Optimization of Polynomial Roots, Eigenvalues and Pseudospectra

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Part I
Globally Optimizing the Roots of a Monic Polynomial subject to One Affine Constraint with

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M. Gürbüzbalaban (NYU)
A. Megretski (MIT)

The Root Radius and the Root Abscissa

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In continuous-time linear dynamical systems, $\alpha(p) < 0$ is the stability condition.

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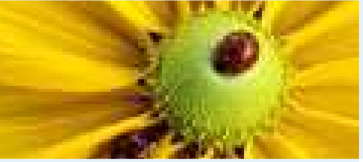
But if an affine family of monic polynomials of degree n has $n-1$ free parameters, this question can be answered efficiently.

Equivalently, there is *just one affine constraint* on the coefficients.

Optimizing the Root Radius, Real Case

Theorem RRR. *Let $B_0, B_1, \ldots, B_n$ be real scalars (with $B_1, \ldots, B_n$ not all zero) and consider the affine family*

$$P = \{z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n \;:\; B_0 + \sum_{j=1}^{n} B_j a_j = 0, a_i \in \mathbf{R}\}.$$



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The optimization problem

$$\rho^* := \inf_{p \in P} \rho(p)$$

has a globally optimal solution of the form

$$p^*(z) = (z - \gamma)^{n-k}(z + \gamma)^k \in P$$

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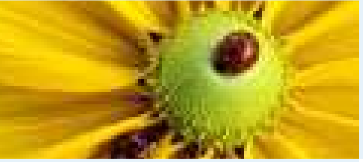
Proof: uses implicit function theorem.



Algorithm to Find γ and k

Corollary RRR. *Let γ be the globally optimal value whose existence is asserted in Theorem RRR, and consider the set*

$$\Xi = \{r \in \mathbf{R} \ : \ g_k(r) = 0 \text{ for some } k \in \{0, 1, \ldots, n\}\}$$

where

$$g_k(z) = B_0 v_0 + B_1 v_1 z + \ldots + B_{n-1} v_{n-1} z^{n-1} + B_n v_n z^n$$

and $(v_0, \ldots, v_n)$ *is the convolution of the vectors*

$$\left(\binom{n-k}{0}, \binom{n-k}{1}, \ldots \binom{n-k}{n-k}\right) \text{ and } \left(\binom{k}{0}, -\binom{k}{1}, \ldots (-1)^k \binom{k}{k}\right)$$

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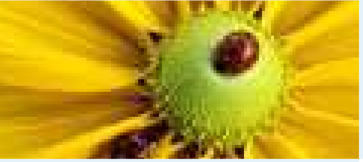

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Proof: immediate.

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Theorem RRC. *Let $B_0, B_1, \ldots, B_n$ be complex scalars (with $B_1, \ldots, B_n$ not all zero) and consider the affine family*

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Proof: Surprisingly, much more complicated than the real case.

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Theorem RAR. *Let $B_0, B_1, \ldots, B_n$ be real scalars (with $B_1, \ldots, B_n$ not all zero) and consider the affine family*

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Let $k = \max\{j : B_j \neq 0\}$. and define the polynomial of degree k

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$$\alpha^* := \inf_{p \in P} \alpha(p).$$

has the infimal value

$$\alpha^* = \min\left\{\beta \in \mathbf{R} \ : \ h^{(i)}(-\beta) = 0 \text{ for some } i \in \{0, \ldots, k-1\}\right\},$$

where $h^{(i)}$ is the i-th derivative of h.

Continuation

Furthermore, the optimal value α^* is attained by a minimizing polynomial p^* if and only if $-\alpha^*$ is a root of h (as opposed to one of its derivatives), and in this case we can take

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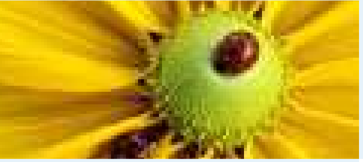
Proof: substantially more complicated than radius case, because optimal value may not be attained.



When the Optimal Abscissa Value is not Attained

Theorem RAR2. Assume that $-\alpha^*$ is not a root of h. Let ℓ be the smallest integer $i \in \{1, \ldots, k-1\}$ for which $-\alpha^*$ is a root of $h^{(i)}$. Then, for all sufficiently small $\epsilon > 0$ there exists a real scalar M_ϵ for which

$$p_\epsilon(z) := (z - M_\epsilon)^m (z - (\alpha^* + \epsilon))^{n-m} \in P$$

where $m = \ell$ or $\ell + 1$, and $M_\epsilon \to -\infty$ as $\epsilon \to 0$.

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There is a formula for M_ϵ, but it is a little complicated and depends on whether $m = \ell$ or $m = \ell + 1$.

In practice, bad idea to make ϵ too small: then $|M_\epsilon|$ becomes large.

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Indeed, this is the case...

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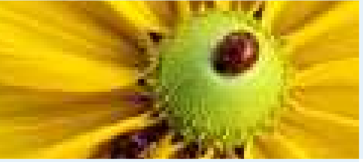
Proof: follows (with some work) from Theorem RRC.

A Simple Example

Minimize the root abscissa over the set of polynomials

$$P = \{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n \ : \ a_1 + a_2 = 0\}.$$



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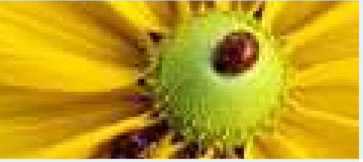
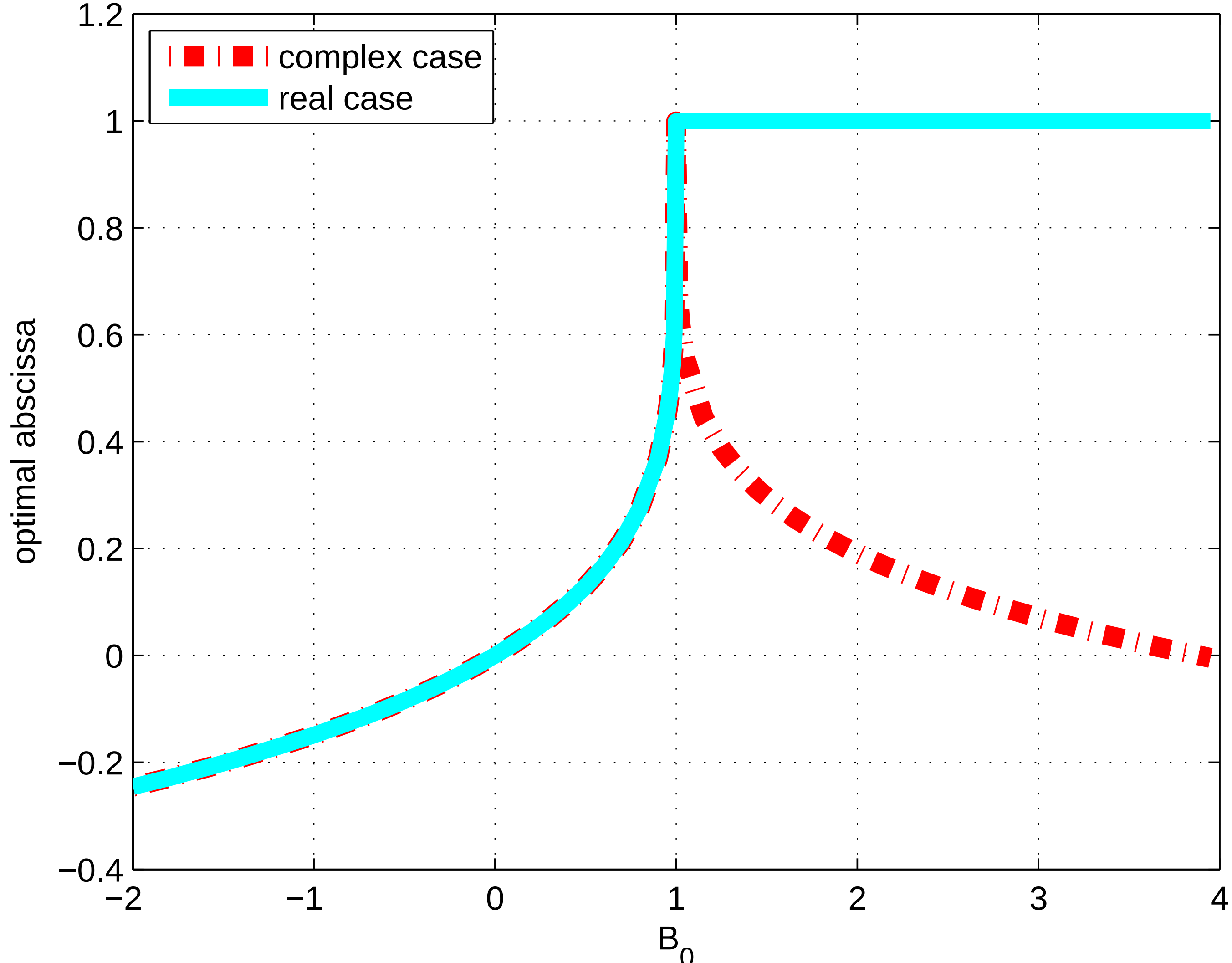
Theorem RAR proves global optimality over $a_i \in \mathbf{R}$ and Theorem RAC proves global optimality over $a_i \in \mathbf{C}$.

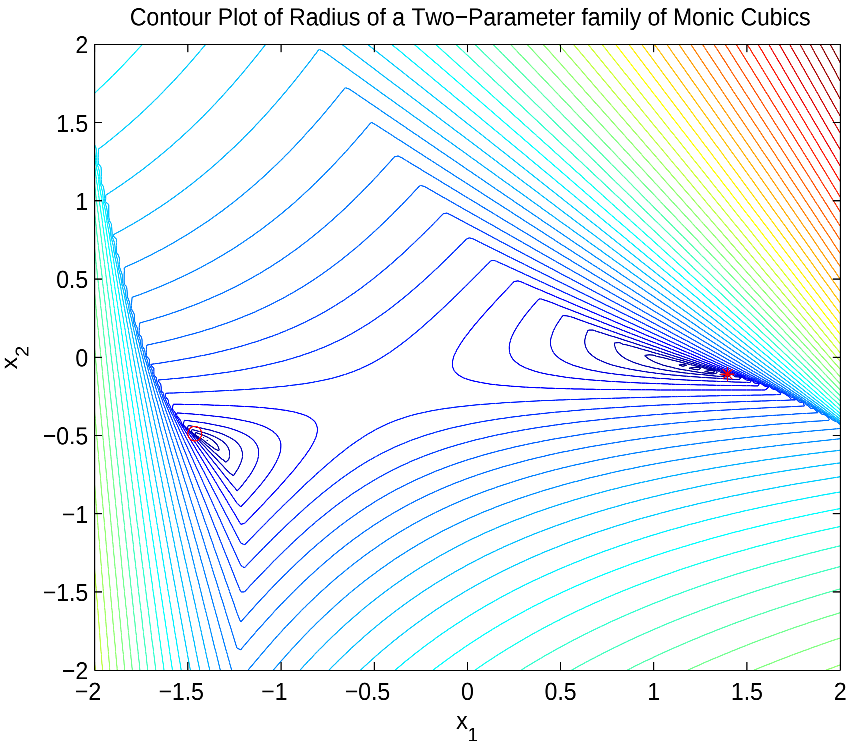


Optimal Root Abscissa for a Quintic Family



$z^5 + a_1 z^4 + \cdots + a_5$ with constraint $B_0 + \sum_{i=1}^5 a_i = 0$

Optimal Root Radius for a Cubic Family



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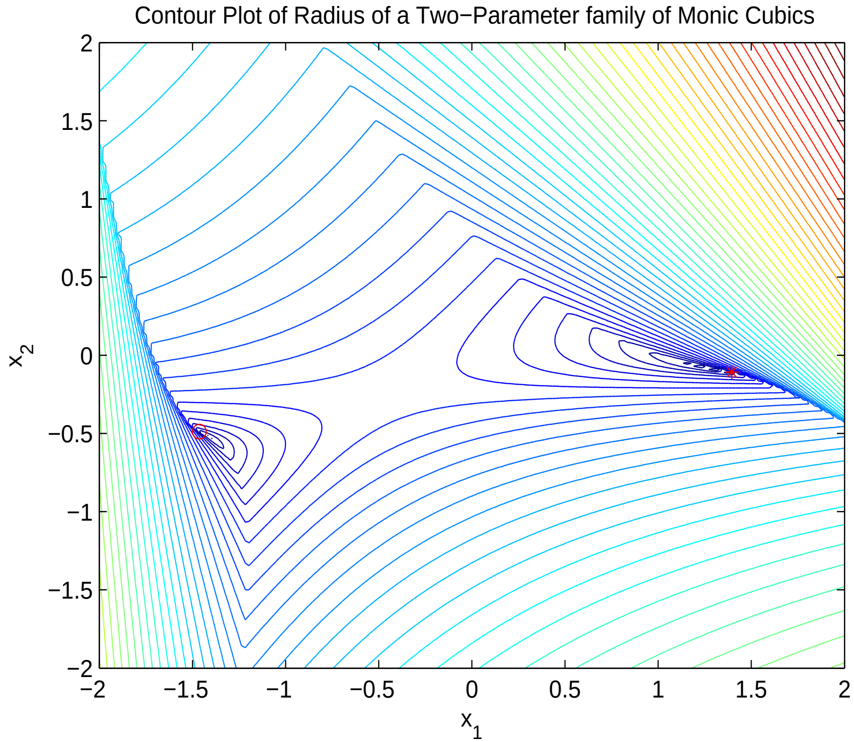

Randomly generated family with coefficients depending affinely on two real parameters x_1, x_2. Above left: contour plot in parameter space. Note the steep contours! '*' marks a global minimizer $(z - \gamma_1)^3$ and 'o' a local minimizer marks a local minimizer $(z - \gamma_2)^3$ where $-\gamma_1 \approx -0.541$ and $-\gamma_2 \approx 0.567$ are *both* roots of g_0 (defined in Corllary RRR).

