# Quadrature Problems for Stochastic Differential Equations

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## OUTLINE

I. Computational Problems for SDEs

II. Deterministic Quadrature on the Lipschitz Class

III. Randomized Quadrature on the Lipschitz Class

IV. Quadrature on the Sequence Space

Joint work with

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*F. Hickernell* (IIT Chicago), *S. Mayer* (Darmstadt),

*T. Müller-Gronbach* (Passau), *Ben Niu* (IIT Chicago),

*S. Toussaint* (Darmstadt), *L. Yaroslavtseva* (Passau).

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# I. Computational Problems for SDEs

**SDE**

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\begin{aligned}
dX_t &= \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \qquad t \in [0,T], \\
X_0 &= x_0
\end{aligned}
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with a Brownian motion $W$. Solution $X = (X_t)_{t\in[0,T]}$ is a stochastic process with continuous paths,

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X : \Omega \to C([0,T]) =: \mathfrak{X}
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1. *Strong approximation*: approximate the solution $X$.
2. *Weak approximation*: approximate the distribution $P_X$ of $X$ on $\mathfrak{X}$.
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Analogously, for the solution $X_T$ at time $T$.

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Reasonable, but not mandatory, $1. \rightsquigarrow 2. \rightsquigarrow 3.$

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$$I(f) = \int_{\mathfrak{X}} f \, dP_X.$$

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$$e^{\mathrm{ran}}(n, F) = \inf_{Q_n} e(Q_n, F).$$

For quadrature and weak approximation of SDEs

1. distribution $P_X$ on $\mathfrak{X}$ given only implicitly,
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**Assumptions**

- scalar SDE (for simplicity),
- $\mu, \sigma \in C_{\mathrm{b}}^2(\mathbb{R})$ (smoothness crucial),
- $\sigma(x_0) \neq 0$ (to exclude deterministic equations).

# II. Deterministic Quadrature on the Lipschitz Class

Here $F = \mathrm{Lip}(1)$, i.e., for $\mathfrak{X} = C([0,T])$ and $f \in F$

$$|f(x) - f(y)| \le \|x - y\|_\infty, \qquad x, y \in \mathfrak{X}.$$

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**Example**: $f$ payoff of an asian or lookback option,

$$f(x) = \max\left(\tfrac{1}{T}\int_0^T x(t)\,dt - K, 0\right),$$

$$f(x) = \max\left(\sup_{t\in[0,T]} x(t) - K, 0\right).$$

## Equivalence of quadrature on $\mathrm{Lip}(1)$ and quantization of $P_X$

$$e^{\det}(n, \mathrm{Lip}(1)) = \inf_{x_1,\ldots,x_n \in \mathfrak{X}} \mathrm{E}(g(X; x_1, \ldots, x_n))$$

for every separable Banach space $\mathfrak{X}$, where

$$g(x; x_1, \ldots, x_n) = \min_{i=1,\ldots,n} \|x - x_i\|.$$

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**Proof of '$\geq$'**

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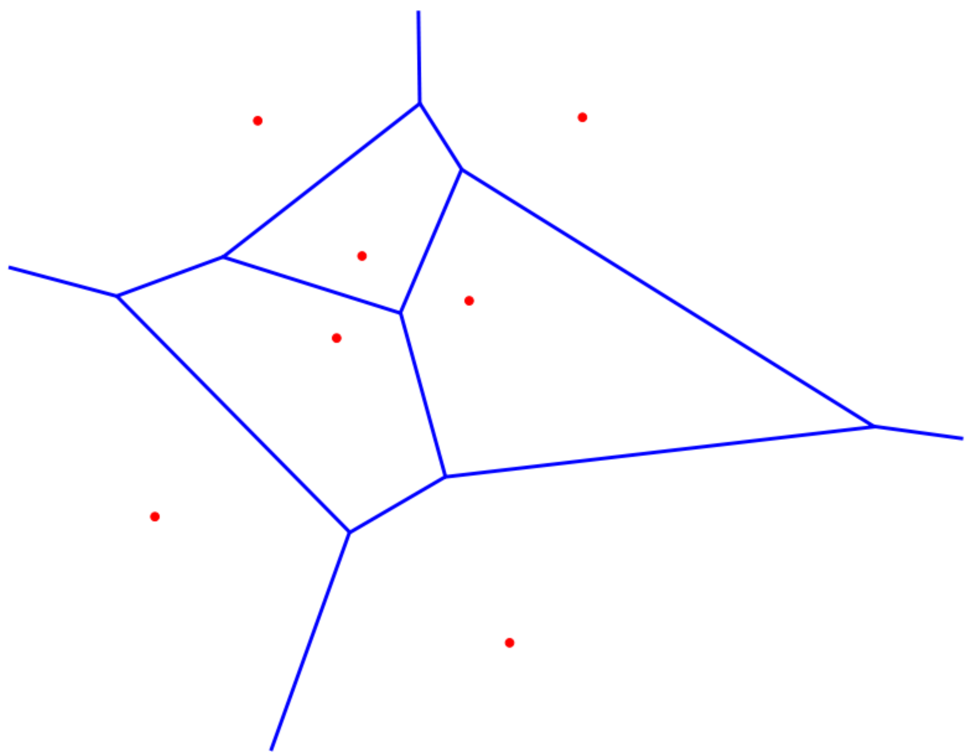

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Quantization for stochastic processes studied since 2000,

*Aurzada, Creutzig, Dereich, Fehringer, Graf, Luschgy, Müller-Gronbach, Matoussi, Pagès, Printems, R, Scheutzow, Wilberts, . . .*

In particular, Gaussian processes, Lévy processes, SDEs.

