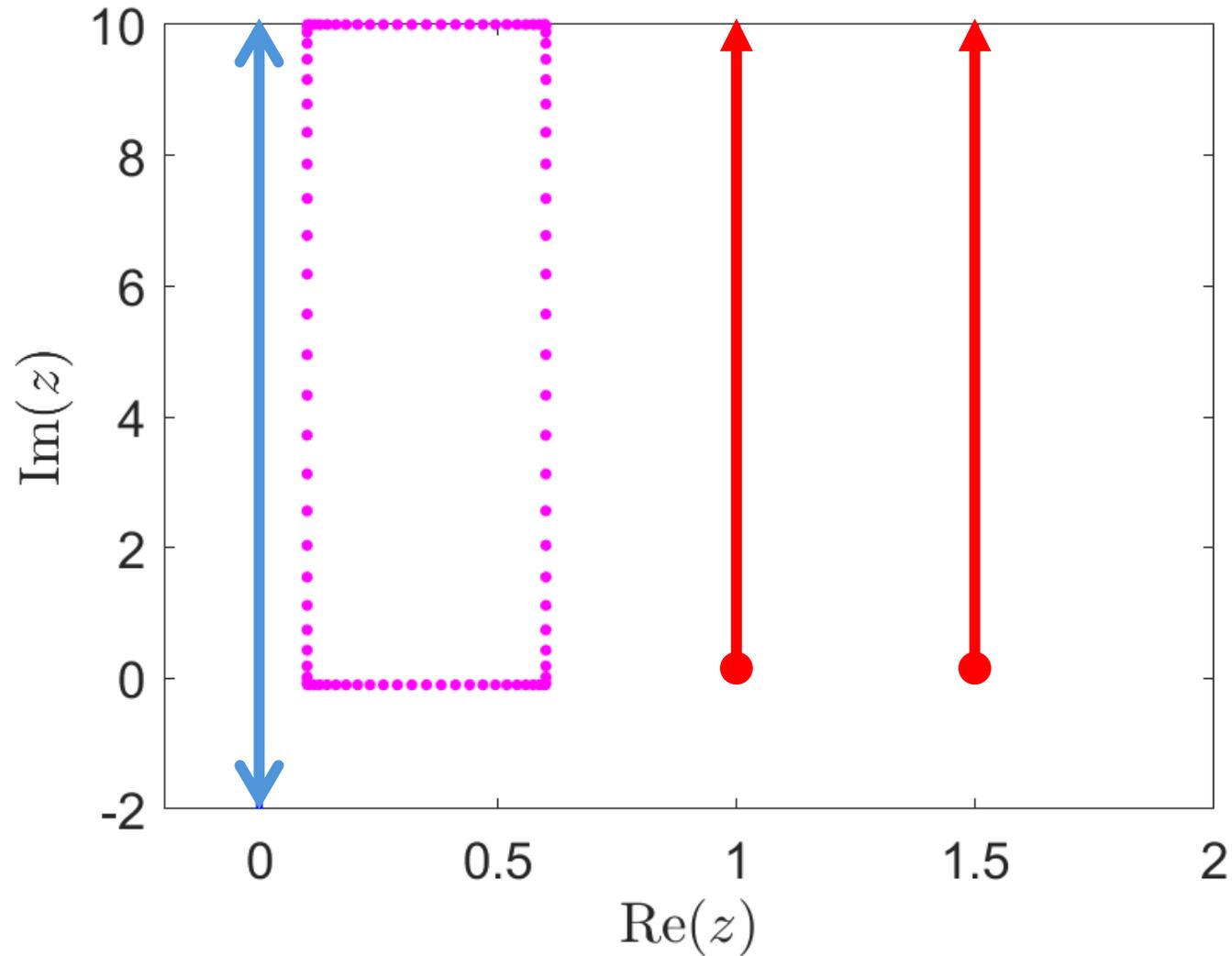
$$\frac{d\phi}{dx}(-1) - i\alpha_L(z)\phi(-1) = 0, \quad \frac{d\phi}{dx}(1) + i\alpha_R(z)\phi(1) = 0.$$

Nonlinearity in boundary conditions!