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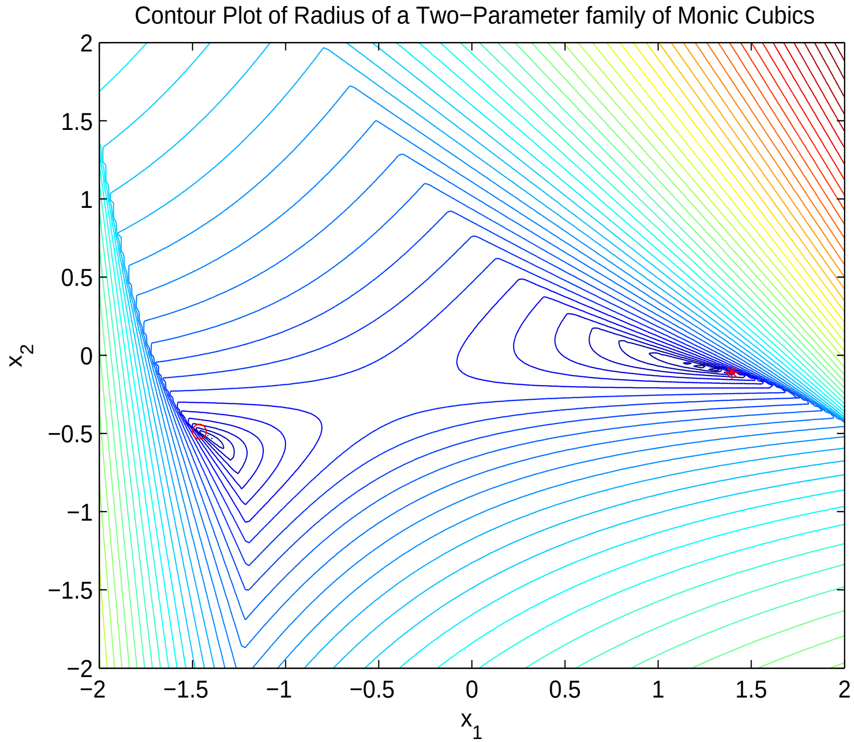

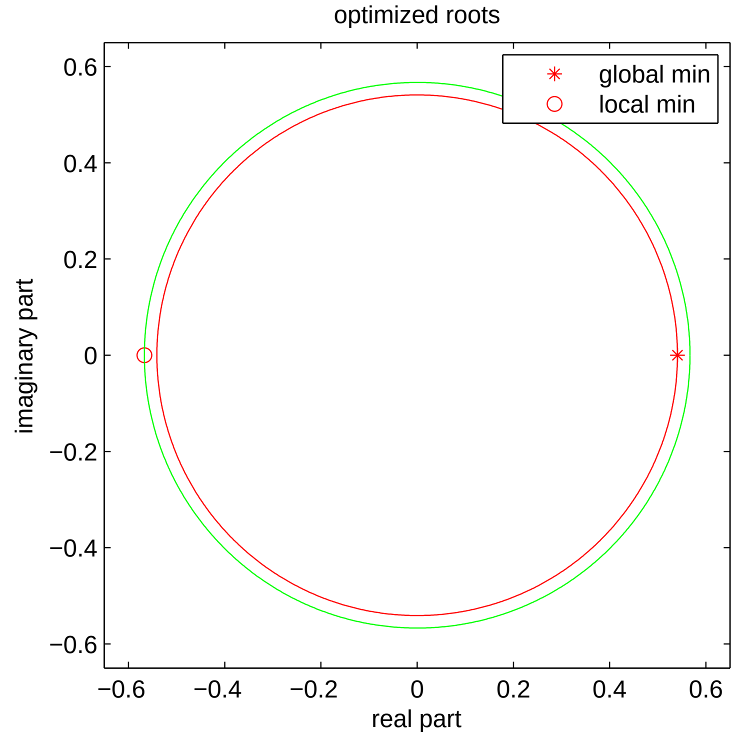

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Right: complex plane plot of optimal roots. '*' is γ_1 and 'o' is γ_2.

Static Output Feedback Stabilization

Given the dynamical system

$$\dot{x} = Fx + Gu, \quad y = Hx$$

where $F \in \mathbf{R}^{n\times n}, G \in \mathbf{R}^{n\times p}$, $H \in \mathbf{R}^{m\times n}$.

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where $F \in \mathbf{R}^{n \times n}, G \in \mathbf{R}^{n \times p}$, $H \in \mathbf{R}^{m \times n}$.

SOF: find a controller $K \in \mathbf{R}^{p \times m}$ so that, setting $u = Ky$

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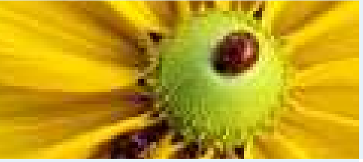
is stable, that is all eigenvalues of $F + GKH$ are in the left half-plane, or prove that this is not possible.



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But, if $p = 1$ and $m = n - 1$ (one input and $n - 1$ outputs)

$$\det(\lambda I - F - GKH) = \det(\lambda I - F) + KH\mathrm{adj}(\lambda I - F)G.$$

This is a monic polynomial with affine dependence on the $n - 1$ entries of $K \in \mathbf{R}^{1\times(n-1)}$ so the SOF problem can be solved explicitly using Theorem RAR.

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Another set of classical problems in control that, in a certain case, can be solved using the theorems given above.

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where

$$P = \{(z^4 + 2z^2)(x_0 + x_1 z + z^2) + y_0 + y_1 z + y_2 z^2 \ : x_0, x_1, y_0, y_1, y_2 \in \mathbf{R}\}$$

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We can minimize the root abscissa explicitly using Theorem RAR as P is a set of monic polynomials with degree 6 whose coefficients depend affinely on 5 real parameters.

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Independently of this, the monomial basis is a poor choice numerically unless the polynomial has very small degree.

Nonetheless, the optimal *value* can be computed accurately even if n is fairly large.

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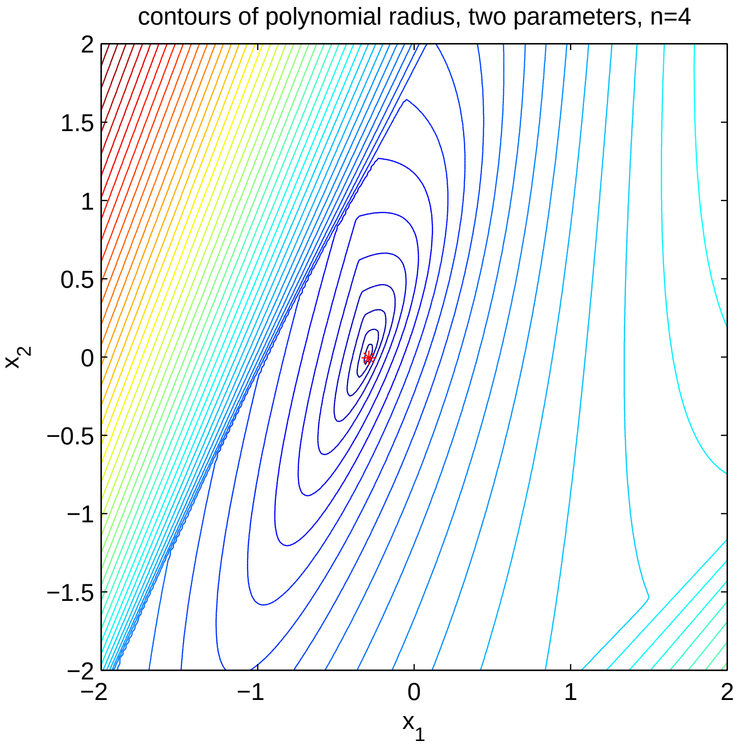
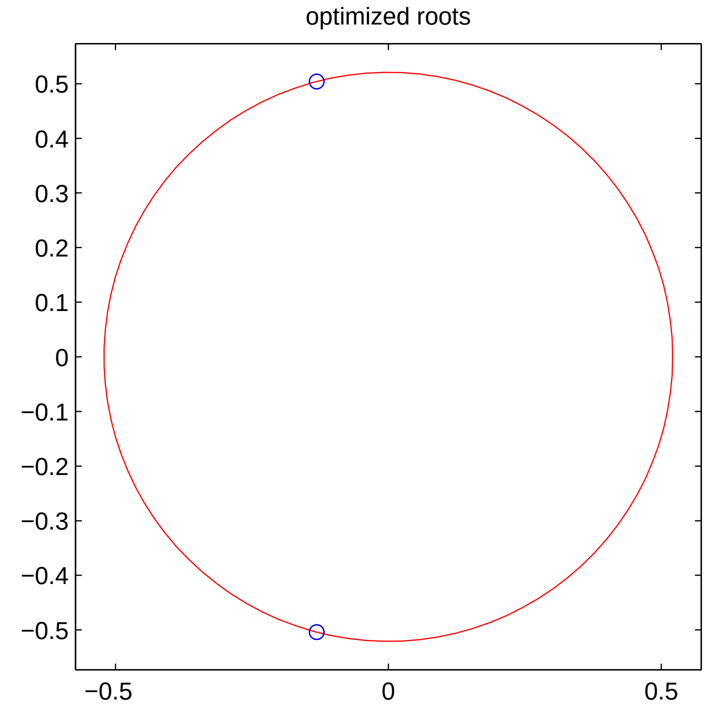
Since there are only two variables, use grid search to approximate global minimizer and then local optimization to refine this.

Degree $n = 4$: one double conjugate pair is active





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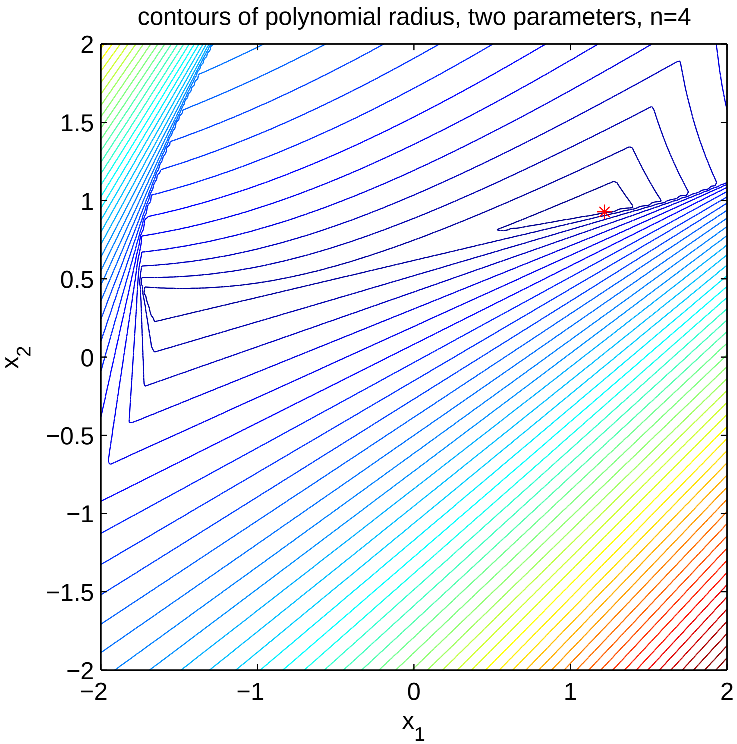

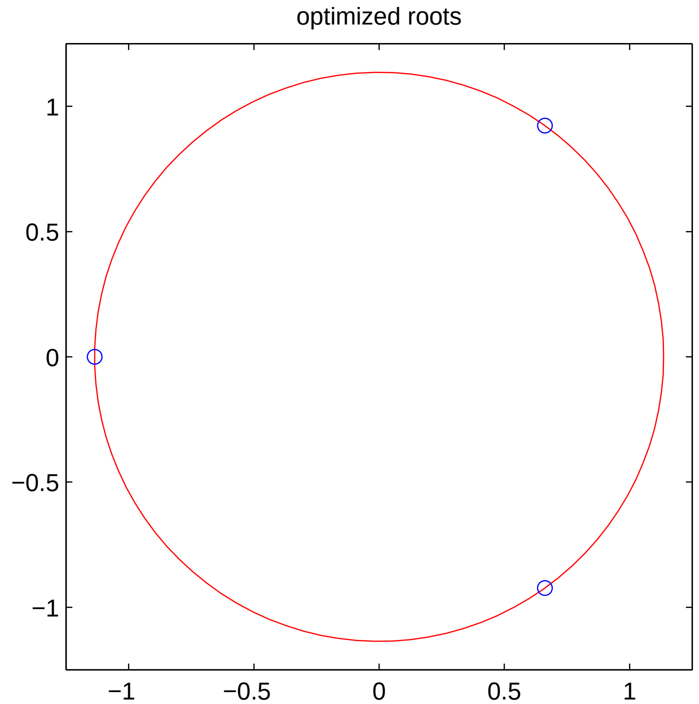

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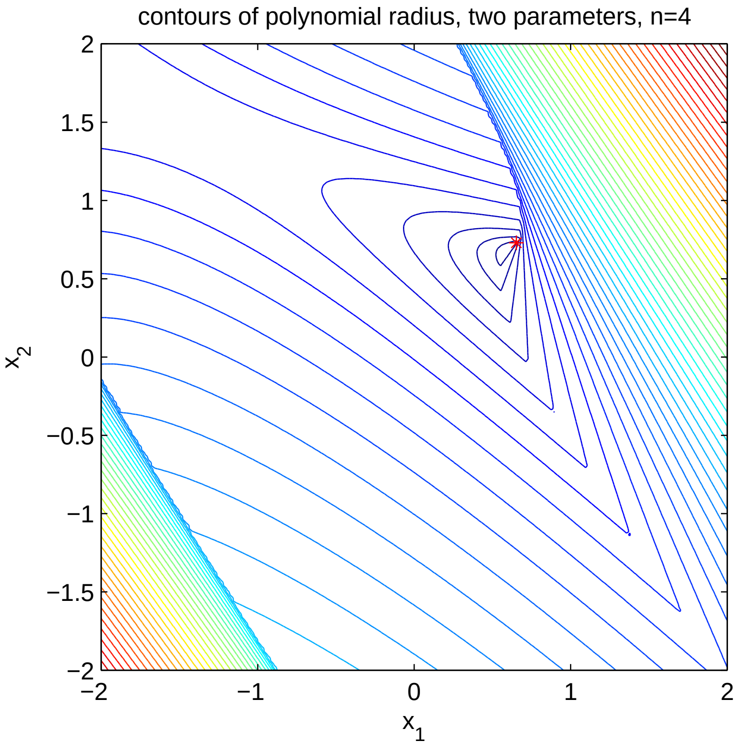