For applications in finance see Pagès, Printems (2008) as well as
`http://www.quantise.maths-fi.com/`

**Theorem** *Dereich* (2008)

$\exists\, c > 0\ \forall \mu, \sigma, x_0$

$$e^{\det}(n, \mathrm{Lip}(1)) \approx c \cdot \mathrm{E} \left( \int_0^T \sigma^2(X_t)\, dt \right)^{1/2} \cdot (\ln n)^{-1/2}.$$

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Formally,

- input: $(\mu, \sigma, x_0)$ as well as $n \in \mathbb{N}$,
- real number model with oracle for function/derivative values of $\mu$ and $\sigma$,
- output: coefficients $a_i \in \mathbb{R}$ and nodes $x_i \in \mathfrak{X}$ (suitably coded).

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**Theorem** *Müller-Gronbach, R* (2011)

$\forall \mu, \sigma, x_0$ construction of $Q_n$ at cost $O(n)$ such that

$$e(Q_n, \mathrm{Lip}(1)) \approx e^{\det}(n, \mathrm{Lip}(1)).$$

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**Remark**

- Analogous results for $\mathfrak{X} = L_p([0,T])$ instead of $C([0,T])$.
- Systems of SDEs require quantization of Lévy areas.
- Distance on the space of probability measures: Wasserstein metric.

**The construction**, illustrated for the square-root diffusion

$$dX_t = \alpha(\kappa - X_t)\,dt + \beta\sqrt{X_t}\,dW_t.$$

Implementation for $L_2([0,T])$ due to Toussaint (2008).

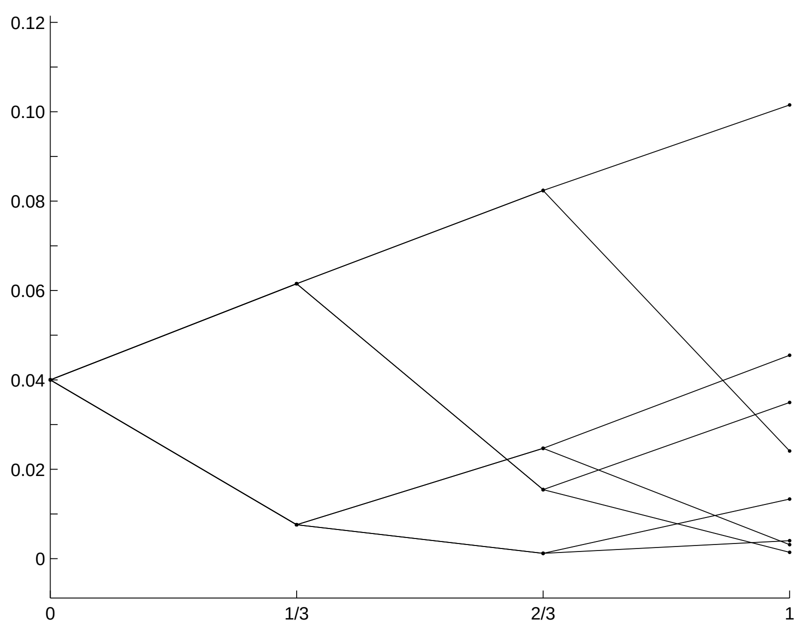
**The construction**, illustrated for the square-root diffusion

$$dX_t = \alpha(\kappa - X_t)\,dt + \beta\sqrt{X_t}\,dW_t.$$

1. For an equidistant time discretization, quantization of the marginal distribution

$$P_{(X_0, X_{T/m}, \dots, X_T)}$$

via quantized version of the Milstein scheme.



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2. Local refinement via quantization of Brownian bridges, taking into account the local smoothness of $X$.

   Basis functions

$$e_k(t) = \sqrt{2T} \cdot \sin(k\pi \cdot t/T)$$

   for $k \in \mathbb{N}$.

   Cf. adaptive step-size control for strong approximation, see Hofmann, Müller-Gronbach, R (2002), Müller-Gronbach (2002).

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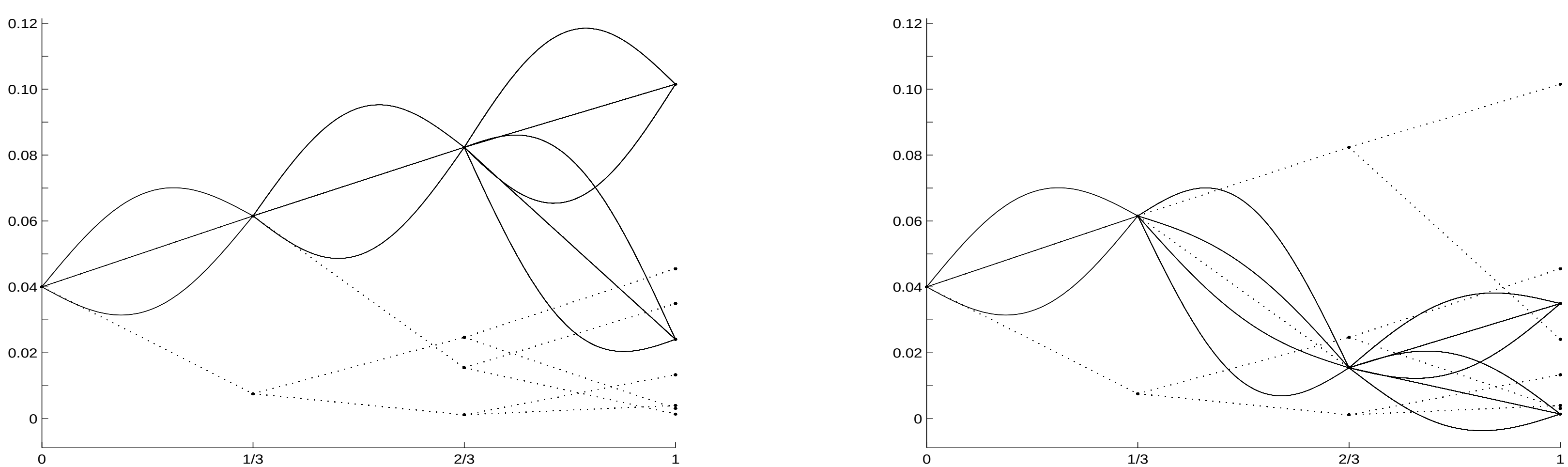

