Optimization in very large graphs

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(joint results with Balázs Szegedy)

Very large graphs

-Internet

-Social networks

-Ecological systems

-VLSI

-Statistical physics

-Brain

What properties to study?

-Does it have an even number of nodes?

-How dense is it (average degree)?

-Is it connected?

-Find connected components.

How is the graph given?

- Graph is HUGE.
- Not known explicitly (not even number of nodes).

How is it given?

- We can sample a uniform random node a bounded number of times, and see edges between sampled nodes.

Works in the dense case only ($\sim cn^2$ edges)

How is it given?

- \- We can sample a uniform random node a bounded number of times, and explore its neighborhood to a bounded depth.

- Works in the sparse case: Bounded degree ($\leq d$).

Different types of algorithmic questions

- Estimate a parameter (triangle density, density of max cut, rank of the adjacency matrix,…)

- Test a property (planar, bipartite, triangle-free,…)

- Find the structure (connected components, max cut, max matching,…)

The distance of two graphs

cut distance

(a) $V(G) = V(G')$

$$d_{\square}(G,G') = \max_{S,T \subseteq V(G)} \frac{|e_G(S,T) - e_{G'}(S,T)|}{n^2}$$

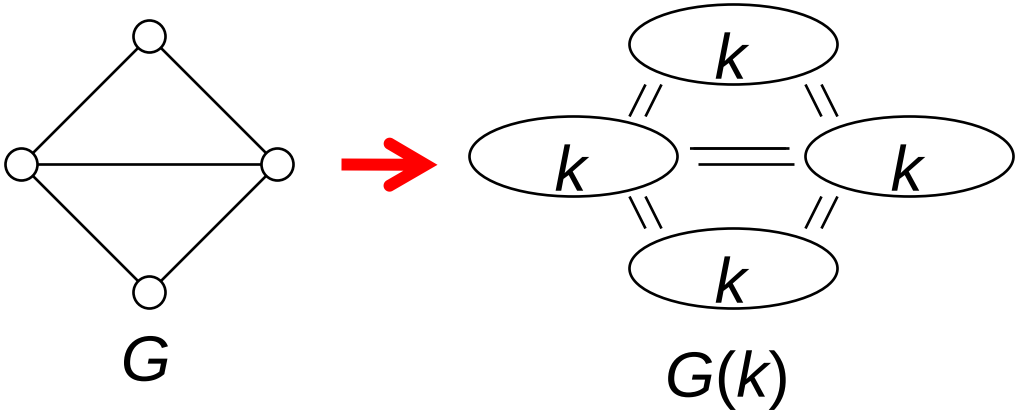
(b) $|V(G)| = |V(G')|$

$$\delta_{\square}^{*}(G,G') = \min_{G \leftrightarrow G'} d_{\square}(G,G')$$

The distance of two graphs

(c) $|V(G)| = n$, $|V(G')| = n'$

Blow up nodes:



$$\delta_\square(G,G') = \lim_{k\to\infty} \delta_\square^*(G(kn'),G'(kn))$$

The distance of two graphs

$$d_{\square}(G,G') = \max_{S,T \subseteq V(G)} \frac{|e_G(S,T) - e_{G'}(S,T)|}{n^2}$$

$$\delta^{*}_{\square}(G,G') = \min_{G \leftrightarrow G'} d_{\square}(G,G')$$

$$\delta_{\square}(G,G') = \ldots_{\square}(G(kn'),G'(kn))$$

fractional overlay

The distance of two graphs

Examples: $$\delta_\square(K_{n,n}, \mathbb{G}(2n,\tfrac{1}{2})) \approx \frac{1}{8}$$

$$\delta_\square(\mathbb{G}_1(n,\tfrac{1}{2}), \mathbb{G}_2(n,\tfrac{1}{2})) = o(1)$$

Two graphs are "close" in the δ distance

$\Leftrightarrow$

their subgraph distributions are "close".