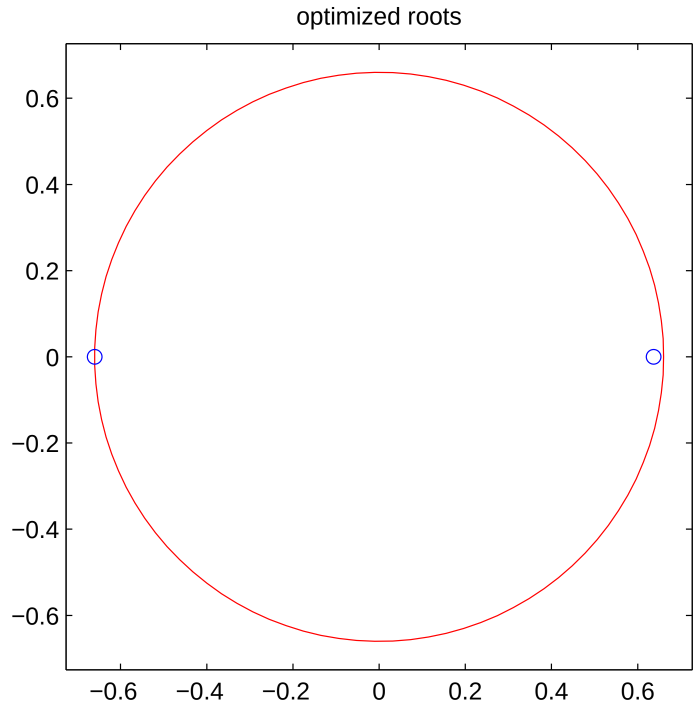

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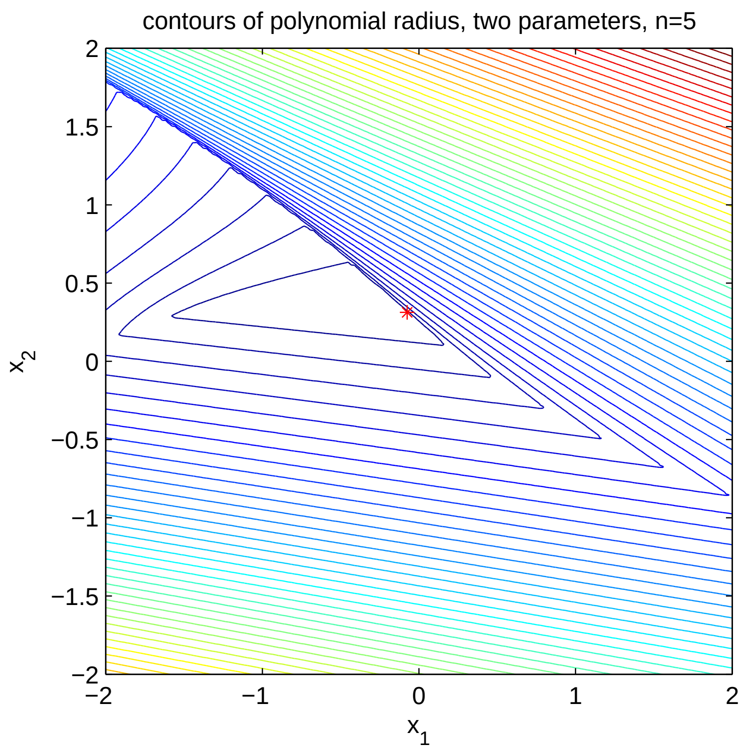
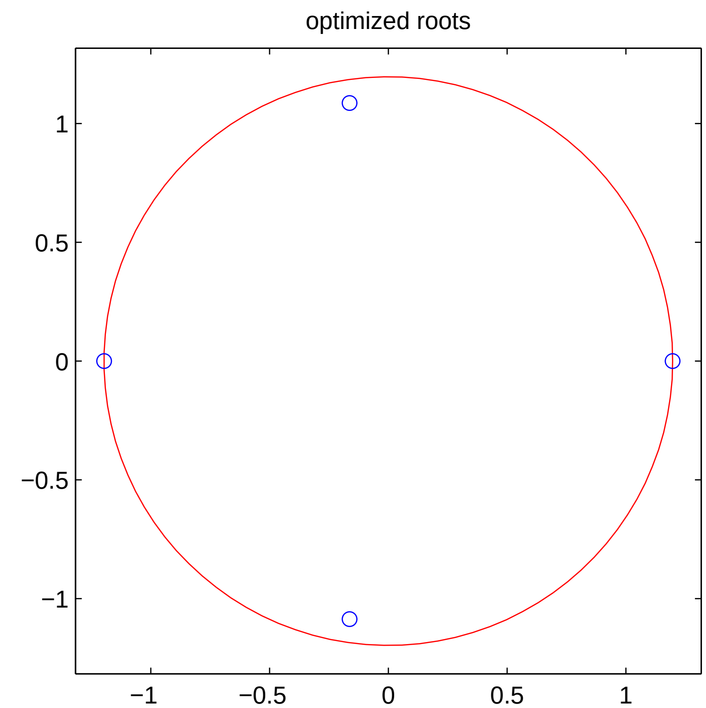
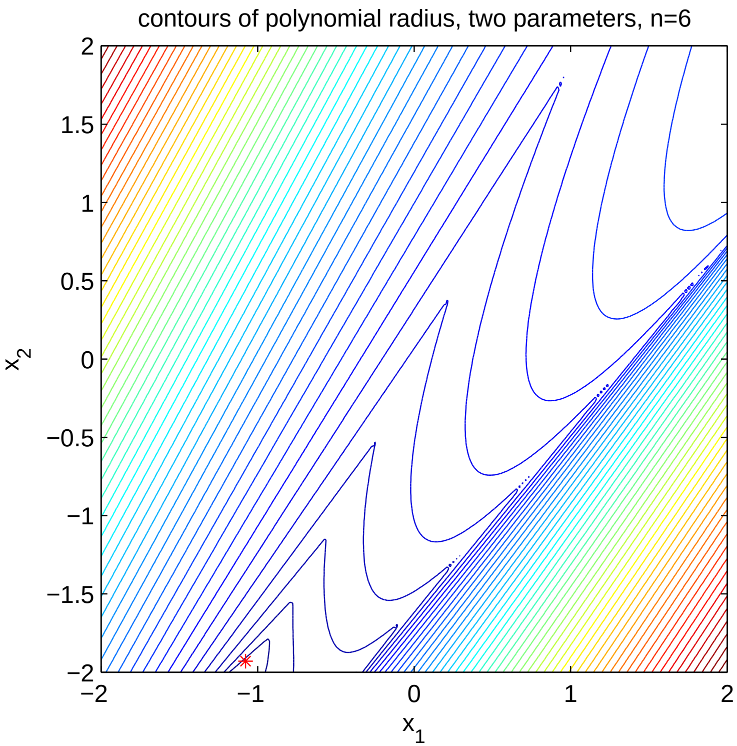
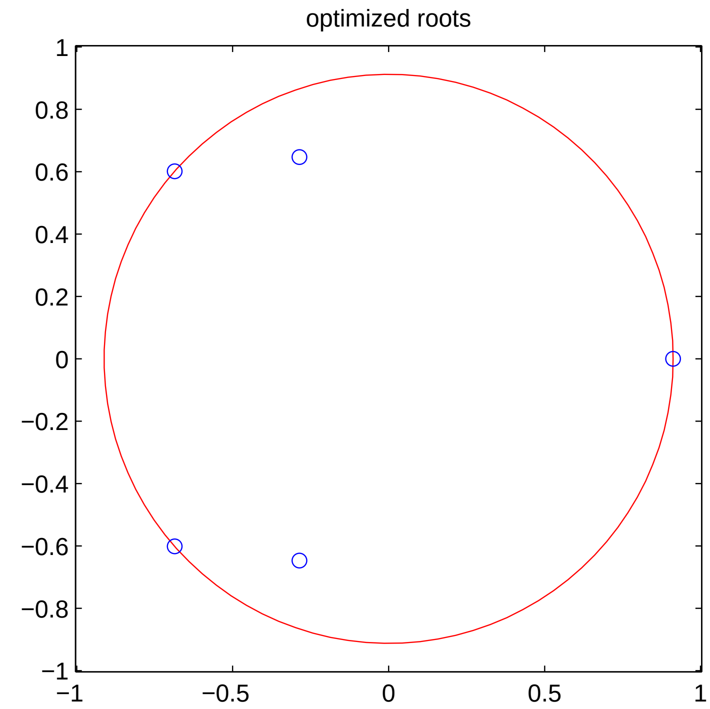




Degree $n = 6$: one double real and one simple conjugate pair are active





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A publicly available MATLAB code implementing the constructive algorithms implicit in the theorems:

www.cs.nyu.edu/overton/software/affpoly/

Part II
Optimization of Eigenvalues
with
J.V. Burke (Wash.)
K.K. Gade (NYU)
S. Grundel (NYU)
A.S. Lewis (Cornell)

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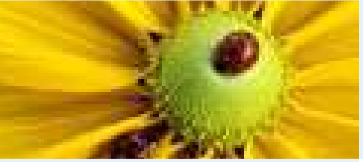

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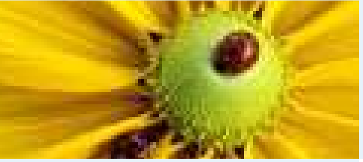

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Hermitian matrices: completely different story!

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For example, consider the matrix family

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The minimal spectral radius of $A(\xi)$ is attained by $\xi = 0$, for which the eigenvalues are $\pm\mathbf{i}$.

The Diaconis-Holmes-Neal Sampler

A nonreversable Markov chain for Monte Carlo simulation. For $x \in (0,1)$, the transition matrix is $A(x) \in \mathbf{R}^{2n\times 2n}$ is

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Diaconis et. al. showed that for $x = 1/n$, the corresponding nonreversible chain reaches a stationary state in $O(n)$ steps, compared to $O(n^2)$ steps for a similar reversible chain.

The Reduced Spectral Radius

The rate of convergence is determined by

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It is easy to prove that this is minimized over $x \in [0,1]$ by

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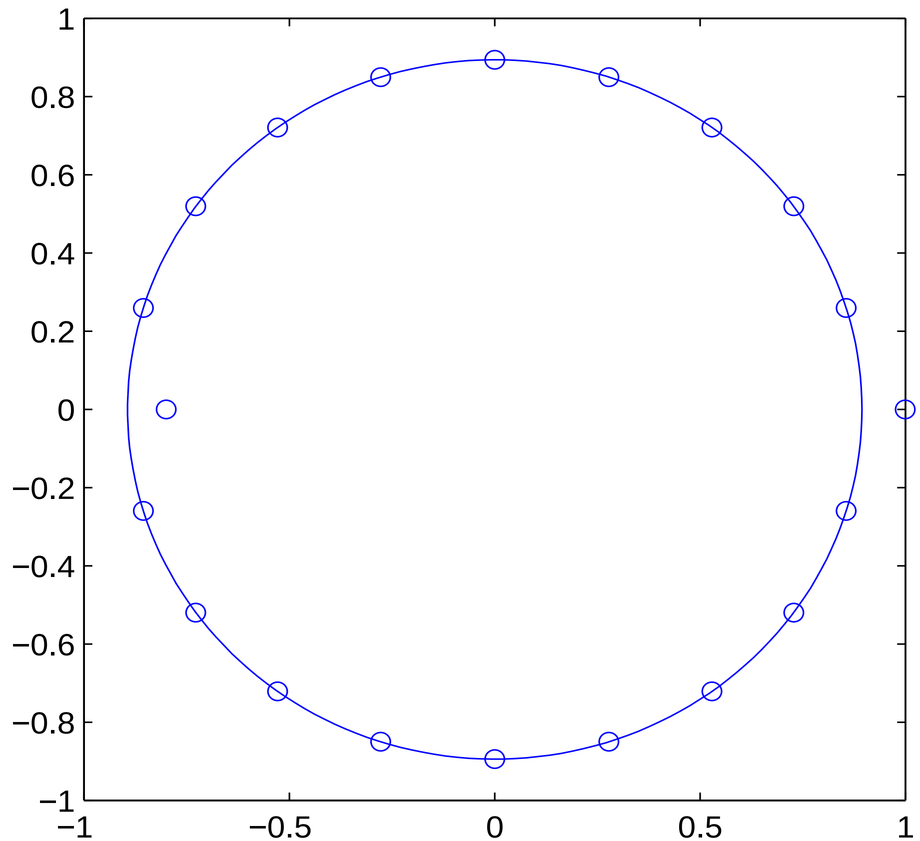
For $x > x_{\rm opt}$, this splits into two real eigenvalues, increasing $\tilde{\rho}$ by $O(|x - x_{\rm opt}|^{1/2})$.

Eigenvalues of the Transition Matrix, $n = 10$



- Blue: eigenvalues when $x = 1/n$ (all complex)

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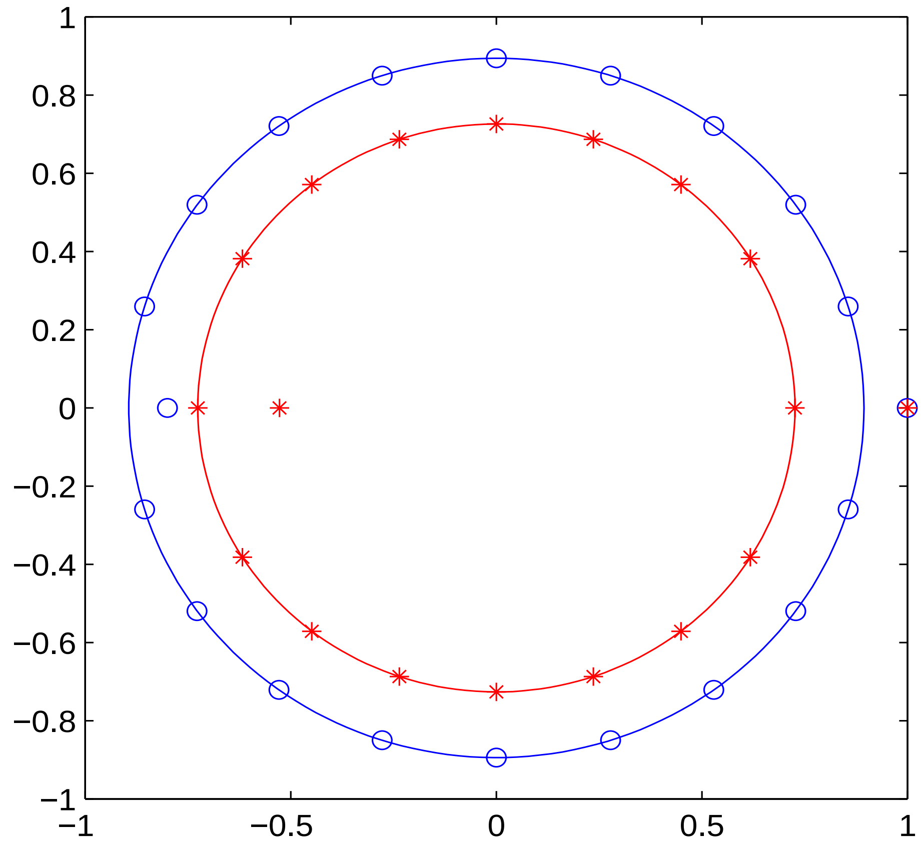