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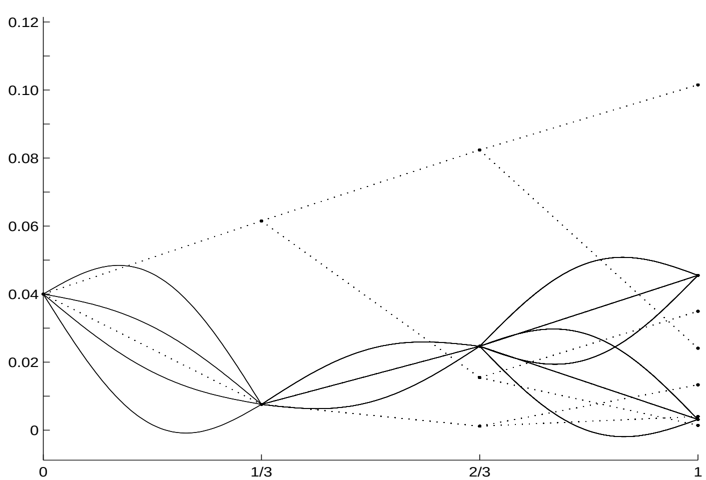


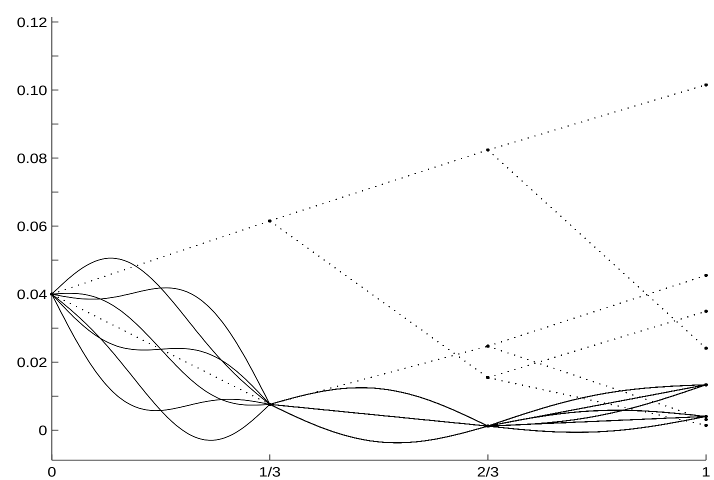

**Remark** Alternative approaches to constructive quantization of SDEs

- ODE-based, using rough paths theory,

  see *Luschgy, Pagès* (2006), *Pagès, Sellami* (2010),

- using series expansions for $X$, see *Luschgy, Pagès* (2008).

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**Remark** Different techniques and results in the marginal case $P_{X_T}$, where

$$Q_n(f) = \sum_{i=1}^{n} a_i \cdot f(x_i)$$

with $a_i \in \mathbb{R}$ and $x_i \in \mathbb{R}^d$.

See *Kusuoka* (2001, 2004), *Lyons, Victoir* (2004), *Crisan, Ghazali* (2007), *Litterer, Lyons* (2010), *Müller-Gronbach, R, Yaroslavtseva* (2011), ...

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**In 'Stochastic Computation' (B7)**

S. Dereich: Constructive Quantization: approximation by empirical measures

L. Yaroslavtseva: A derandomization of the Euler scheme

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**Question**: Evaluation of integrands $f \in F$ anywhere in $\mathfrak{X}$ at cost one?

# III. Randomized Quadrature on the Lipschitz Class

Randomized quadrature formula $Q_n(f) = \sum_{i=1}^n a_i \cdot f(X_i)$.

**Variable subspace sampling**: for any scale of finite-dim. subspaces

$\mathfrak{X}_0 \subset \mathfrak{X}_1 \subset \ldots \subset \mathfrak{X}$

$$X_1(\omega), \ldots, X_n(\omega) \in \bigcup_{m=0}^{\infty} \mathfrak{X}_m,$$

$$\operatorname{cost}(Q_n) = \mathrm{E}\left(\sum_{i=1}^{n} \inf\{\dim \mathfrak{X}_m : X^{(i)} \in \mathfrak{X}_m\}\right).$$

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**Example**

$$\mathfrak{X}_m = \{x \in \mathfrak{X} \mid x \text{ piecewise linear with breakpoints } \ell/2^m \cdot T\}.$$

Classical Euler-MC algorithm vs. multi-level Euler-MC algorithm.