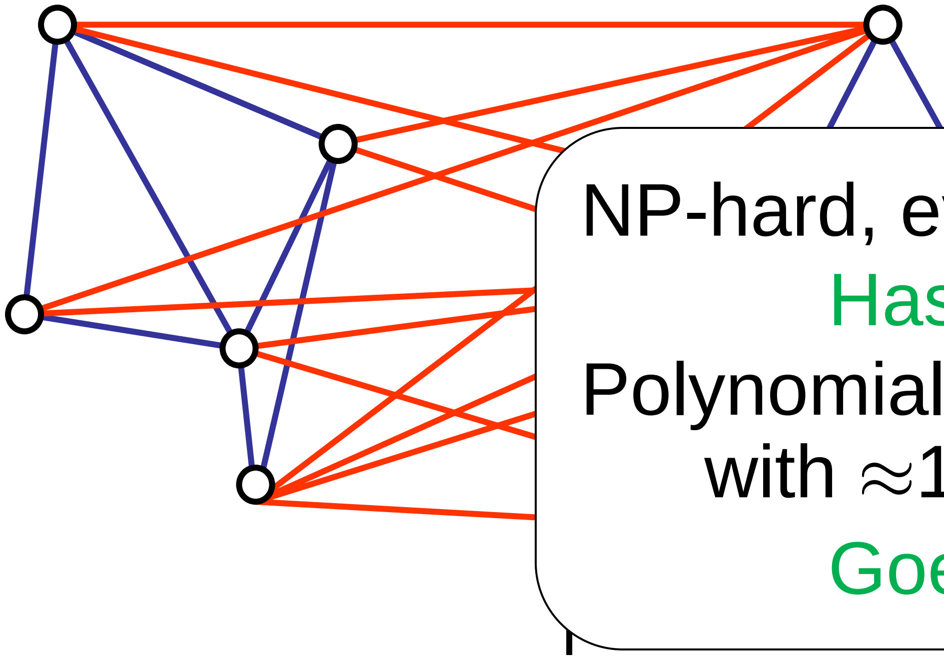
Borgs-Chayes-L-Sós-Vesztergombi

Parameter estimation (dense case)

Triangle density: easy

Maximum cut: nontrivial

The maximum cut problem



NP-hard, even with 6% error

Hastad

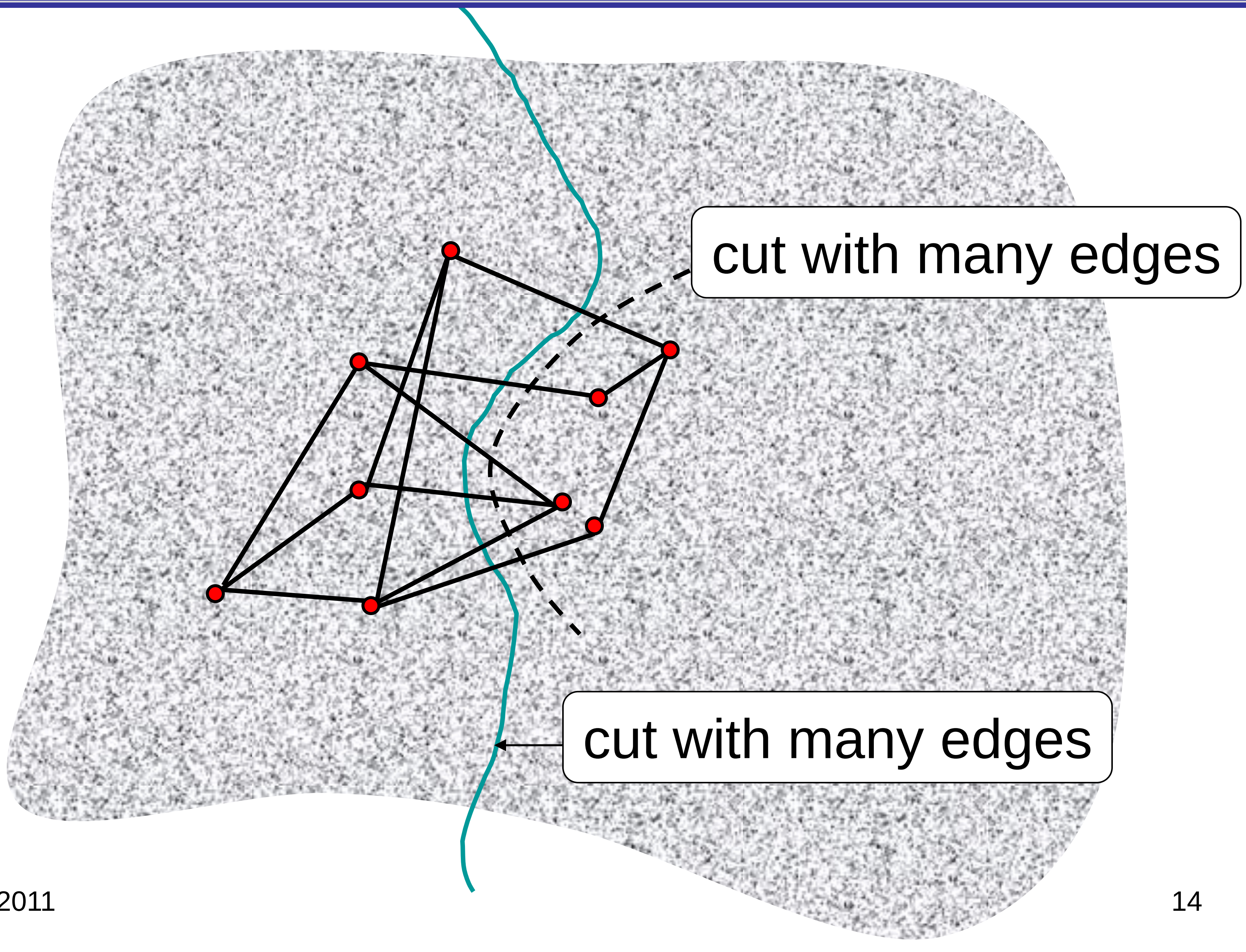
Polynomial-time computable with ≈13% error