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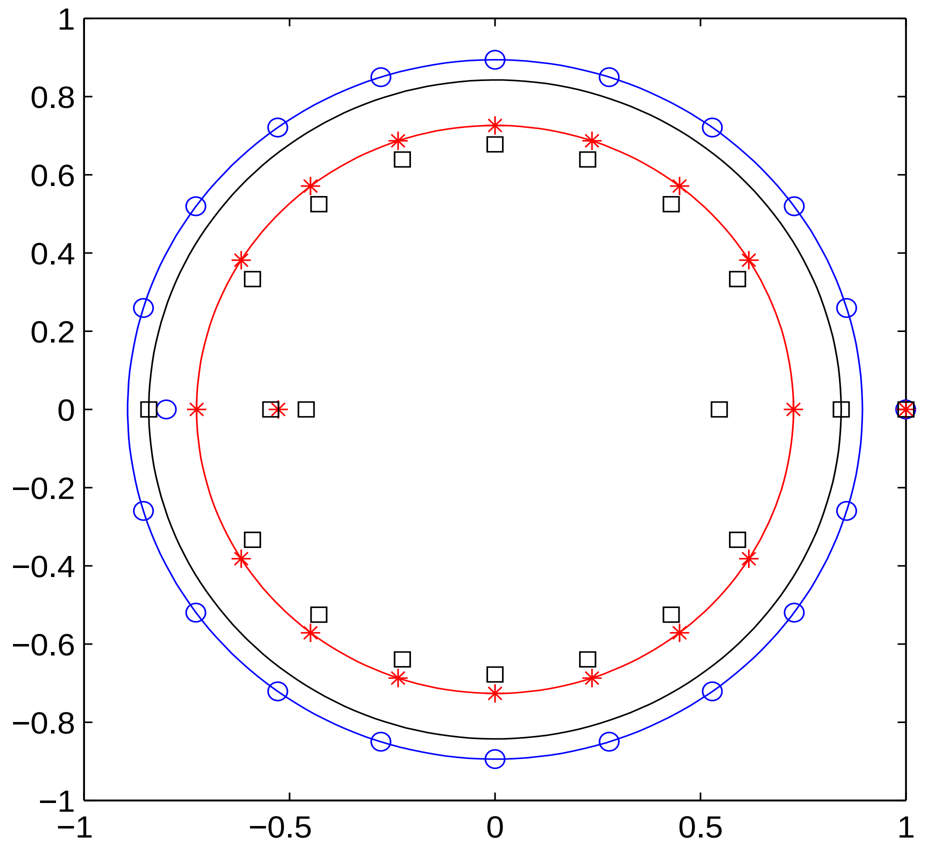

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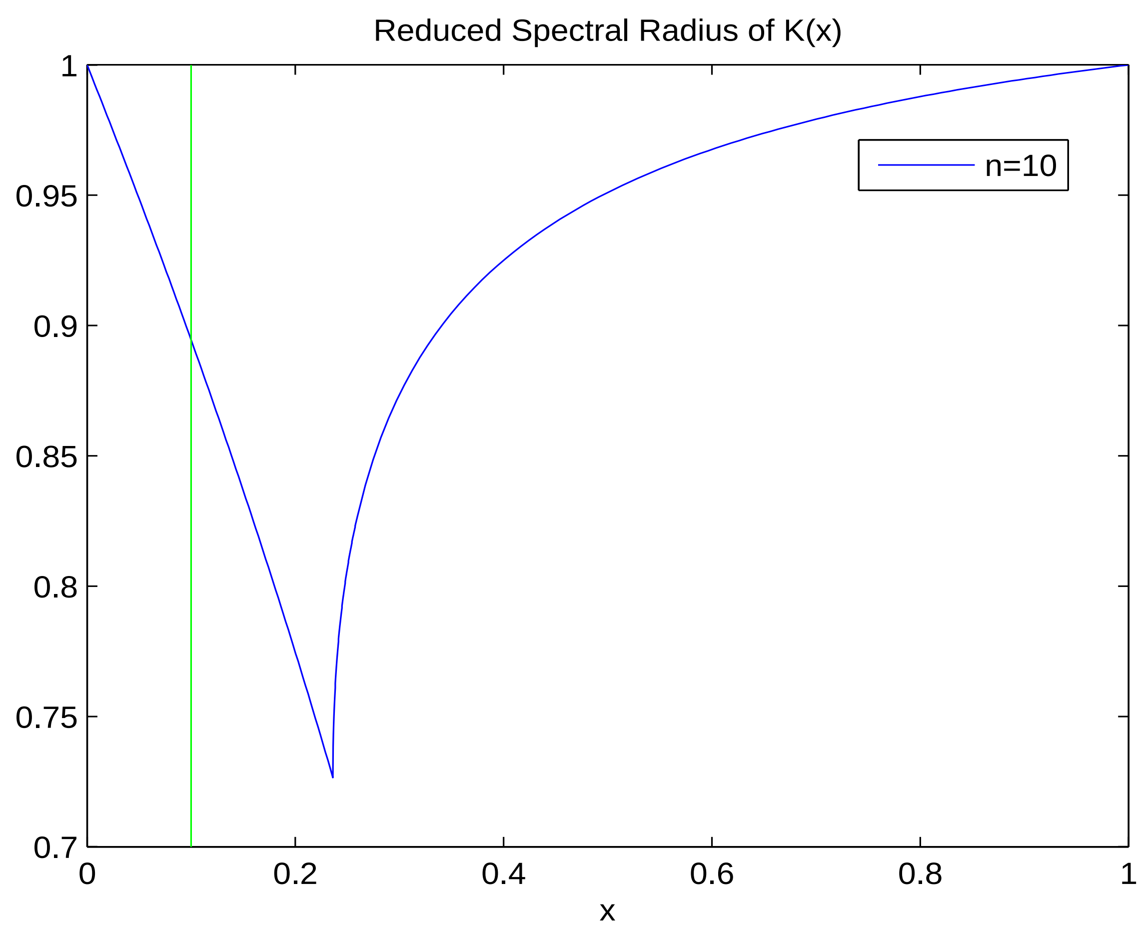
- Black: eigenvalues when $x > x_{\text{opt}}$ ($\tilde{\rho}$ increases rapidly)

Reduced Spectral Radius as a Function of x



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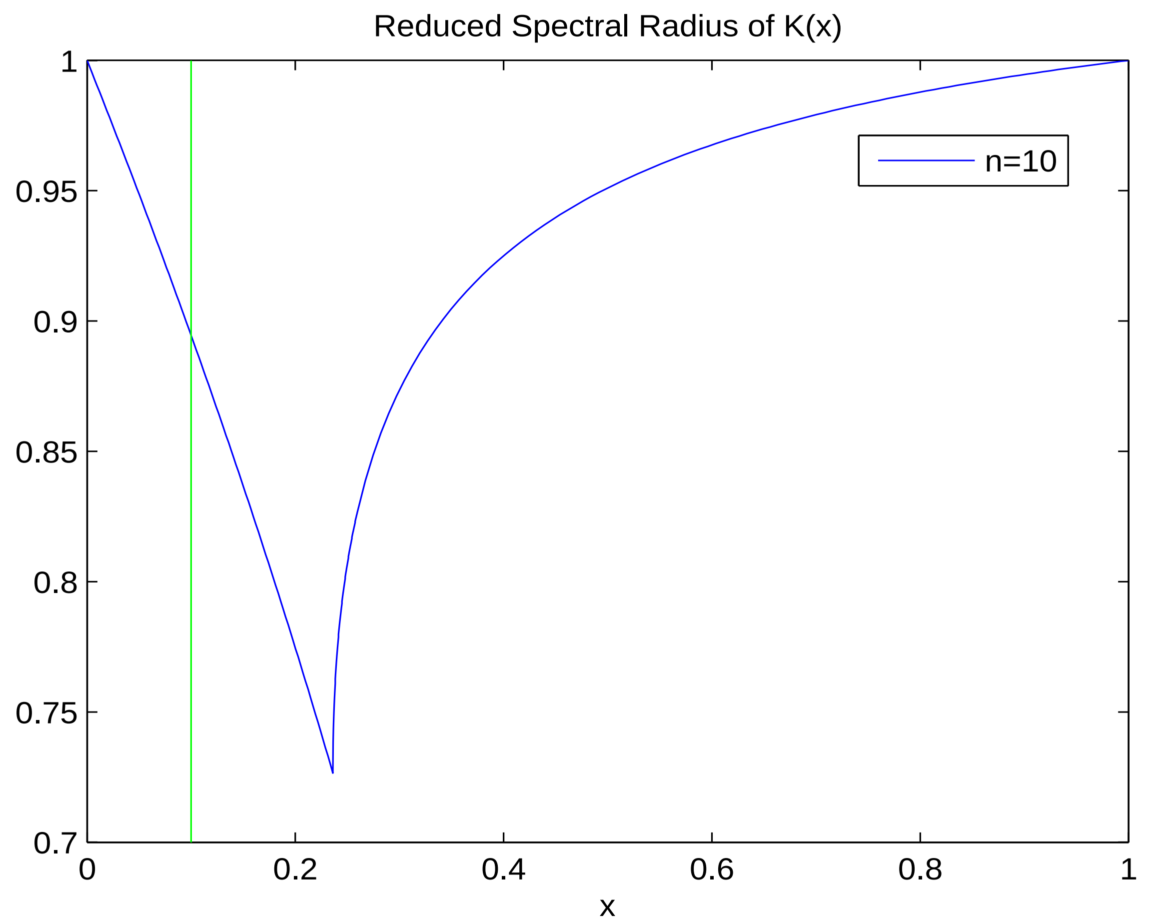

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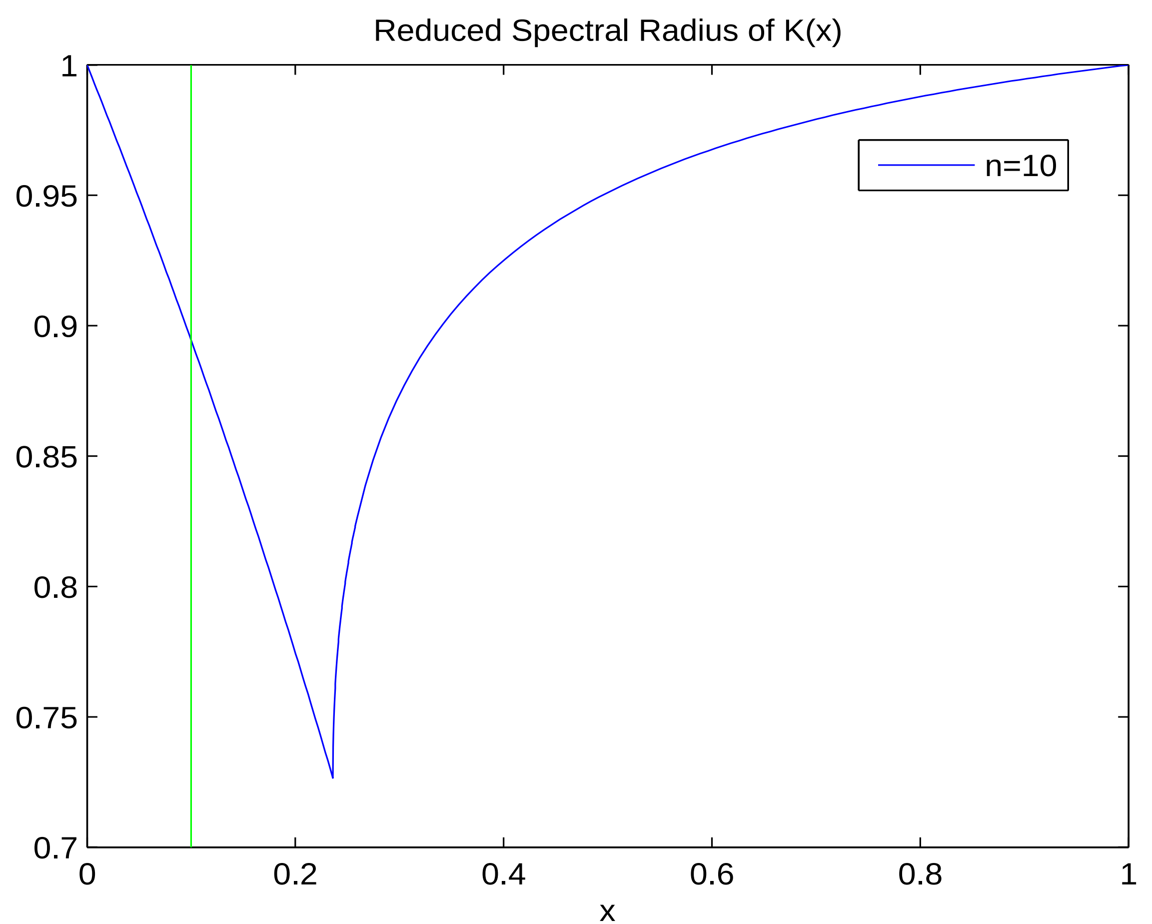

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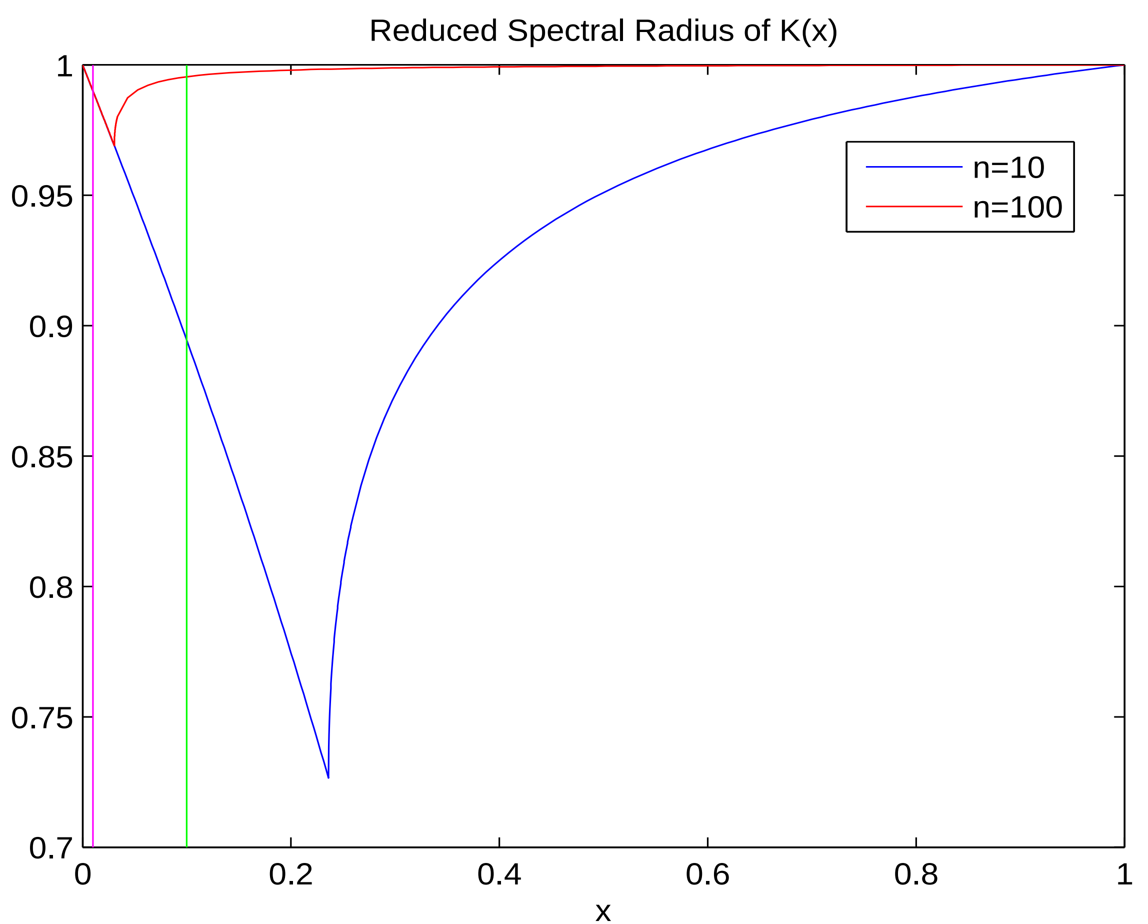
Much better to underestimate x_{opt} than overestimate. Similar plots apply to optimal damping for one-dimensional wave equation, optimal choice of parameter for SOR (successive over-relaxation), etc etc.