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$N$**-th minimal error**, redefined,

$$e^{\mathrm{ran}}(N, F) = \inf_{\operatorname{cost}(Q) \leq N} e(Q, F).$$

**Theorem** *Creutzig, Dereich, Müller-Gronbach, R* (2009)

$$N^{-1/2} \preceq e^{\mathrm{ran}}(N, \mathrm{Lip}(1)) \preceq N^{-1/2} \cdot \ln N.$$

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**Remark**

- Deterministic algorithms only yield $(\ln N)^{-1/2}$.
- Upper bound via multi-level Euler-MC algorithm.
- Fixed subspace sampling only yields $N^{-1/4}$, up to $\ln$'s.
- Lower bound valid for the class of randomized algorithms.

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**Remark**

- General result for
  - every probability measure $P$ on any separable Banach space $\mathfrak{X}$ and
  - $F = \mathrm{Lip}(1)$.

  Upper and lower bounds for $e^{\mathrm{ran}}(N, F)$ in terms of
  - quantization numbers of $P$,
  - Kolmogorov widths of $P$, see *Mathé* (1990), ....

# IV. Quadrature on the Sequence Space

**Motivation**: As previously, $\mathfrak{X} = C([0,T])$,

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \qquad t \in [0,T],$$
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and $f : \mathfrak{X} \to \mathbb{R}$. Approximate $I(f) = \int_{\mathfrak{X}} f\,dP_X$.

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$X = \Gamma(\xi_1, \xi_2, \dots) =$ **Euler expansion** of $X$ with step-sizes $1/2^k$,

based on **Lévy-Ciesielski decomposition** of $W$,

with $\xi_1, \xi_2, \dots$ iid and $P_{\xi_1} = N(0,1) =: \rho$.

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Truncation

$$\int_{\mathbb{R}^d} f \circ \Gamma(z_1, \dots, z_d, 0, \dots)\,d\rho^{\otimes d}(z_1, \dots, z_d).$$

**The general formulation**: Given

- a probability measure $\rho$ on $D \subseteq \mathbb{R}$ and
- a class $G$ of functions $g : \mathfrak{Z} \to \mathbb{R}$ on $\mathfrak{Z} = D^{\mathbb{N}}$.

Compute

$$I(g) = \int_{\mathfrak{Z}} g \, d\rho^{\otimes \mathbb{N}}.$$

**The general formulation**: Given

- a probability measure $\rho$ on $D \subseteq \mathbb{R}$ and
- a class $G$ of functions $g : \mathfrak{Z} \to \mathbb{R}$ on $\mathfrak{Z} = D^{\mathbb{N}}$.

Compute

$$I(g) = \int_{\mathfrak{Z}} g \, d\rho^{\otimes \mathbb{N}}.$$

We study

- quadrature formulas

$$Q_n(g) = \sum_{i=1}^{n} a_i \cdot g(\mathbf{z}_i), \qquad a_i \in \mathbb{R}, \mathbf{z}_i \in \mathfrak{Z},$$

  and variable subspace sampling,

- unit balls $G$ in Hilbert spaces with a reproducing kernel.

**Variable subspace sampling**, based on

$$\mathfrak{Z}_{d,y} = \{\mathbf{z} \in \mathfrak{Z} \mid z_{d+1} = z_{d+2} = \cdots = y\}$$

for any $y \in D$. Thus

$$\mathbf{z}_1, \ldots, \mathbf{z}_n \in \bigcup_{d=1}^{\infty} \mathfrak{Z}_{d,y}$$

and

$$\mathrm{cost}(Q_n) = \sum_{i=1}^{n} \inf\{d \mid \mathbf{z}_i \in \mathfrak{Z}_{d,y}\}.$$

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**$N$-th minimal error**

$$e^{\det}(N, G) = \inf_{\operatorname{cost}(Q) \leq N} e(Q, G).$$

**Weighted tensor product spaces**, a specific case: $D = [0, 1]$ and

$$K_{\boldsymbol{\gamma}}(\mathbf{y}, \mathbf{z}) = \prod_{j=1}^{\infty} (1 + \gamma_j \cdot \min(y_j, z_j)), \qquad \mathbf{y}, \mathbf{z} \in \mathfrak{Z},$$

for weights $\gamma_1 \geq \gamma_2 \geq \cdots > 0$ such that

$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$

See *Hickernell, Wang* (2002).

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Counterpart for functions $g : D^d \to \mathbb{R}$ of finitely many variables

- if $\gamma_1 = \cdots = \gamma_d = 1$ then $H(K_{\boldsymbol{\gamma}}) = W_2^{(1,\ldots,1)}([0,1]^d)$,
- for the weighted case see *Sloan, Woźniakowski* (1998), **. . .** ,
  *Novak, Woźniakowski* (2008, 2010, . . . ).

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**Example** For

$$g(\mathbf{z}) = \sum_{j=1}^{\infty} \eta_j \cdot z_j^2$$

we have

$$g \in H(K_{\boldsymbol{\gamma}}) \quad \Leftrightarrow \quad \sum_{j=1}^{\infty} \frac{\eta_j^2}{\gamma_j} < \infty.$$

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**Class of integrands**

$$G_{\boldsymbol{\gamma}} = \{g \in H(K_{\boldsymbol{\gamma}}) \mid \|g\|_{\boldsymbol{\gamma}} \leq 1\}.$$

**Theorem** *Kuo, Sloan, Wasilkowski, Woźniakowski* (2010)

*Hickernell, Müller-Gronbach, Niu, R* (2011), *Gnewuch* (2011),

*Plaskota, Wasilkowski* (2011)

Assume that $\rho$ is the uniform distribution on $D = [0, 1]$ and

$$\gamma_j \asymp j^{-(1+2q)}$$

with $q > 0$.