Goemans-Williamson

maximize

Applications: optimization, statistical mechanics…

Density of maximum cut



Parameter estimation (dense case)

A graph parameter f can be estimated from samples if and only if

(i) $\forall\varepsilon>0\ \exists\delta>0$ s.t. $V(G)=V(G')$ and $d\ (G,G')<\delta$

$$\Rightarrow$$

$$|f(G)-f(G')|<\varepsilon.$$

(ii) $|f(G)-f(G-v)|\to 0 \quad (|V(G)|\to\infty)$

(iii) $\forall G$: $f(G(m))$ is convergent as $m\to\infty$.

Borgs, Chayes, L, Sós, Vesztergombi

Property testing (dense case)

„Property testing": Arora-Karger-Karpinski

Goldreich-Goldwasser-Ron

Rubinfeld-Sudan

Fischer

Frieze-Kannan

Alon-Shapira

Regularity Lemma

The key to algorithmic results in the dense case

Original Regularity Lemma Szemerédi 1976

"Weak" Regularity Lemma Frieze-Kannan 1999

"Strong" Regularity Lemma
Alon – Fisher – Krivelevich - M. Szegedy

Regularity Lemma

G: graph

$P=\{V_1,\ldots,V_k\}$: partition of $V(G)$

G_P : edge-weighted complete graph on $V(G)$, where the weight of edge uv ($u \in V_i$, $v \in V_j$) is

$$p_{ij} = e_G(V_i,V_j)/|V_i||V_j|$$

Regularity Lemma

"Weak" Regularity Lemma (Frieze-Kannan):

$\forall k \geq 1$, $\forall$ graph G $\exists$partition P={V_1,…,V_k} such that

$$d_{\square}(G,G_P) \leq \frac{4}{\sqrt{\log k}}$$

Similarity distance of nodes

Two nodes are "similar", if they are connected.

Does not measure what we need…

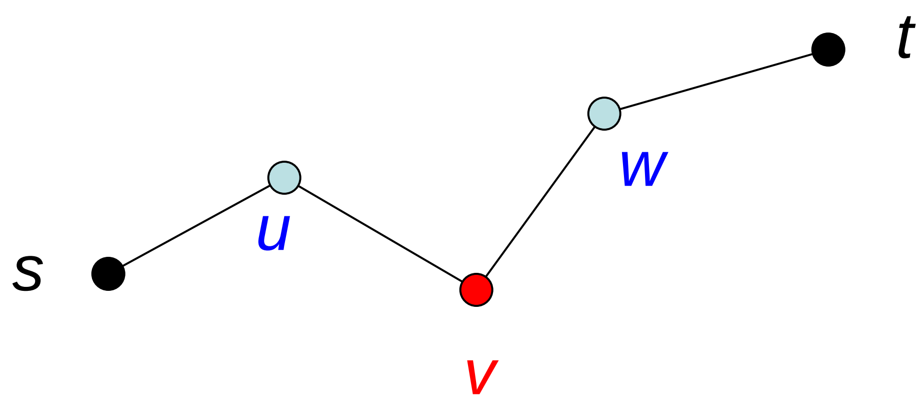
They are similar, if their neighborhoods are (almost) the same.