Reduced Spectral Radius as a Function of x



Note the big improvement changing x from $1/n$ to x_{opt}.

Much better to underestimate x_{opt} than overestimate. Similar plots apply to optimal damping for one-dimensional wave equation, optimal choice of parameter for SOR (successive over-relaxation), etc etc.

Convergence rate deteriorates as n increases.

Adding More Parameters

Not surprising that with one free parameter, we can only make one pair of eigenvalues coalesce.

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Let's change $K(x)$ to have multiple parameters:

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Still doubly stochastic. Can we now reduce $\tilde{\rho}$ further?

No! It appears that $\mathbf{x}_{\text{opt}} = [x_{\text{opt}}, \ldots, x_{\text{opt}}]^T$ is locally optimal.

Checking Local Optimality

Numerically: by running an optimization method suitable for nonsmooth objectives at randomly generated points near $\mathbf{x}_{\mathrm{opt}}$. We used the "gradient sampling" method, and repeatedly obtained convergence to $\mathbf{x}_{\mathrm{opt}}$.

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- $\mathbf{x}_{\text{opt}}$ satisfies a *necessary* condition for local optimality
- if we remove some redundancy by setting $x_j = x_{n-1-j}$ for $j = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor$ and $x_{n-1} = x_n$, we find $\mathbf{x}_{\text{opt}}$ satisfies a *sufficient* condition for local optimality.

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Because the analysis of this example is complicated, we explain the concepts on a much simpler example.

Subdifferential Analysis

Given a Euclidean space $\mathbf{E}$ and a continuous function $f : \mathbf{E} \to \mathbf{R}$, the *regular subdifferential* of f at a point x in $\mathbf{E}$, denoted $\hat{\partial} f(x)$, is

$$\{y \in \mathbf{E} : f(x+z) - f(x) \geq \langle y, z \rangle + o(\|z\|) \ \text{ for small } z \in \mathbf{E}\}.$$

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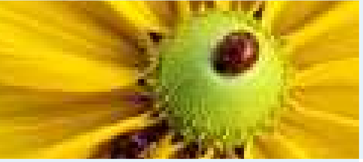

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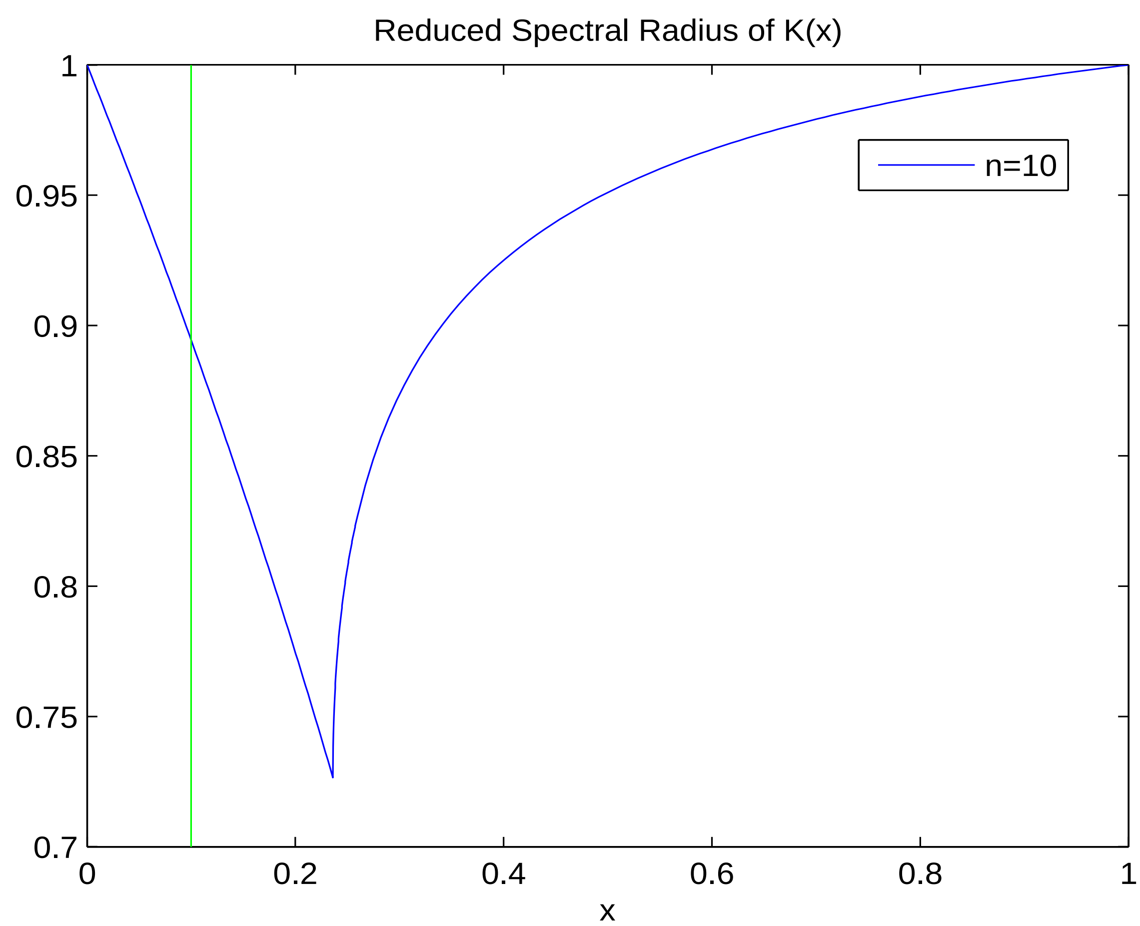
We say f is *subdifferentially regular* at x if $\partial f(x) = \hat{\partial} f(x)$ and $\partial^\infty f(x)$ is the recession cone of $\hat{\partial} f(x)$, hence $\{0\}$ if $\hat{\partial} f(x)$ is compact.

One Variable Example



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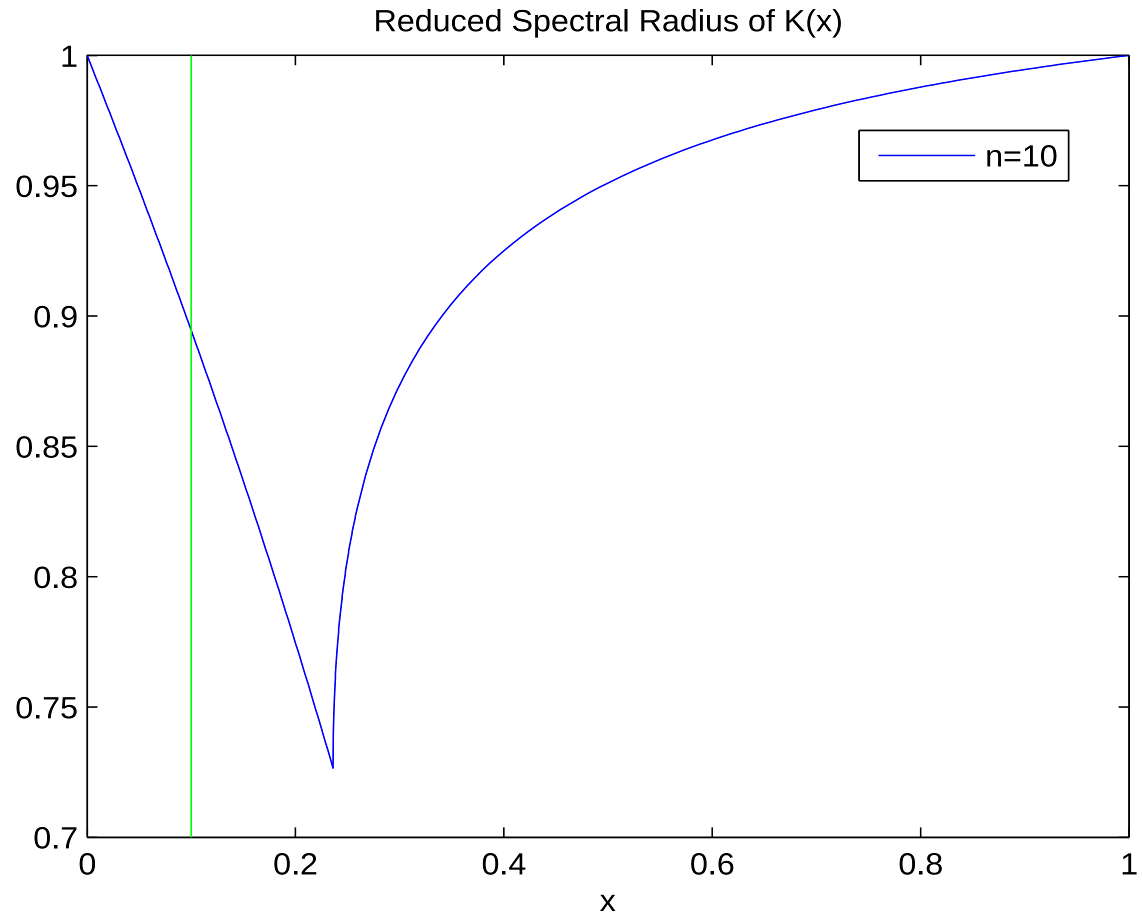

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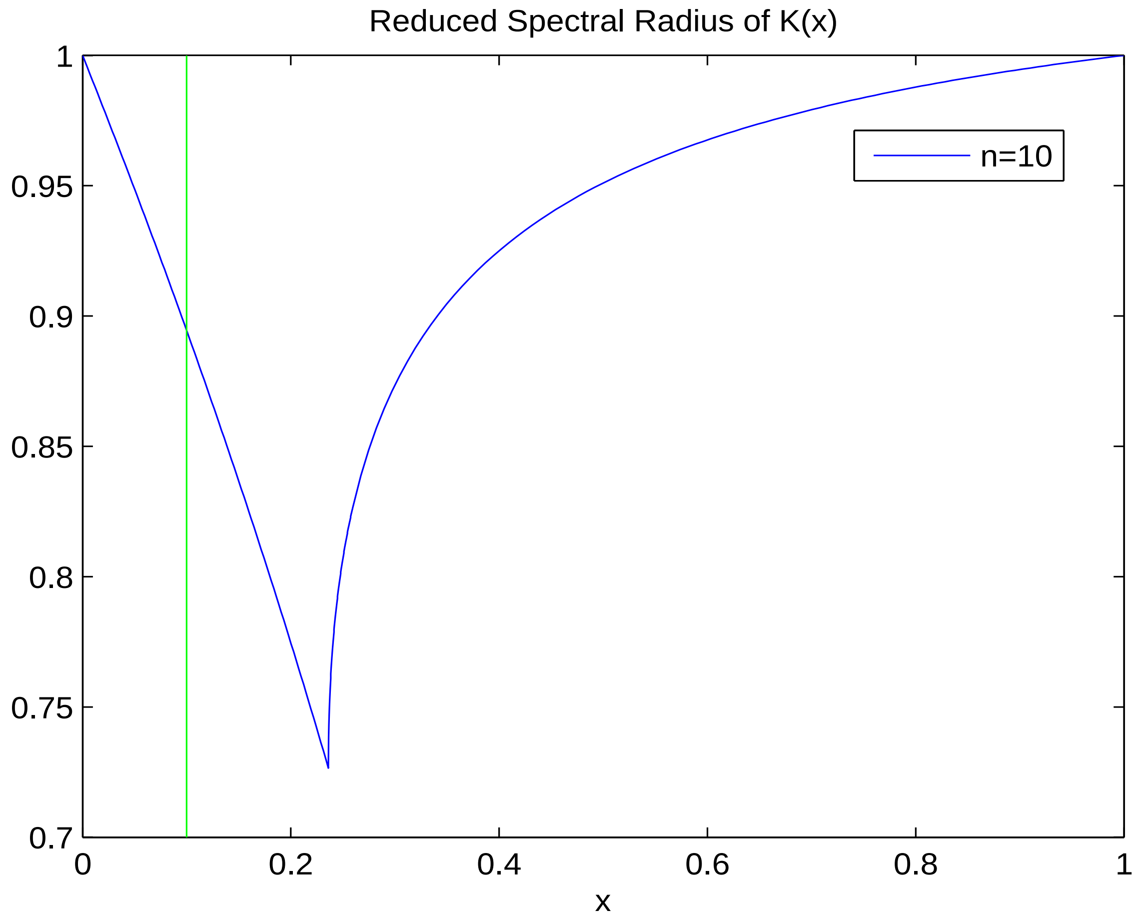

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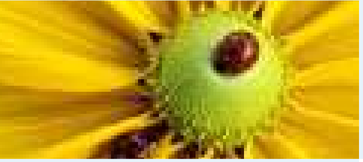