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with $q > 0$. Then, for every $\varepsilon > 0$,

$$N^{-\min(q,1)} \preceq e^{\det}(N, G_{\boldsymbol{\gamma}}) \preceq N^{-\min(q,1)+\varepsilon}$$

if $|q - 1| \geq 1/2$ and

$$N^{-\min(q,1)} \preceq e^{\det}(N, G_{\boldsymbol{\gamma}}) \preceq N^{-\frac{q+1/2}{2}+\varepsilon}$$

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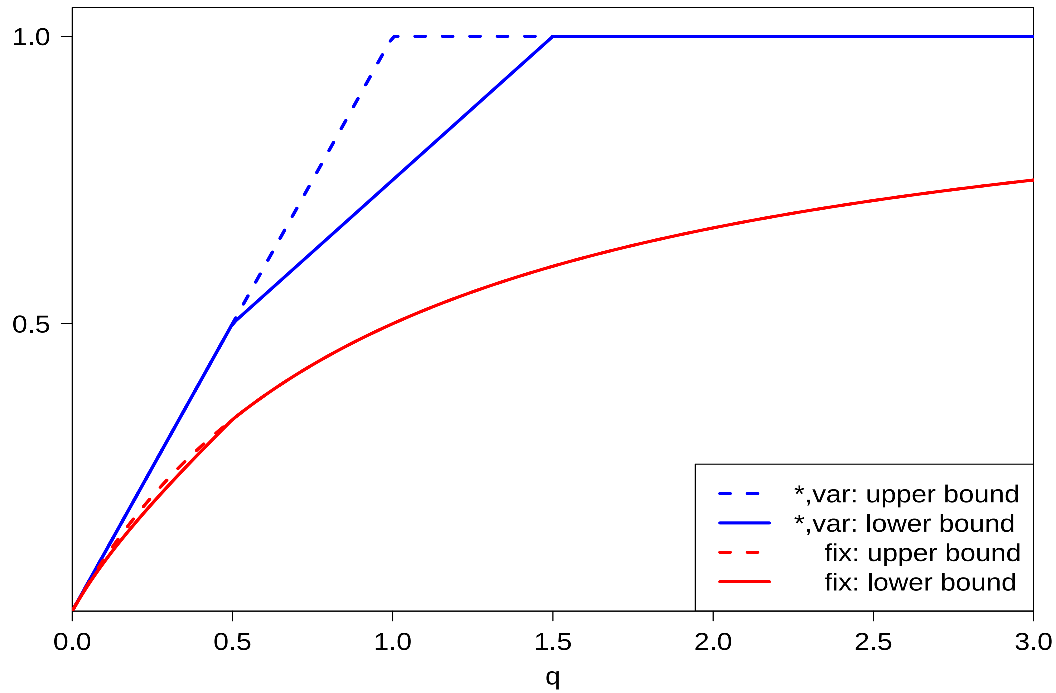
**Remark** General results for product measures and weighted tensor product spaces.

**Order of convergence** of minimal errors in terms of decay of weights



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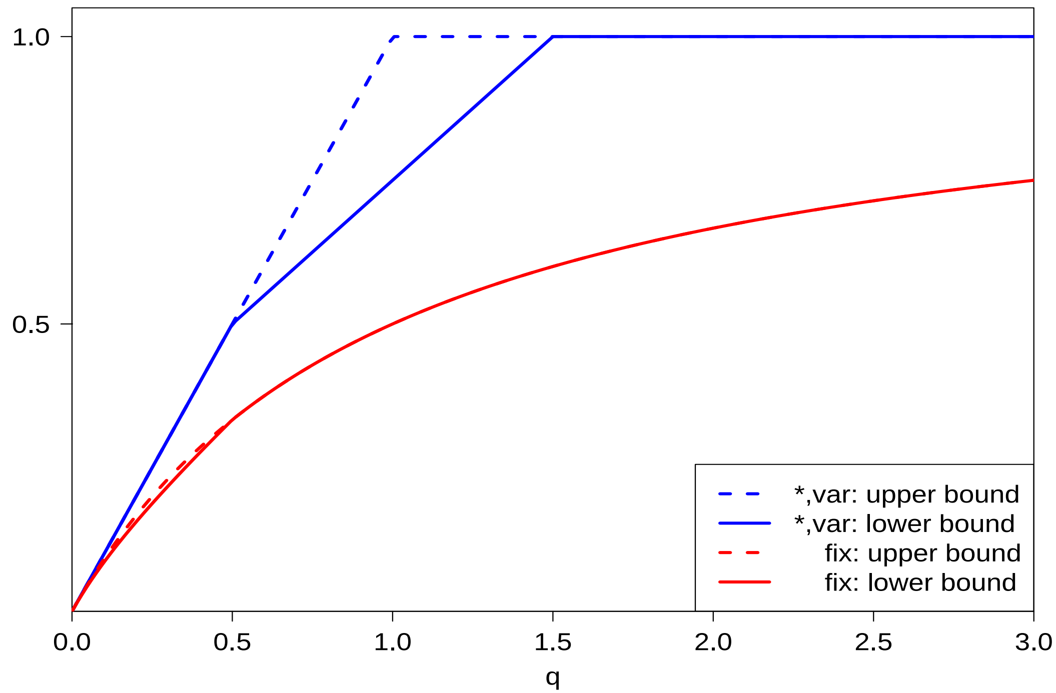
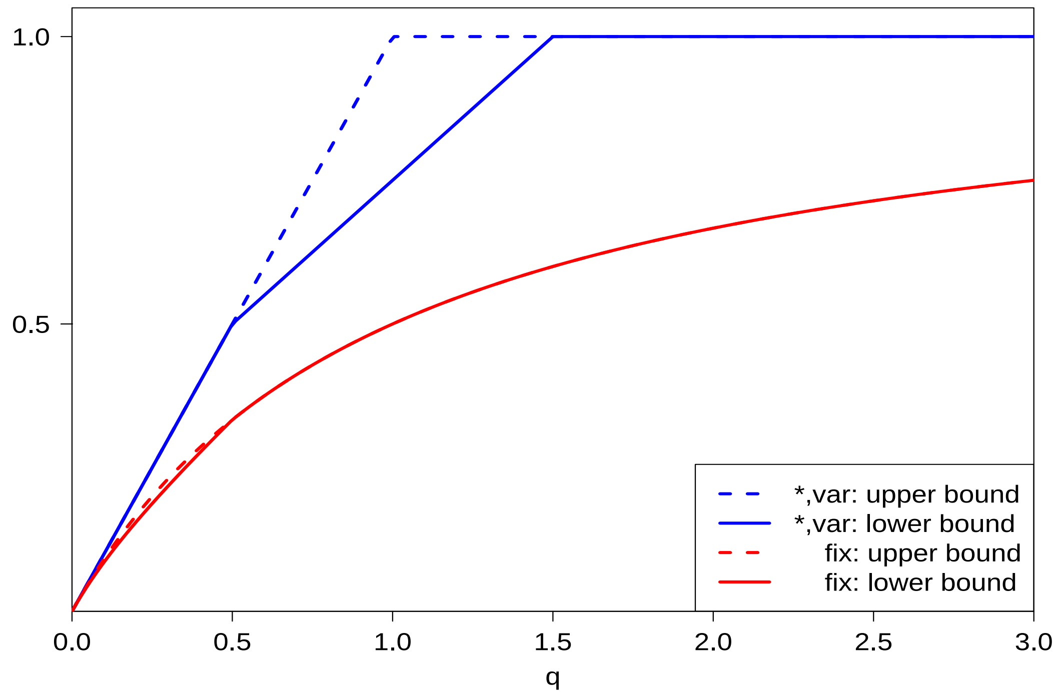