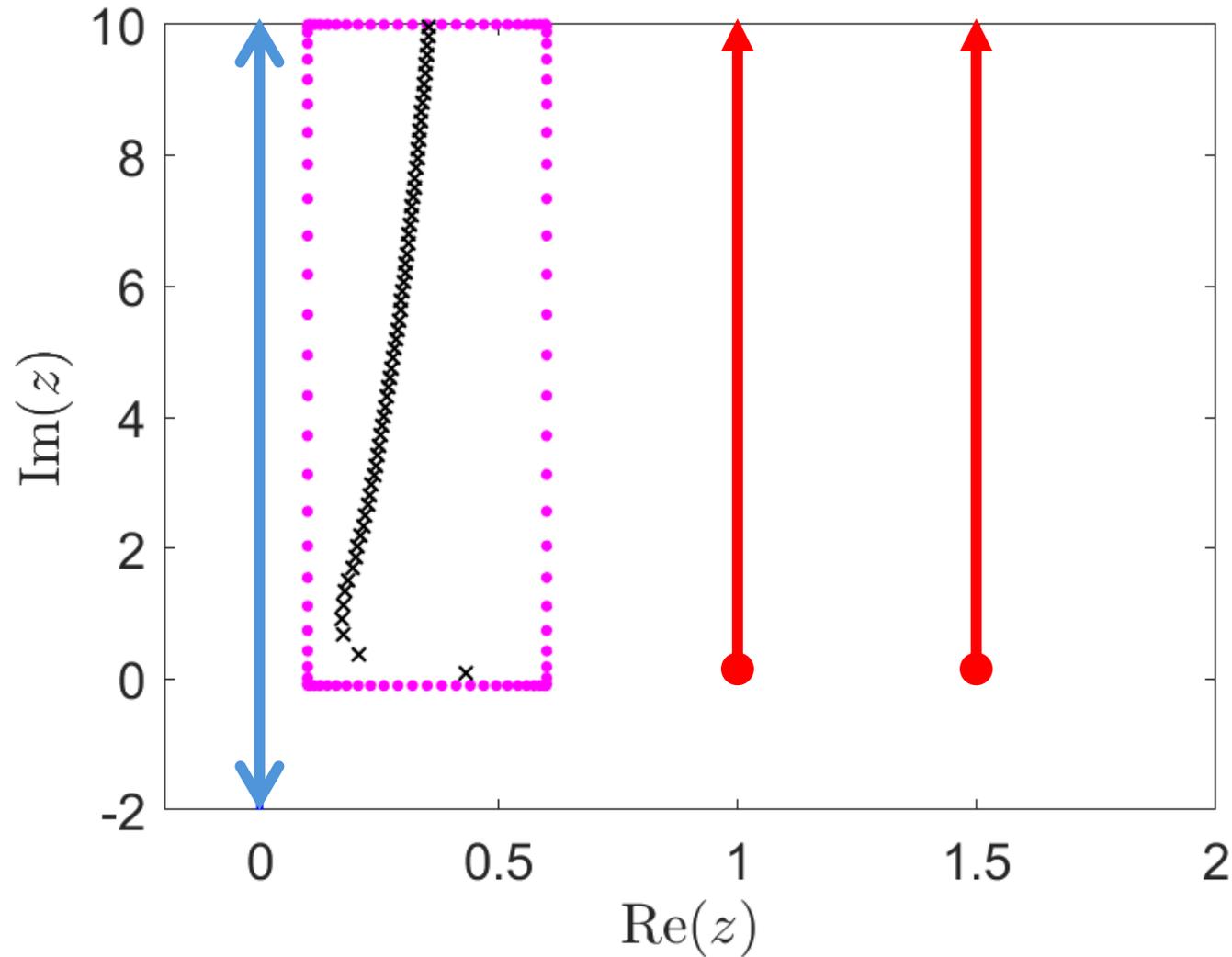
Example 5: Resonances



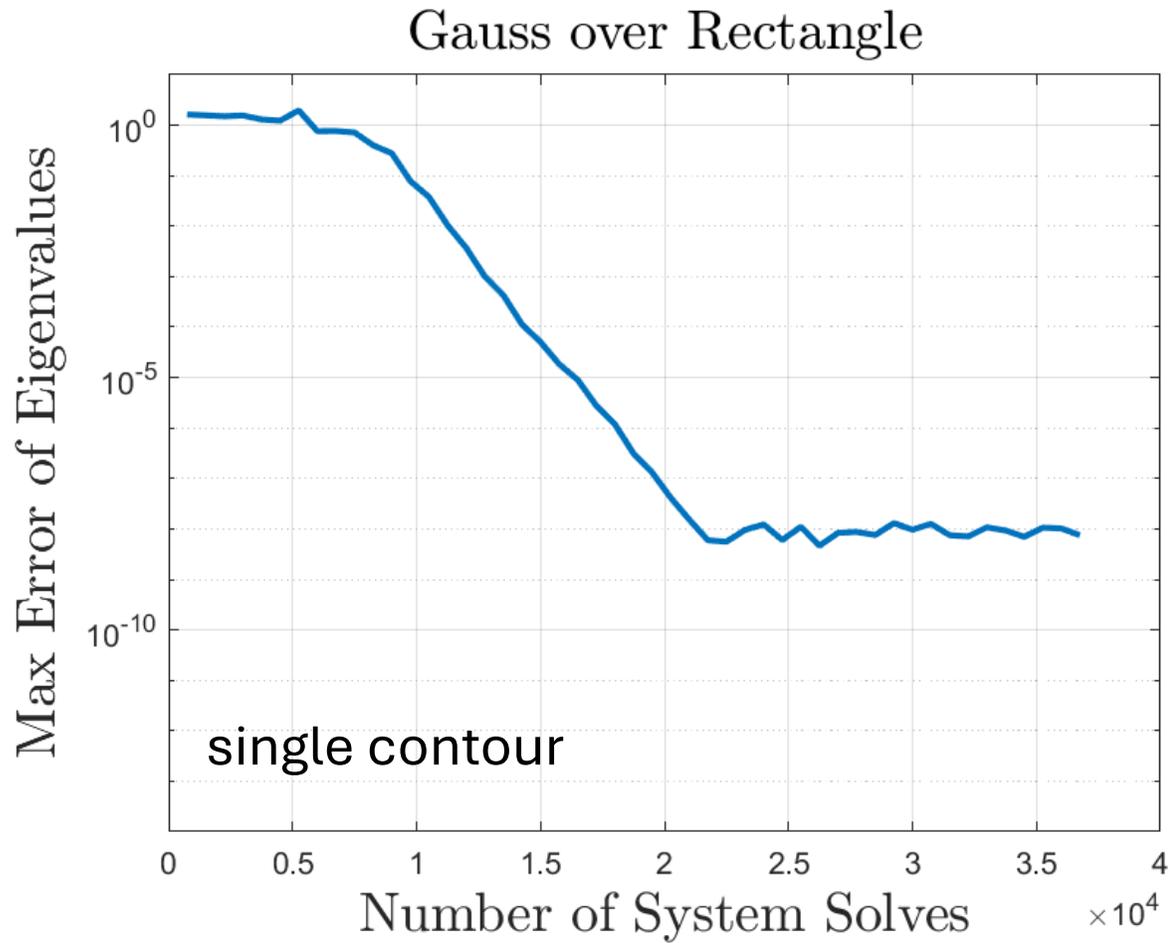
Example 5: Resonances



Example 5: Resonances

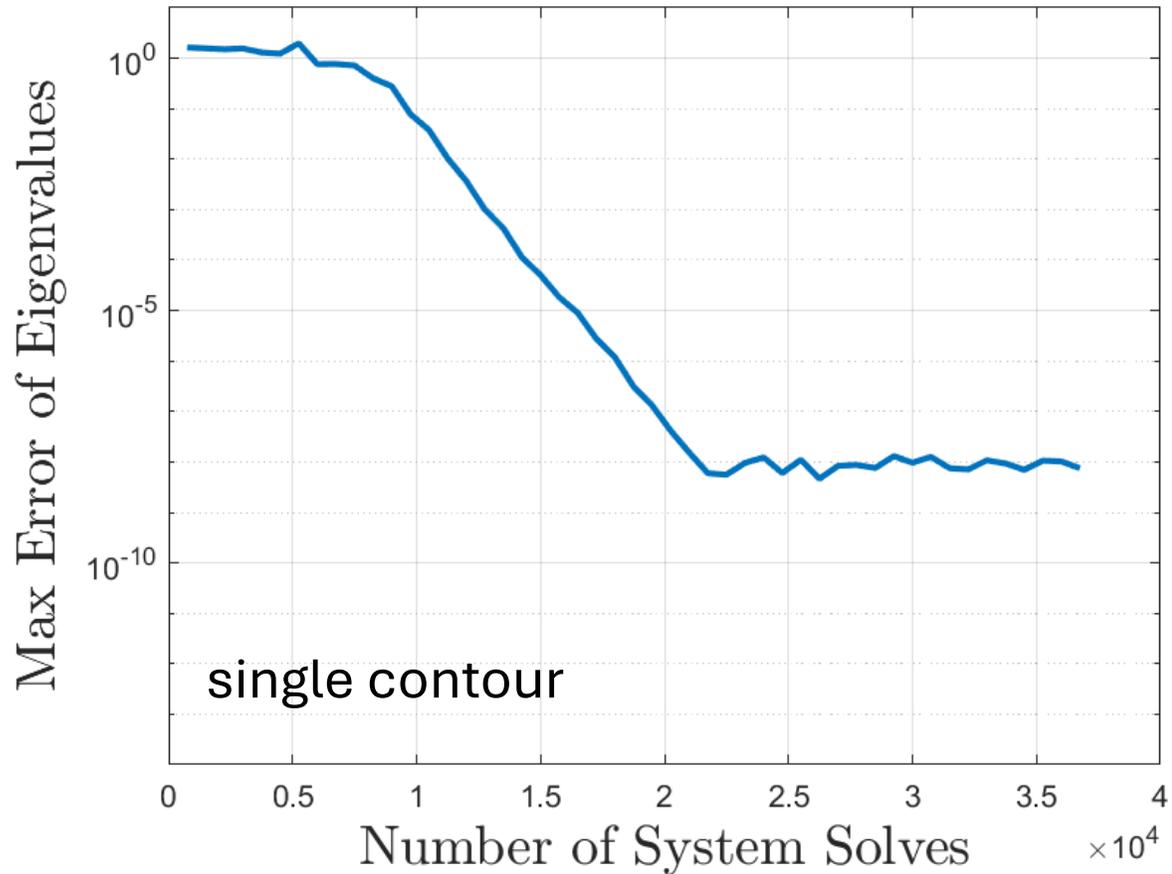


Example 5: Resonances

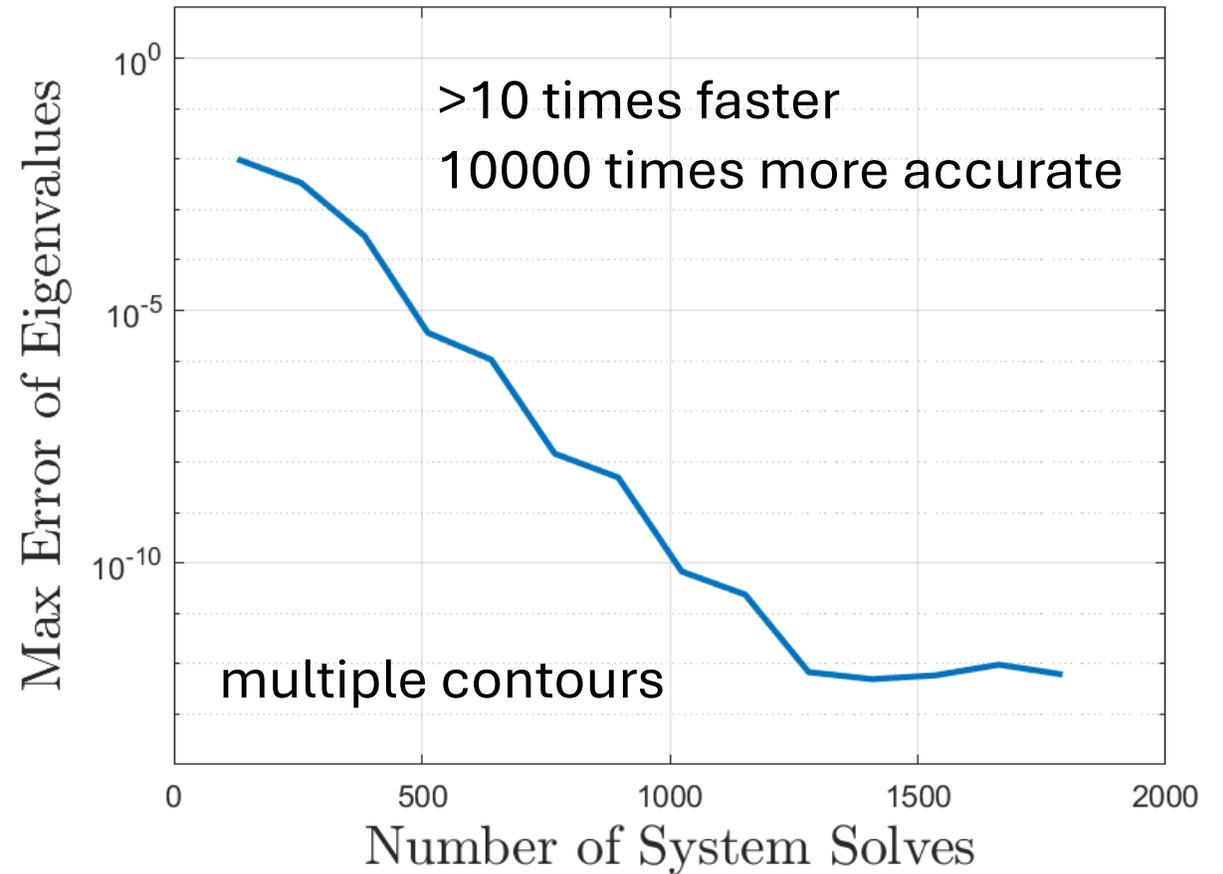


Example 5: Resonances

Gauss over Rectangle



Trapezoidal around Eigenvalues



Conclusion: Systematic treatment of nonlinear spectral problems in infinite dimensions

Discretization often causes issues. Bullet proof + practical algorithms exist.