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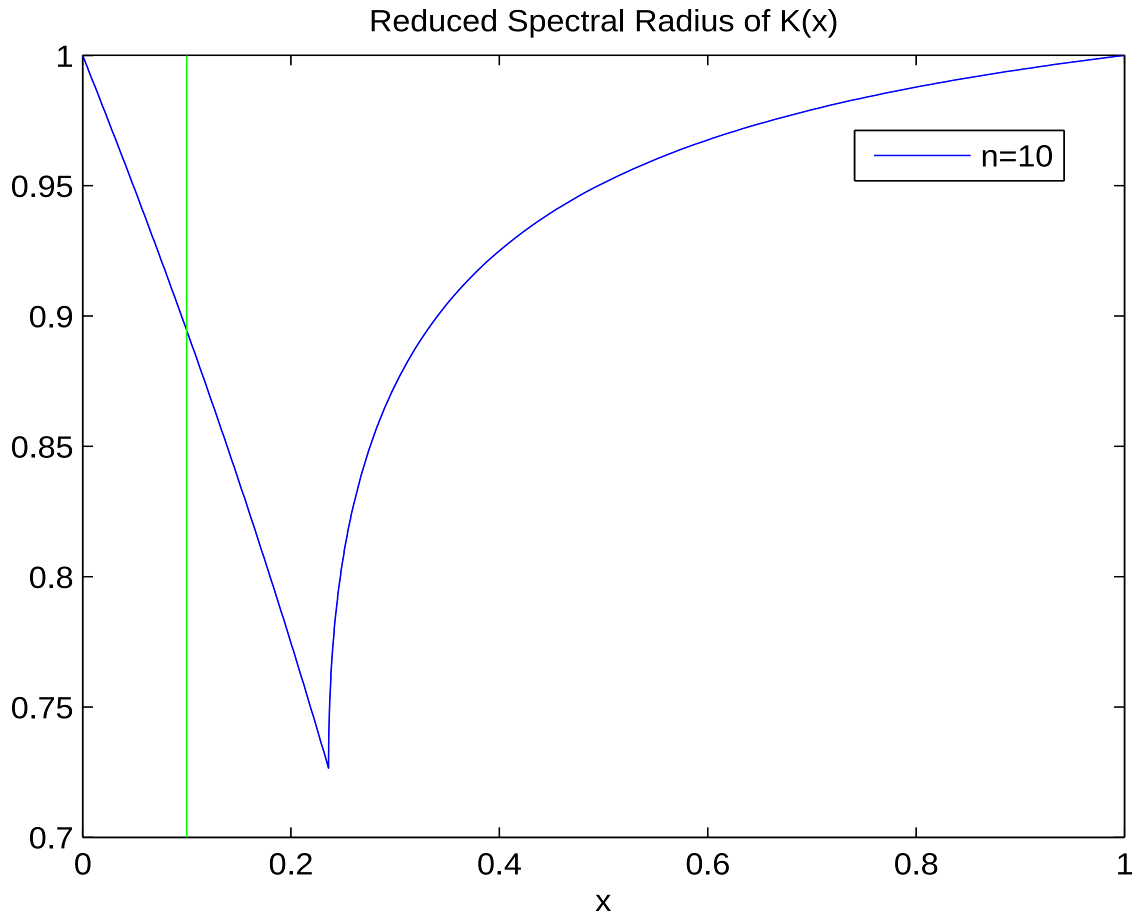

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Hence, this function of one variable is subdifferentially regular.

The Subdifferential of the Spectral Abscissa

Theorem SSA. *The spectral abscissa $\alpha : \mathbf{C}^{n\times n} \to \mathbf{R}$ is subdifferentially regular at $X \in \mathbf{C}^{n\times n}$ if and only if all the active eigenvalues are nonderogatory. When this is the case, let $X = PJP^{-1}$, where $J = \mathrm{Diag}(J_1, \ldots, J_k, J_{k+1}, \ldots)$,*

$$J_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & \lambda_i \end{bmatrix} \quad (m_i \times m_i), \quad i = 1 \ldots, k$$

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$$\partial\alpha(X) = \Big\{P^{-*}WP^* : W = \mathrm{Diag}(W_1, \ldots, W_k, 0, \ldots, 0),$$

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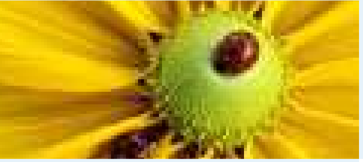

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$$\partial^\infty \alpha(X) :\ \text{same, except } \theta_1^i \equiv 0.$$

When All Active Eigenvalues are Simple

Then $X = PJP^{-1}$, where $J = \mathrm{Diag}(\lambda_1, \ldots, \lambda_k, J_{k+1}, \ldots)$ with Re $\lambda_i = \alpha(X), i = 1, \ldots, k$ (the active eigenvalues),

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If $k = 1$, we have

$$\partial\alpha(X) = \{u_1 v_1^*\} = \{\nabla\alpha(X)\}.$$

Optimality Conditions for a Matrix Family

Let $A : \mathbf{C}^m \to \mathbf{C}^{n\times n}$ be a smooth map satisfying the regularity condition

$$\mathcal{N}\big((DA(\bar{x}))^*\big) \cap \partial^\infty \alpha\big(A(\bar{x})\big) = 0,$$

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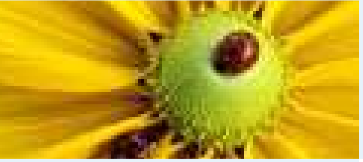

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Furthermore, if the last condition holds, small perturbations to the data make only small perturbations in the minimizer.

A Simple Example

Define the affine map $A : \mathbf{C}^{n-1} \to \mathbf{C}^{n\times n}$ by

$$A(x) = \begin{pmatrix} -x_1 & 1 & 0 & \dots & 0 \\ x_1 & 0 & 1 & \dots & 0 \\ x_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ x_{n-1} & 0 & 0 & \dots & 0 \end{pmatrix}.$$

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So, by Theorem SSA and the chain rule,

$$\hat{\partial}(\alpha \circ A)(0) = \partial(\alpha \circ A)(0) = (DA)^*\partial\alpha(A(0))$$

$$= \left\{(\theta_2 - 1/n, \theta_3, \theta_4, \dots, \theta_n)^T : \text{ Re } \theta_2 \geq 0\right\},$$

and since the interior of this set contains 0, we see $\bar{x} = 0$ is a sharp local minimizer of the spectral abscissa of $A(x)$.

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- quadrilateral mesh case: numerically reduced moduli of eigenvalues to about 10^{-4} and determined that the derogatory zero eigenvalue has Jordan block sizes 5, 3, 2, 2.

Active Eigenvalue Structures at Optimizers

This example: one active eigenvalue, nonderogatory, with multiplicity n.

Markov Chain example: $n-2$ active eigenvalues, all nonderogatory, with multiplicities $2, 1, 1, \ldots, 1$: same kind of analysis applies.

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Any structure is possible, but expect nonderogatory eigenvalues generically.

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Numerical Optimization Methods

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BFGS quasi-Newton method with line search: very effective in practice.

In both cases, make no attempt to *predict active eigenvalues and their multiplicities*: simply use gradients which exist a.e.:

$$\nabla(\alpha \circ A)(x) = \big(DA(x)\big)^* \nabla\alpha\big(A(x)\big) = \big(DA(x)\big)^* v_1 u_1^*$$

so

$$\big(\nabla(\alpha \circ A)(x)\big)_k = \langle \frac{\partial A}{\partial x_k}(x), v_1 u_1^* \rangle = u_1^* \frac{\partial A}{\partial x_k}(x) v_1$$

where u_1 and v_1 are left and right eigenvectors for the dominant eigenvalue λ_1.

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Optimizing Matrix Stability
J.V. Burke, A.S. Lewis and M.L. Overton
Proceedings of the American Mathematical Society (2001)

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The subdifferential we use is due to Mordukhovich. It coincides with the Clarke generalized gradient in the subdifferentially regular case.

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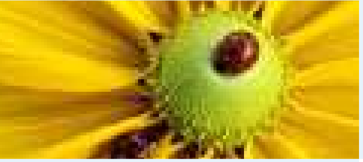

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Eigenvalue Optimization in C^2 Subdivision and Boundary Subdivision
Sara Grundel, Ph.D. thesis, NYU, 2011.

Part III
Optimization of Pseudospectra
with
M. Gürbüzbalaban (NYU)
A.S. Lewis (Cornell)

Pseudospectra

The area swept out in the complex plane by the eigenvalues under perturbation.

$$\Lambda_\epsilon(A) = \{z \in \mathbf{C} \ : \ \det(A+E-zI) = 0 \text{ for some } E \text{ with } \|E\| \le \epsilon\}$$

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Motivation: a more robust measure of system behaviour than eigenvalues.

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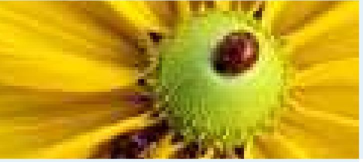
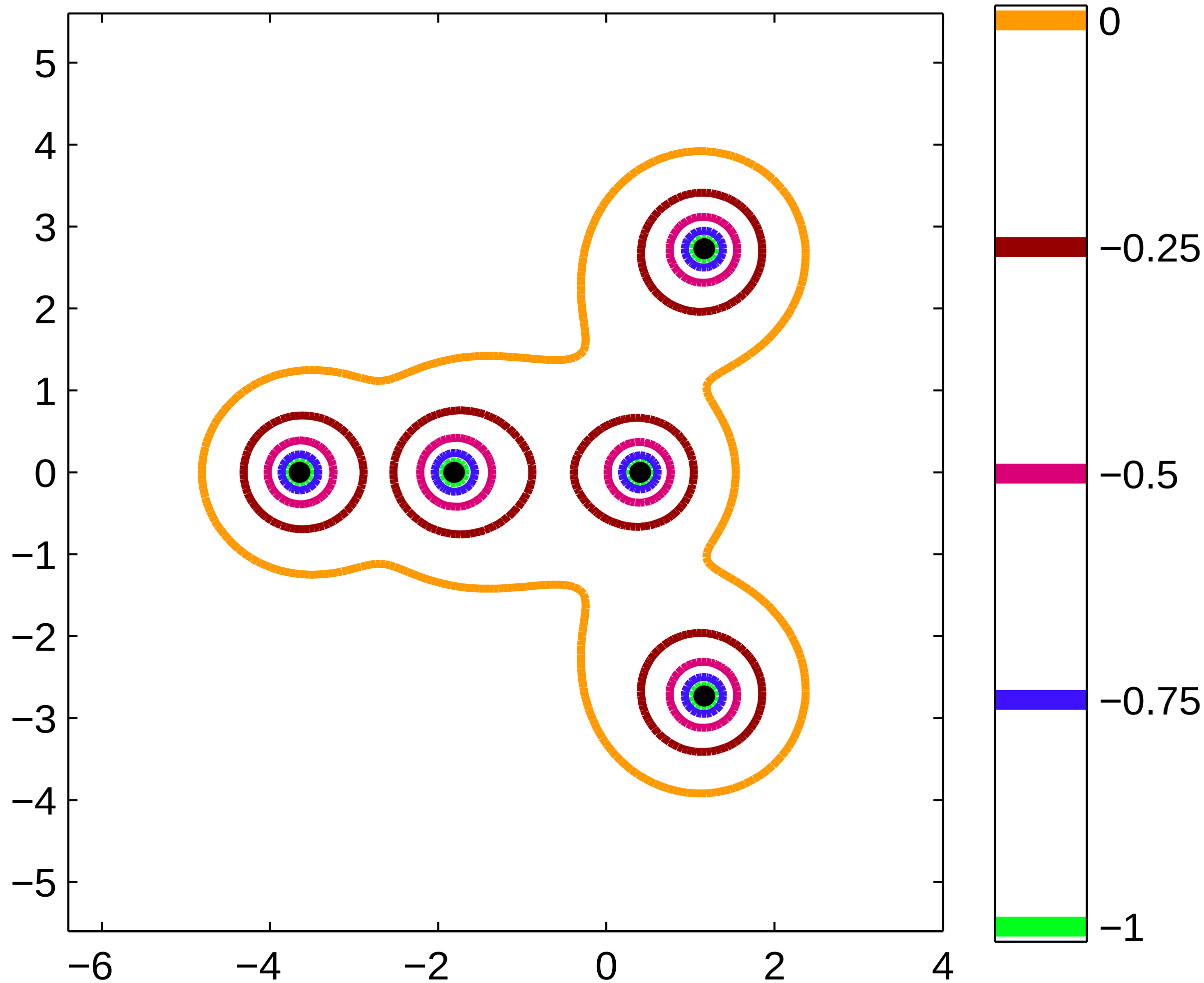
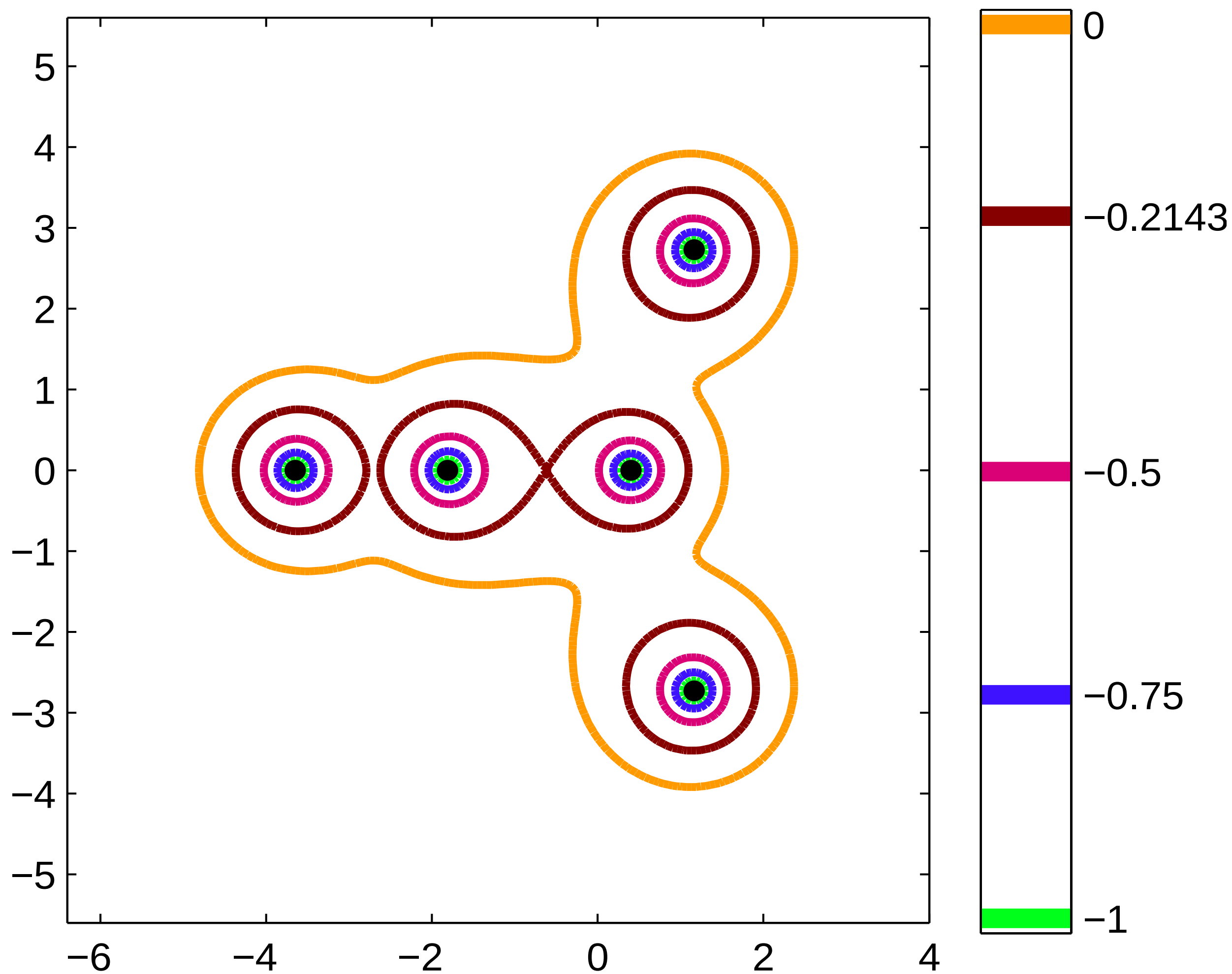
Let $f(x, y) = \sigma_n(A - (x + iy)I)$. Then pseudospectra are lower level sets of f.