**Remark**

- Variable subspace sampling superior to fixed subspace sampling.
- $W_2^1([0,1]) \hookrightarrow H(K_{\boldsymbol{\gamma}})$, and $e^{\det}(N) \asymp N^{-1}$ on $W_2^1([0,1])$.
- Similar results for randomized (Monte Carlo) algorithms.

**Order of convergence** of minimal errors in terms of decay of weights




## Remark

- Much stronger results concerning the computational cost in *Plaskota, Wasilkowski* (2011).

## A proof of the upper bound for $e^{\mathrm{det}}(N, G)$

- **Construction of a multi-level algorithm**

  Put $g_d(\mathbf{z}) = g(z_1, \ldots, z_d, 0, \ldots)$. Clearly

$$g_{d_L}(\mathbf{z}) = g_{d_1}(\mathbf{z}) + \sum_{\ell=2}^{L} \Big( g_{d_\ell}(\mathbf{z}) - g_{d_{\ell-1}}(\mathbf{z}) \Big).$$

  Use rank-1 lattice rules for integration of $g_{d_1}$ and $g_{d_\ell} - g_{d_{\ell-1}}$.

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  Use rank-1 lattice rules for integration of $g_{d_1}$ and $g_{d_\ell} - g_{d_{\ell-1}}$.

- **Analysis**

  Choose suitable weights $\widetilde{\gamma}_j \gg \gamma_j$, derive estimates for

$$\sup_{\|g\|_{\boldsymbol{\gamma}} \le 1} \|g_{d_\ell} - g_{d_{\ell-1}}\|_{\widetilde{\boldsymbol{\gamma}}}.$$

  Employ tractability result for rank-1 lattice rules,

  see *Hickernell, Sloan, Wasilkowski* (2004).

# A Numerical Example

## Black-Scholes Model

$$dX_t = \alpha \cdot X_t \, dt + \beta \cdot X_t \, dW_t,$$
$$X_0 = x_0$$

with $t \in [0, 1]$, $\alpha = 0.05$, $\beta = 0.5$, and $x_0 = 2.0$.

## Asian option

$$f(X) = \left( \int_0^1 X(t) \, dt - K \right)_+$$

with $K = 2.0$. See Giles, Waterhouse (2009).