- Singular values $\rightarrow \text{Sp}(T)$ (verified $\text{Sp}_\epsilon(T)$) computed when $\text{graph}(T(z))$ cts in z
- Keldysh's theorem: Laurent expansion when $z \mapsto T(z)$ holomorphic
- Contour methods use Keldysh + Cauchy to compute e-values and e-vectors:
 - 1) Stable up to intrinsic properties of T and scalar functions h .  Good choice largely open problem!
 - 2) Sketching does not degrade stability (not surprising).
 - 3) Must be done with trace class covariance (surprising).  1,2,4 apply to finite dimensions too!
 - 4) Often better to use multiple smaller contours to post-process.

*Also in this programme: Classify problem difficulty + prove algorithms are optimal.
Holomorphic operators of type A allow Newton's method.*

- C., Drysdale, "Universal Methods for Nonlinear Spectral Problems," **Journal of Spectral Theory**, to appear.
- C, "INFINITE-DIMENSIONAL SPECTRAL COMPUTATIONS: Foundations, Algorithms, and Modern Applications," **CUP**, to appear.

Shameless plug...

Upcoming book with CUP:

INFINITE-DIMENSIONAL SPECTRAL COMPUTATIONS

Foundations, Algorithms, and Modern Applications

100s of: classifications, algorithms, examples (webpage: full code), figures, exercises (webpage: full solutions).

****Out early 2026 (hopefully!)...****

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If something interests you, please speak to me after.

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Bonus example: Nonlinear shift on $\ell^2(\mathbb{Z})$

$$T(z) = S - f(z)S^* = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & 0 & 1 & & \\ & -f(z) & 0 & 1 & \\ & & -f(z) & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Exercise (for Marcus Webb):

a) $\text{Sp}(T) = \{z: |f(z)| = 1\}$.