A Randomly Generated Example, $n = 5$



The black dots are the eigenvalues and the colored curves are the boundaries of the ϵ-pseudospectra. The scale on the right is log 10.



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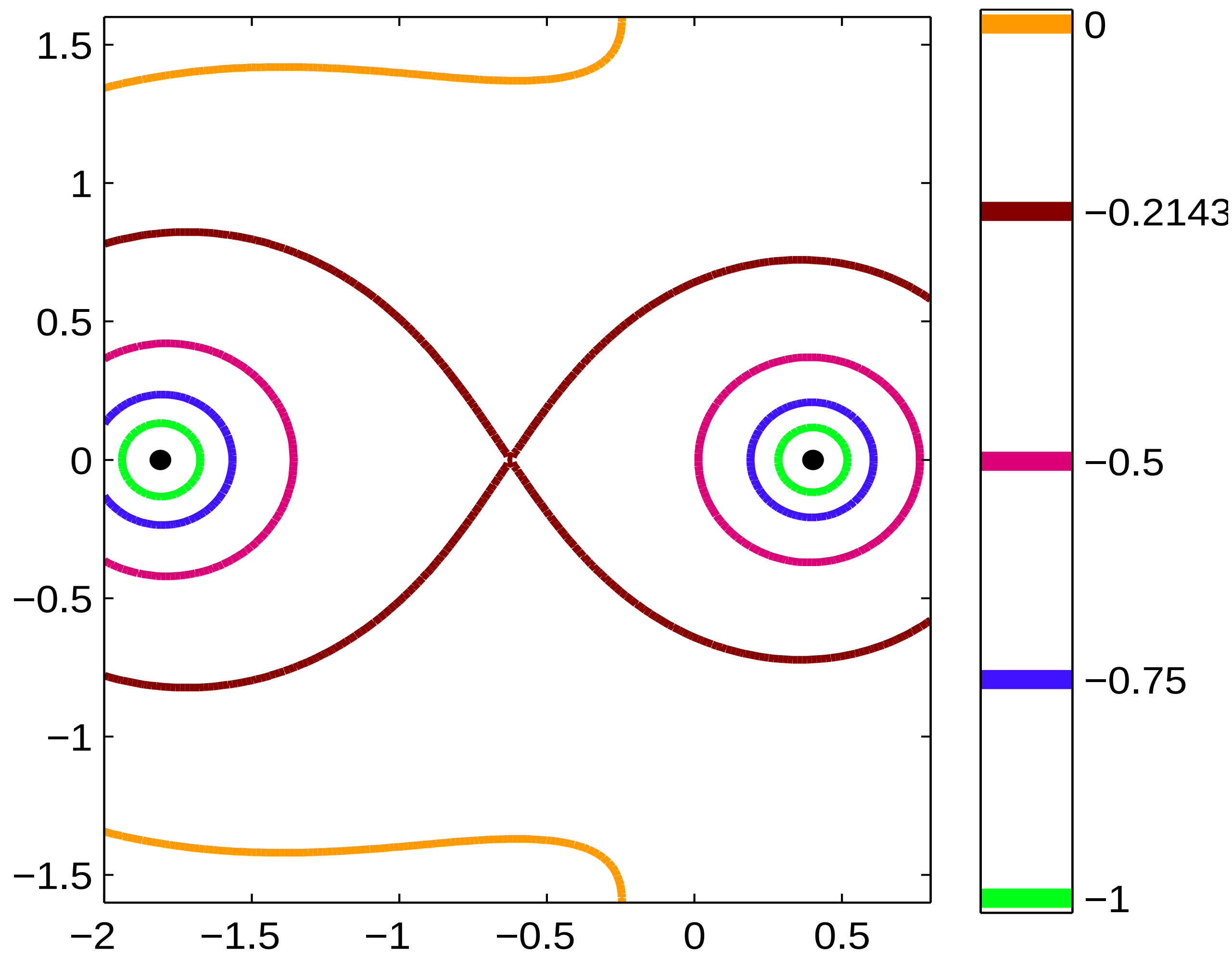
Same, Close-up



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The set-valued map $\Lambda_\epsilon(A)$ is not Lipschitz continuous w.r.t. either A or ϵ near critical values of ϵ.

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What about pseudospectral max functions?

Pseudospectral abscissa:

$$\alpha_\epsilon(A) = \max\{\text{Re } z \ : \ z \in \Lambda_\epsilon(A)\}$$

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Pseudospectral radius:

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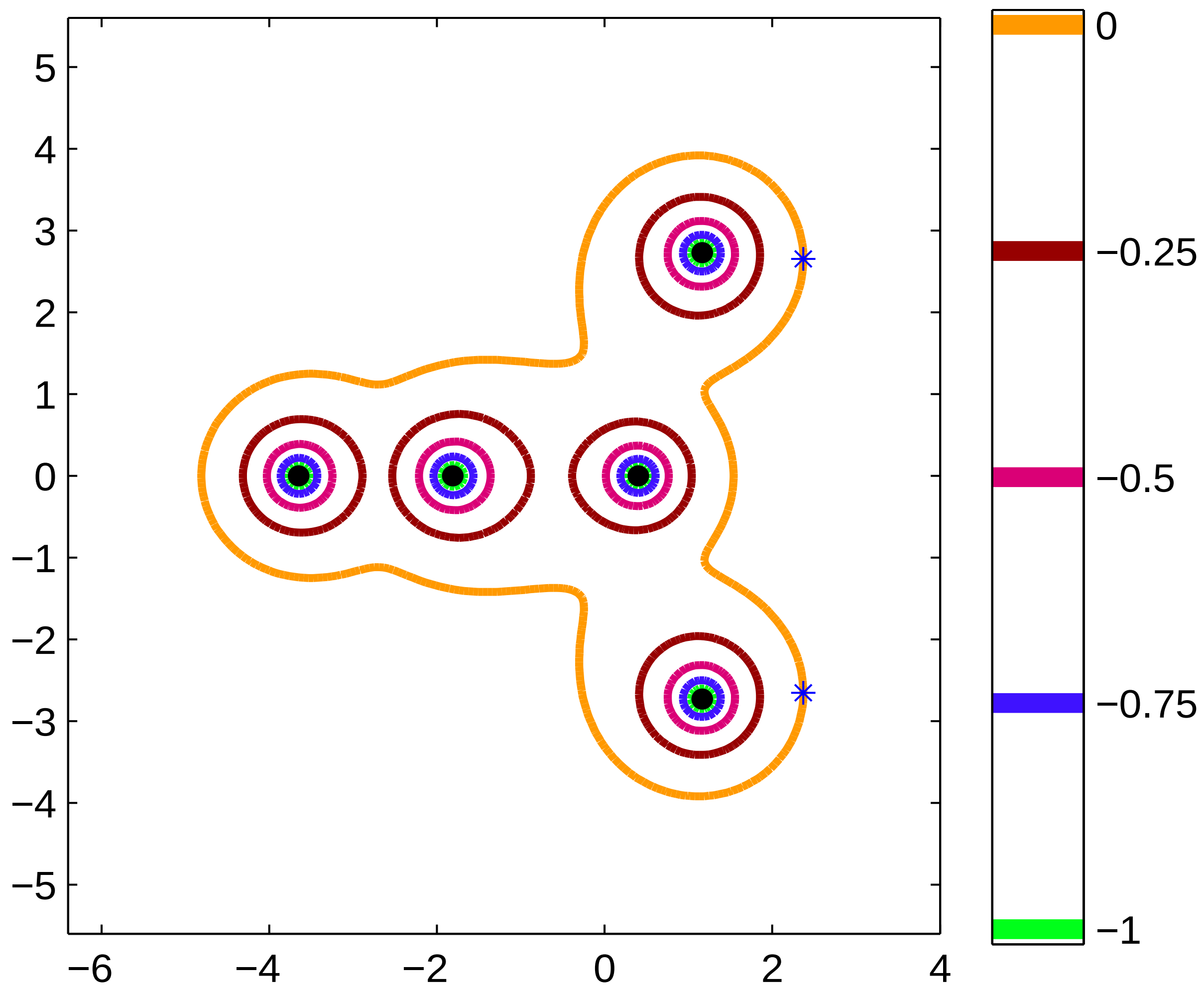
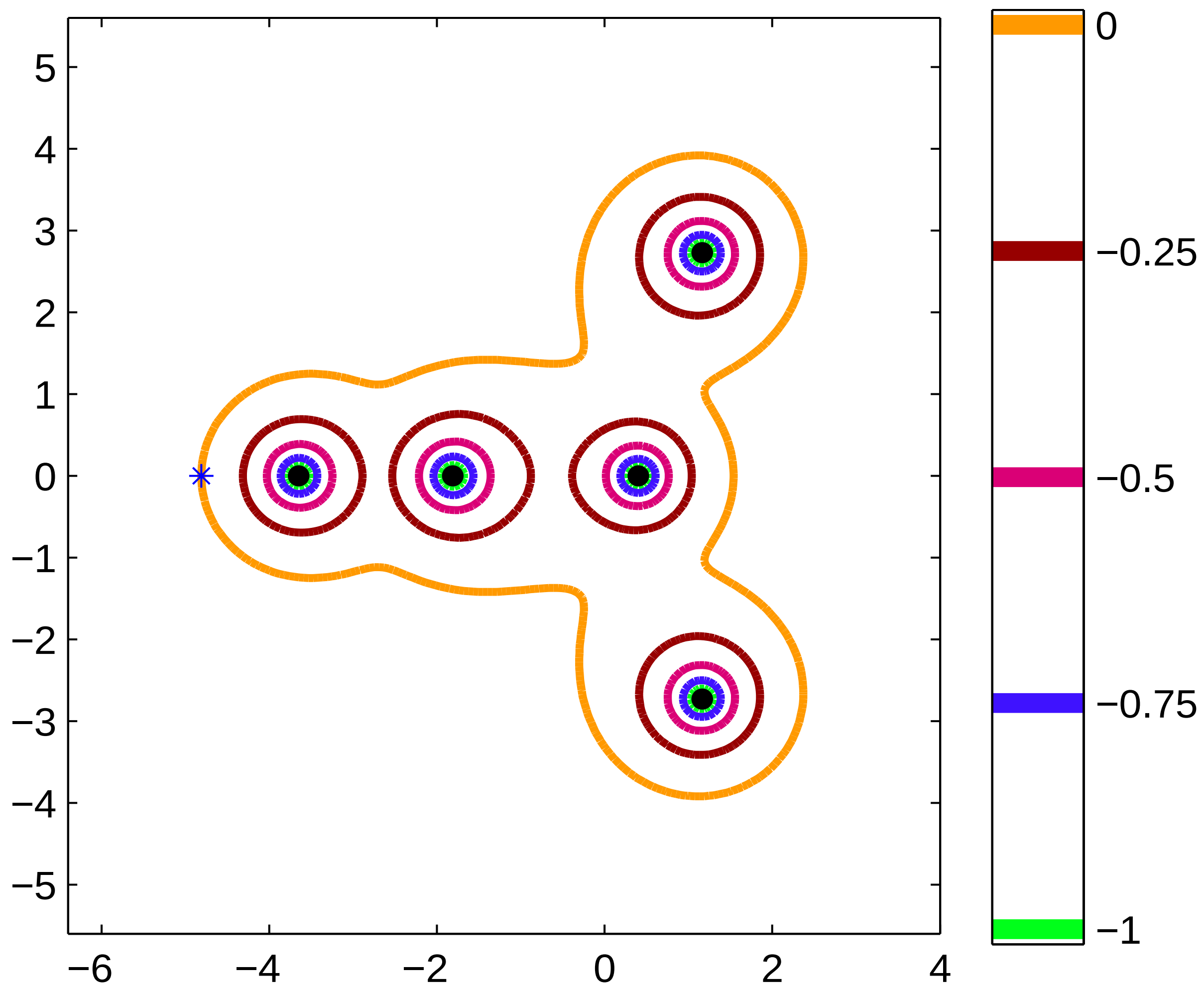
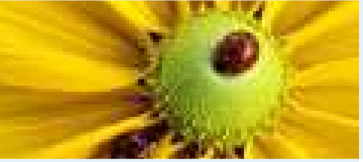
Same Example: Pseudospectral Abscissa, $\epsilon = 10^0$



Same Example: Pseudospectral Radius, $\epsilon = 10^0$





Pseudospectral Max Functions are Locally Lipschitz

Theorem PMFLL. *The pseudospectral abscissa and radius functions α_ϵ and ρ_ϵ are locally Lipschitz at all matrices A and at all $\epsilon > 0$.*

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Computing α_ϵ and ρ_ϵ: requires solving nonconvex global optimization problem in two real variables. Specialized algorithms are based on computing eigenvalues of Hamiltonian matrices and checking whether any are imaginary.