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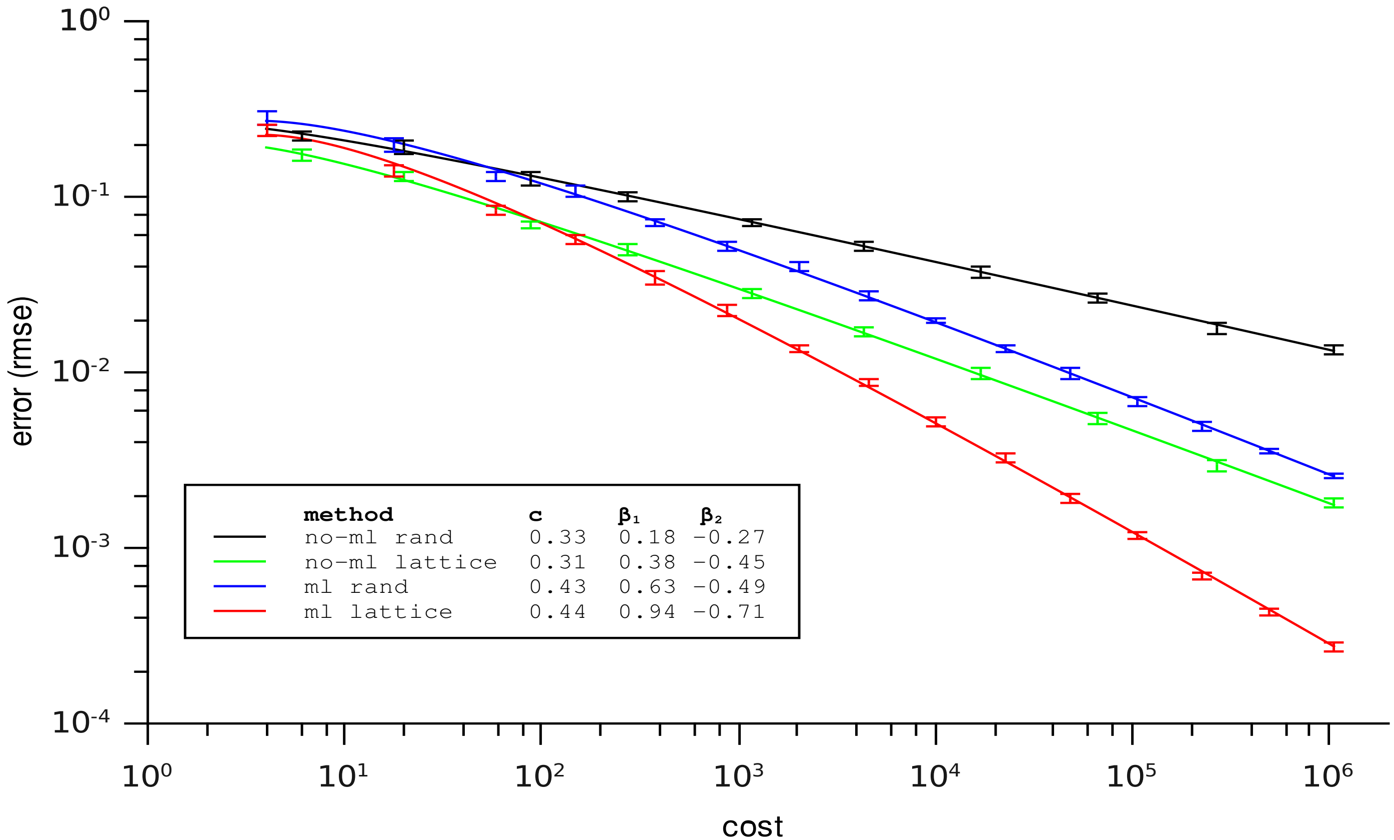
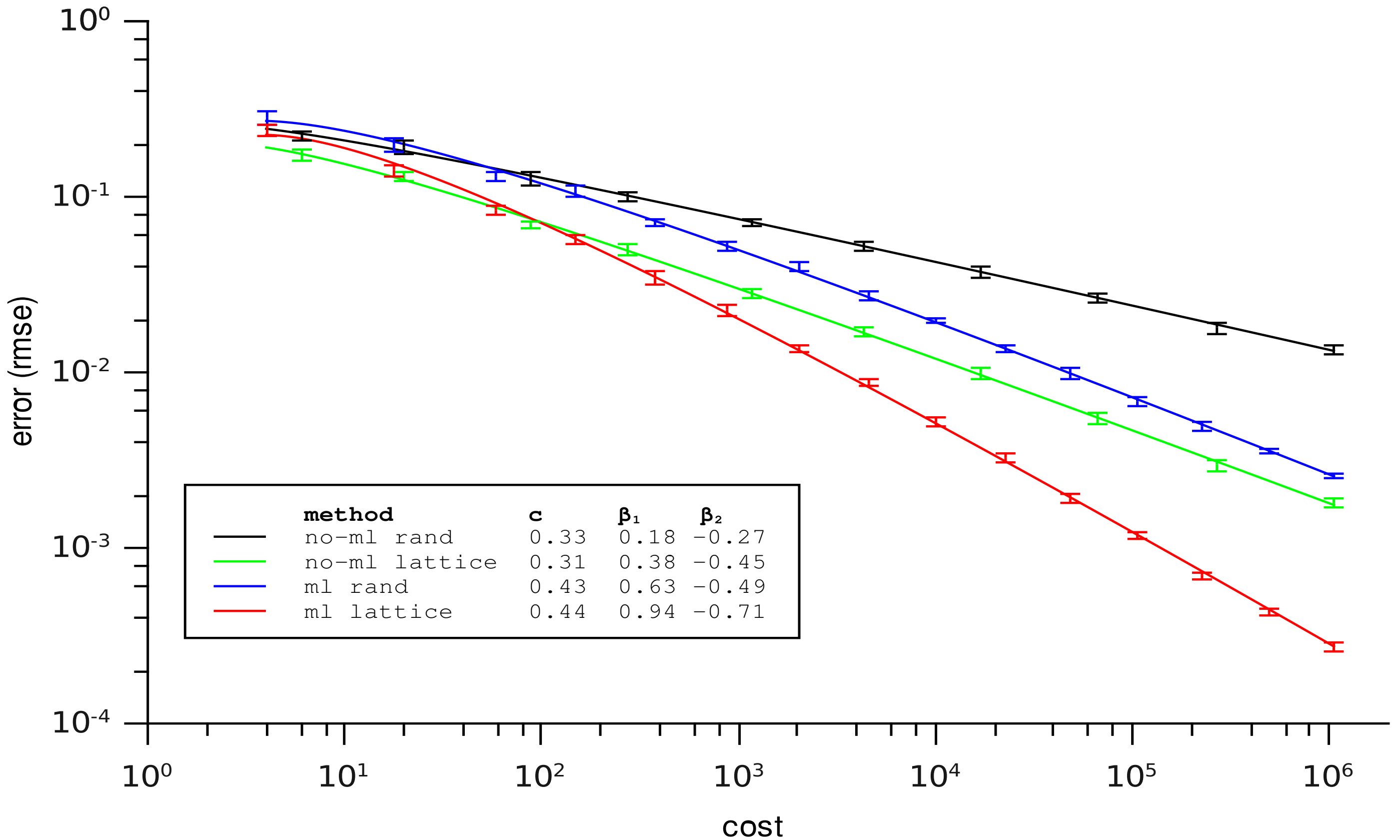
Estimation of the root mean square error based on $1000$ replications, see Mayer (2011). Log-linear regression, assuming that