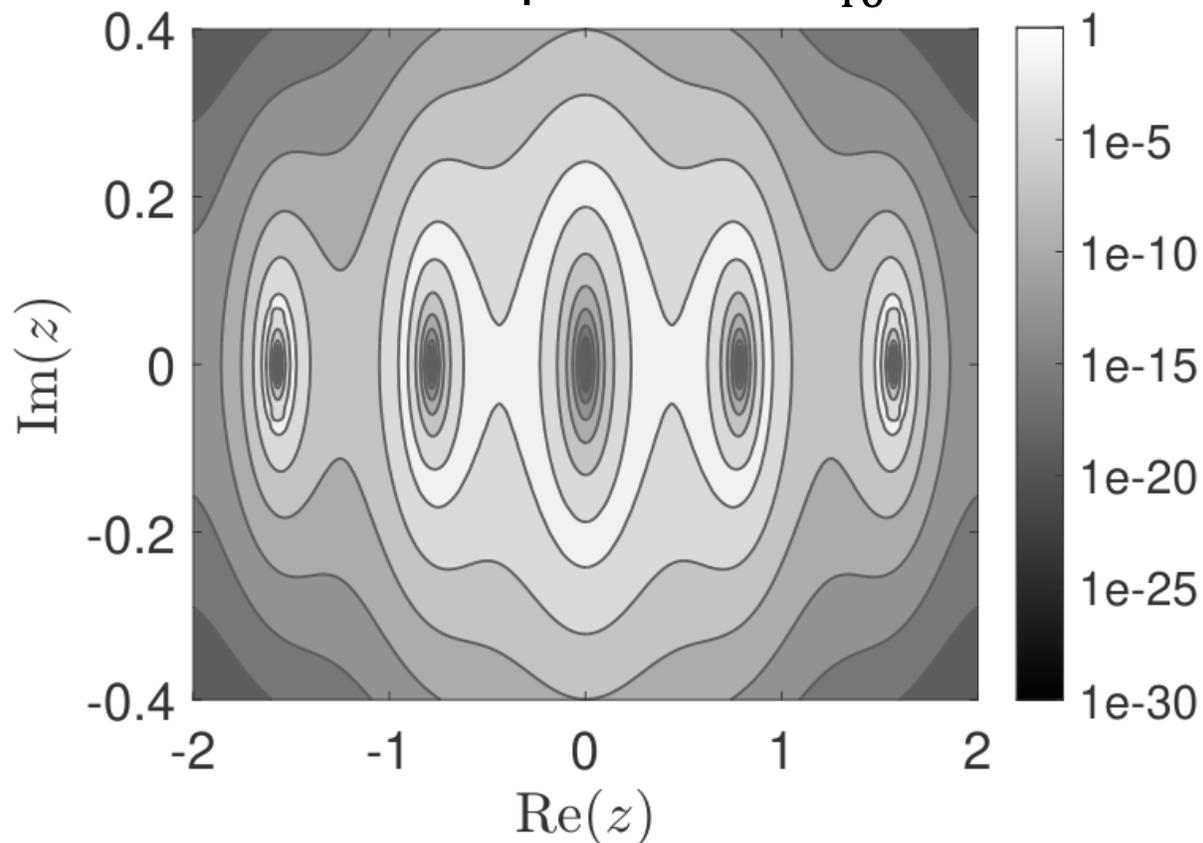
b) Truncate to $k \times k$ Toeplitz operator T_k :

$$\text{Sp}(T_k) = \begin{cases} \mathbb{C}, & \text{odd } k \\ \{z: f(z) = 0\}, & \text{even } k \end{cases}$$

Bonus example: Nonlinear shift on $\ell^2(\mathbb{Z})$

$$f(z) = \sin(4z)(|z|^2 + 1)$$
$$V_N = \text{span}\{e_{-N}, \dots, e_N\}$$

Pseudospectra of T_{40}



Bullet proof algorithm, using

$$\sigma_{\text{inf}}(T(z)\mathcal{P}_N) = \sigma_{\text{inf}}(\mathcal{P}_{N+1}T(z)\mathcal{P}_N)$$