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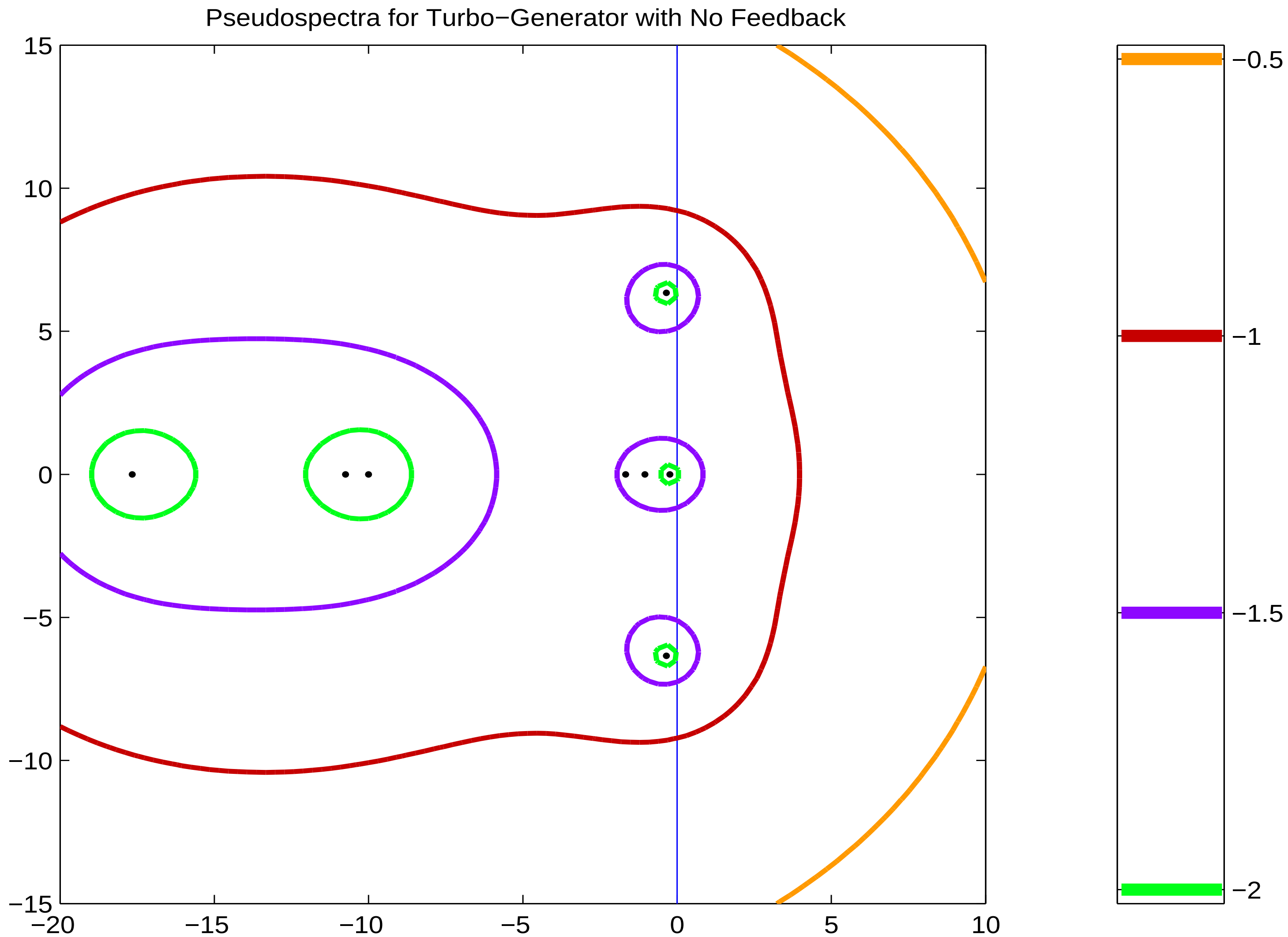
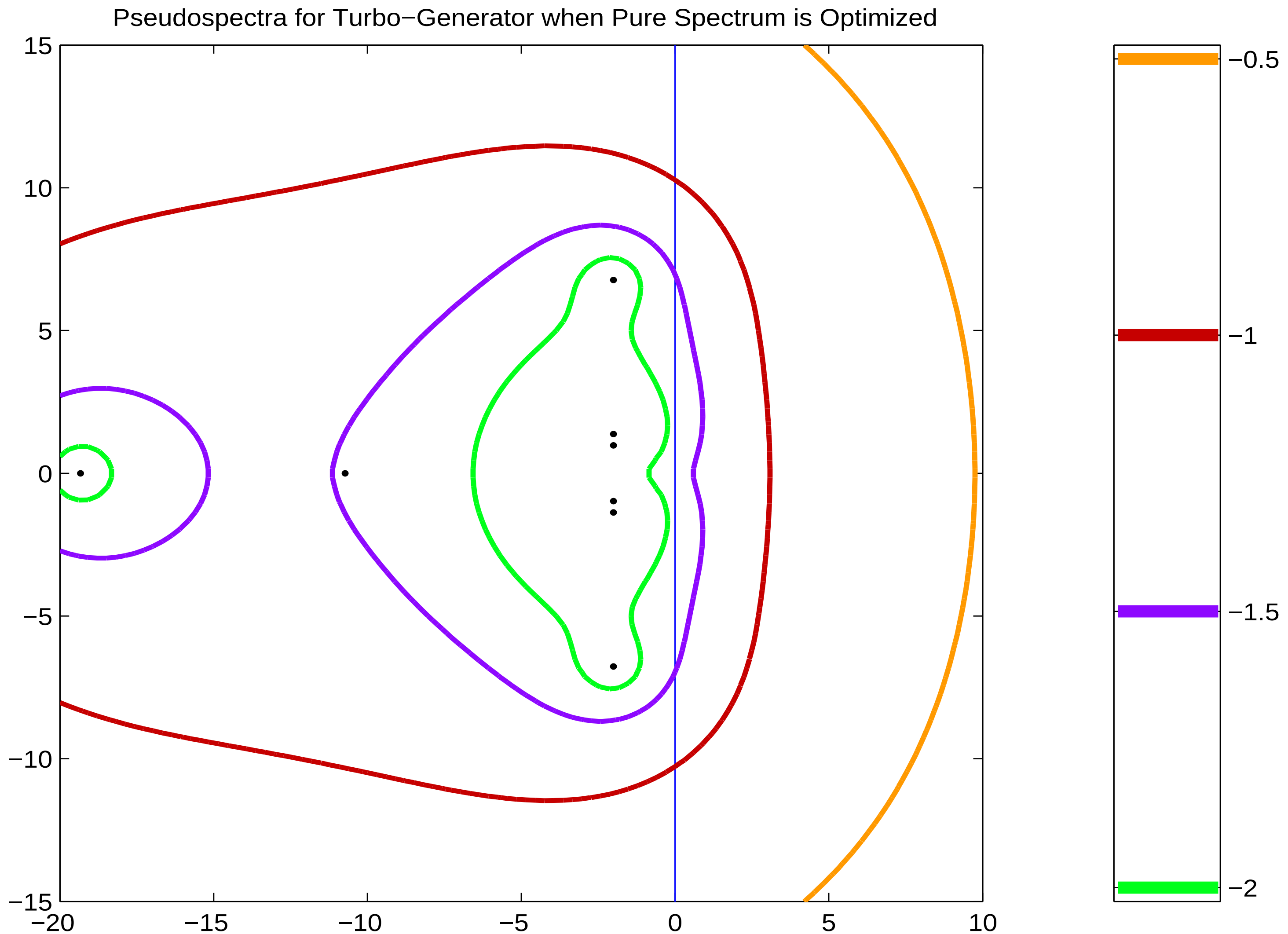
Numerical optimization using gradients: now left and right singular vectors appear instead of left and right eigenvectors.

A Turbo Generator Control Problem



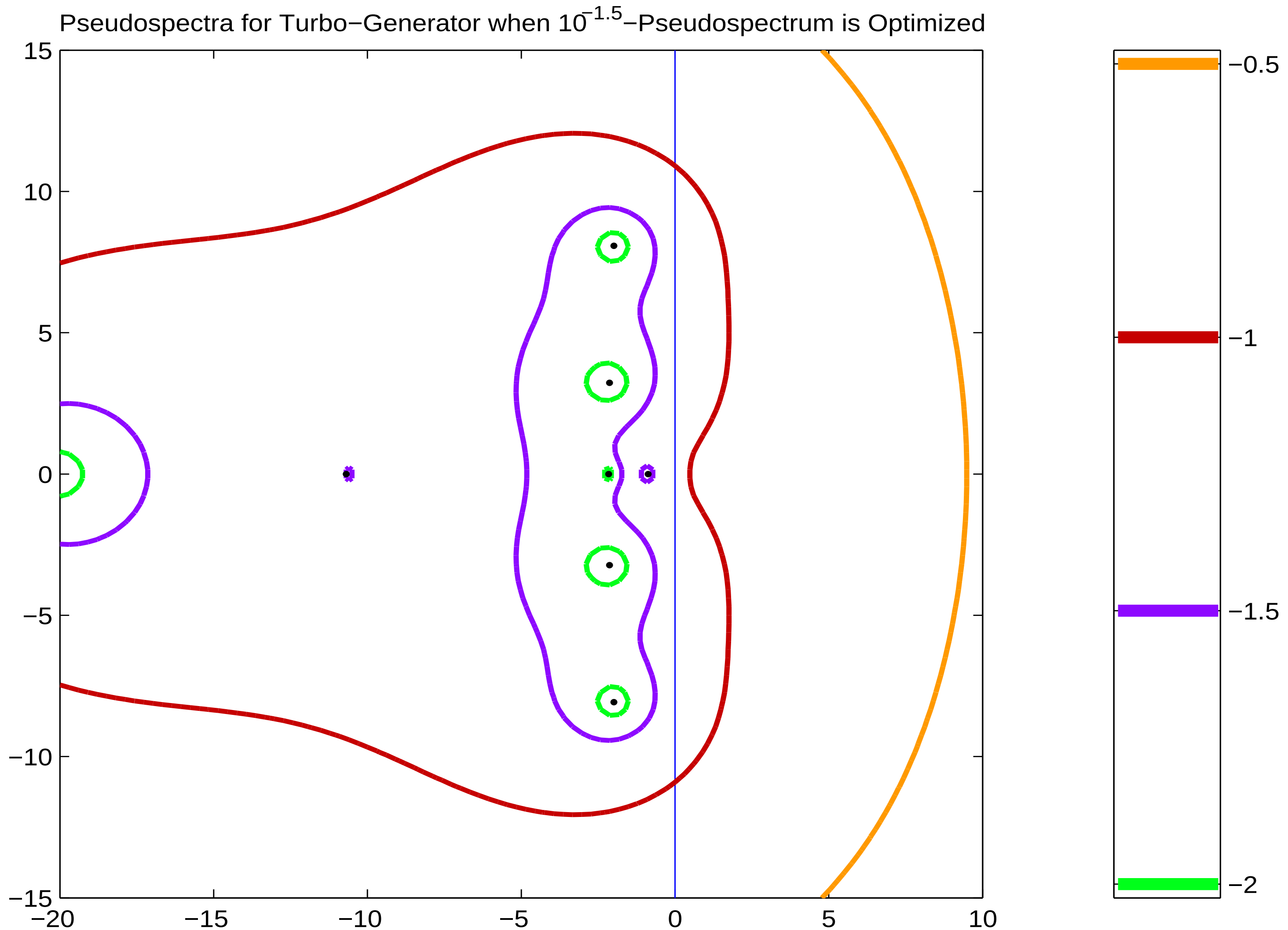
Pseudospectra for open-loop turbo generator plant with no feedback.

Turbo Generator with Optimized Eigenvalues



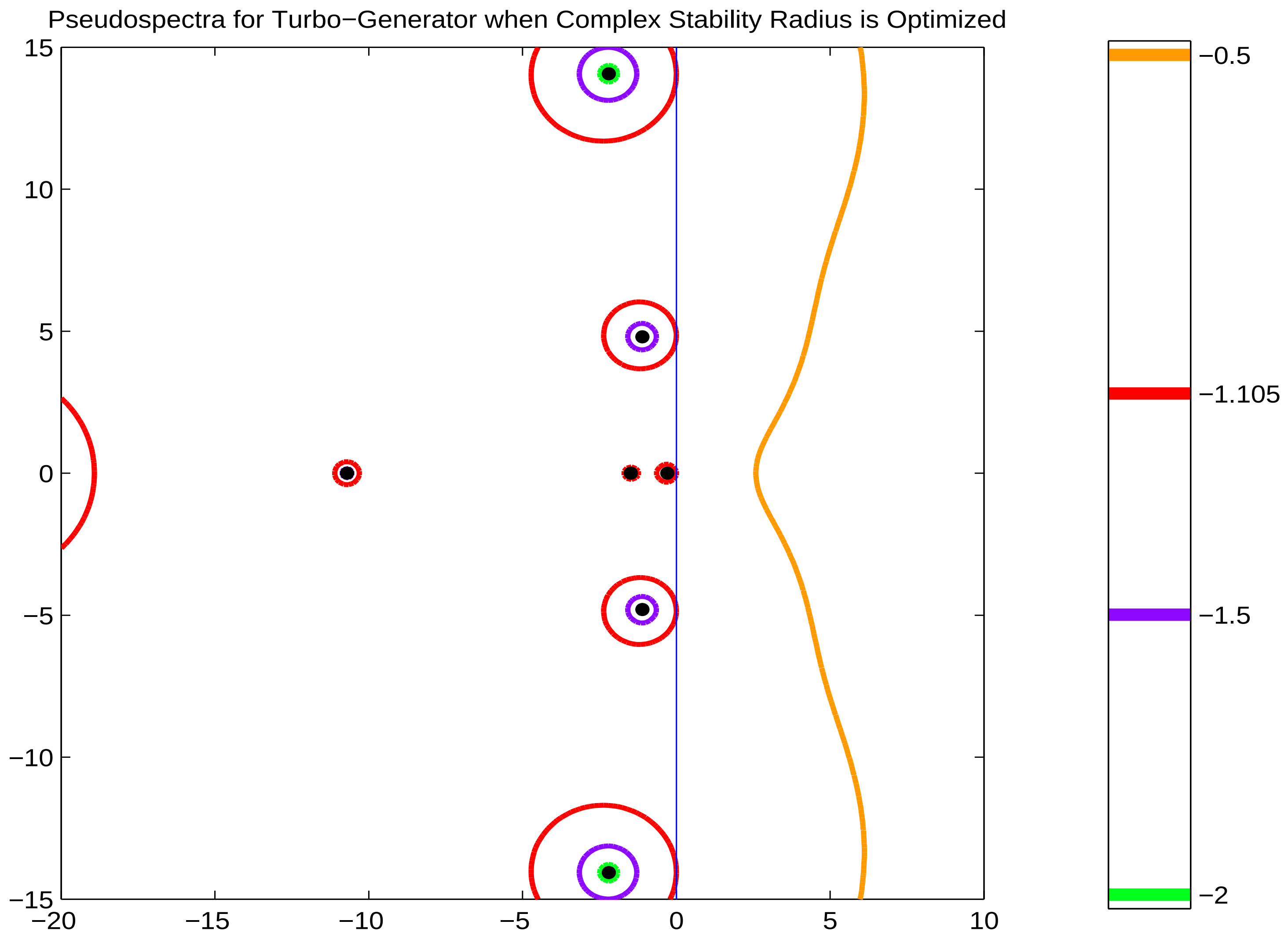
Pseudospectra for turbo generator plant with feedback computed by minimizing the spectral abscissa

Turbo Generator with Optimized ϵ-Pseudospectrum



Pseudospectra for turbo generator plant with feedback computed by minimizing α_ϵ with $\epsilon = 10^{-1.5}$

Turbo Generator with Optimized Dist. to Instability



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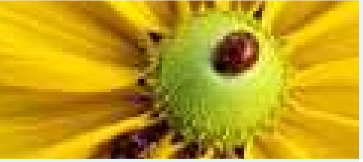

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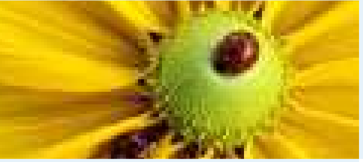

Thanks a lot for your attention!