$$\text{rms error} = c \cdot \frac{(\ln(\text{cost}))^{\beta_1}}{\text{cost}^{\beta_2}}.$$

# Root mean square error vs. cost



## Root mean square error vs. cost




## Asymptotic results

| | |
|---|---|
| Euler | $1/4$ |
| multi-level Euler | $1/2$ |

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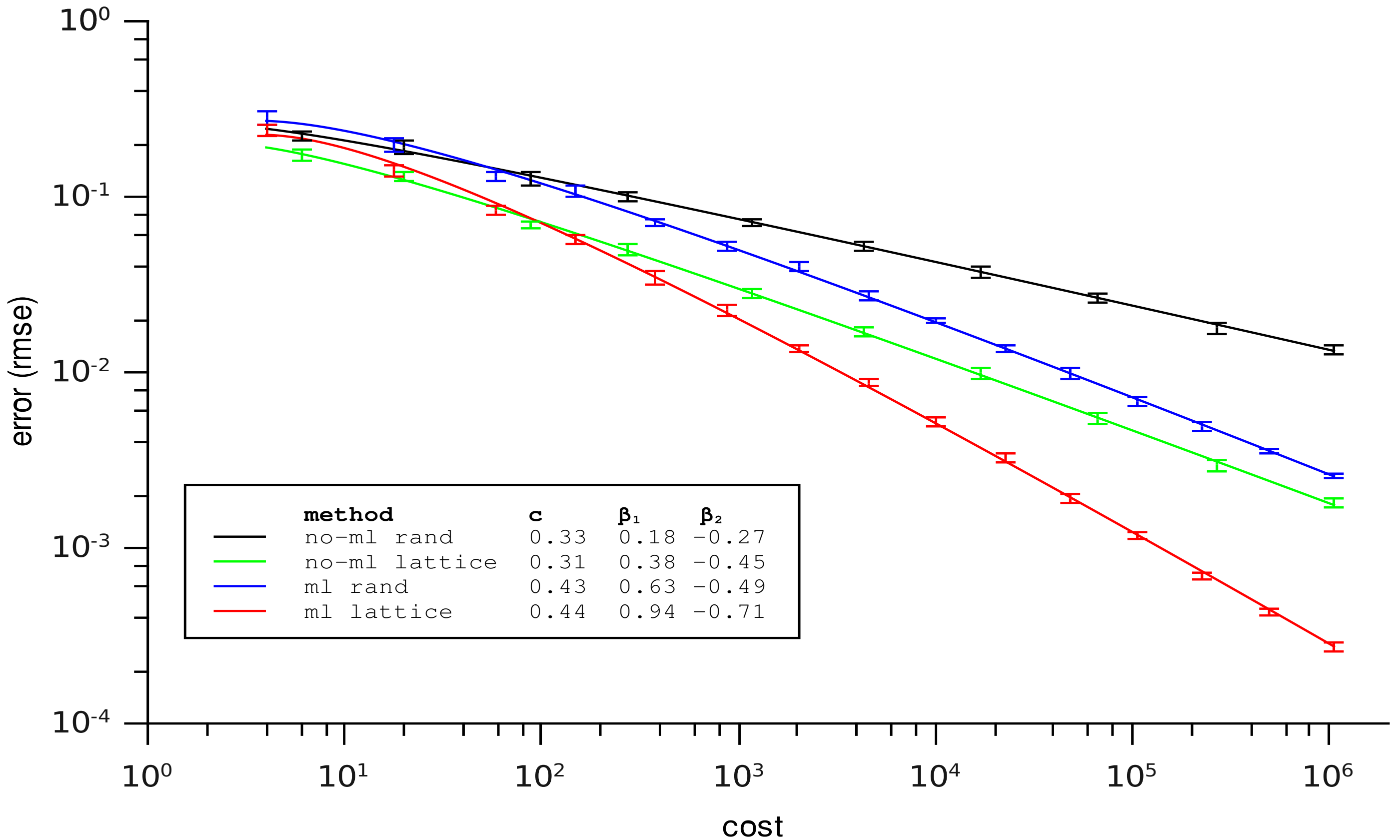


## Asymptotic results

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| Euler | $1/4$ |
| multi-level Euler | $1/2$ |
| Lévy-Ciesielski, shifted rank-one | ? |
| Lévy-Ciesielski, shifted rank-one, multi-level | ? |