Too strong…

See: random graph

Similarity distance of nodes

$$d_2(s,t) := \mathrm{E}_v \left| \mathrm{P}_u(s, v \in N(u)) - \mathrm{P}_w(t, v \in N(w)) \right|$$



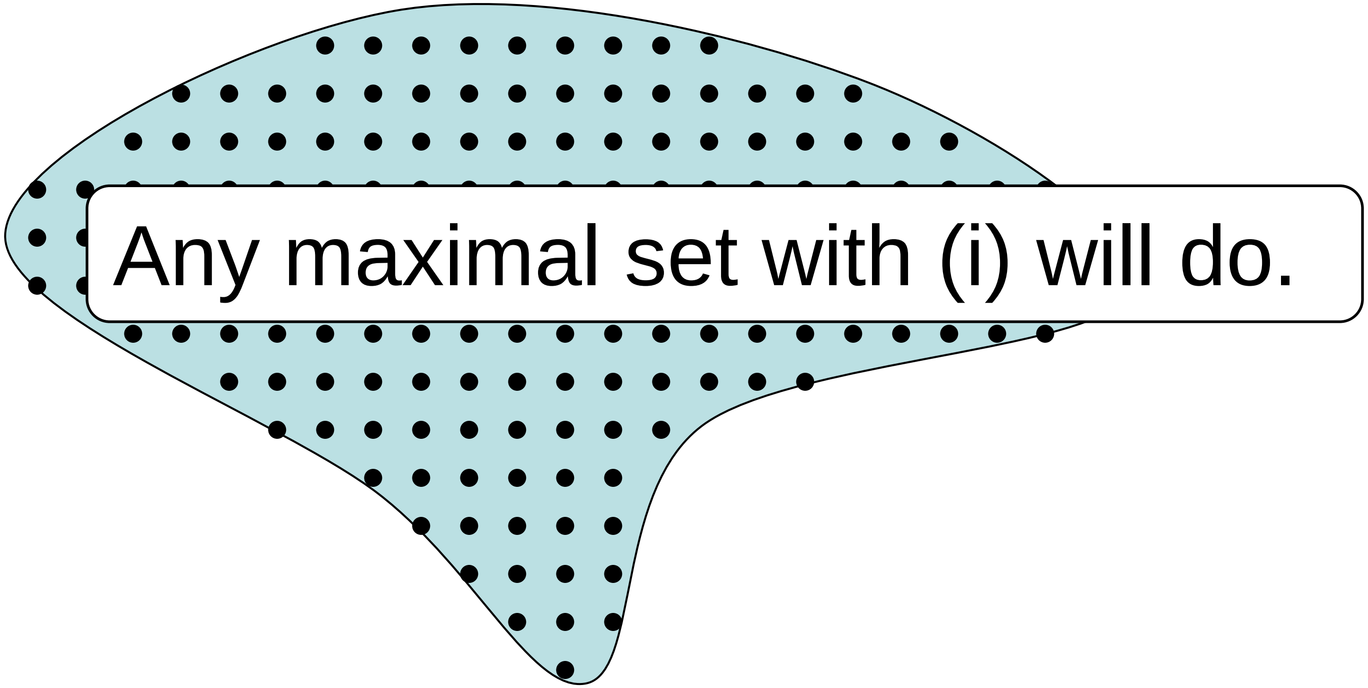
Fact 1: This is a metric.

Fact 2: Can be computed by sampling.

Representative set of nodes

(i) $u,v \in R \Rightarrow d_2(u,v) > \varepsilon$

(ii) $u \in V(G) \Rightarrow d_2(u,R) \leq \varepsilon$



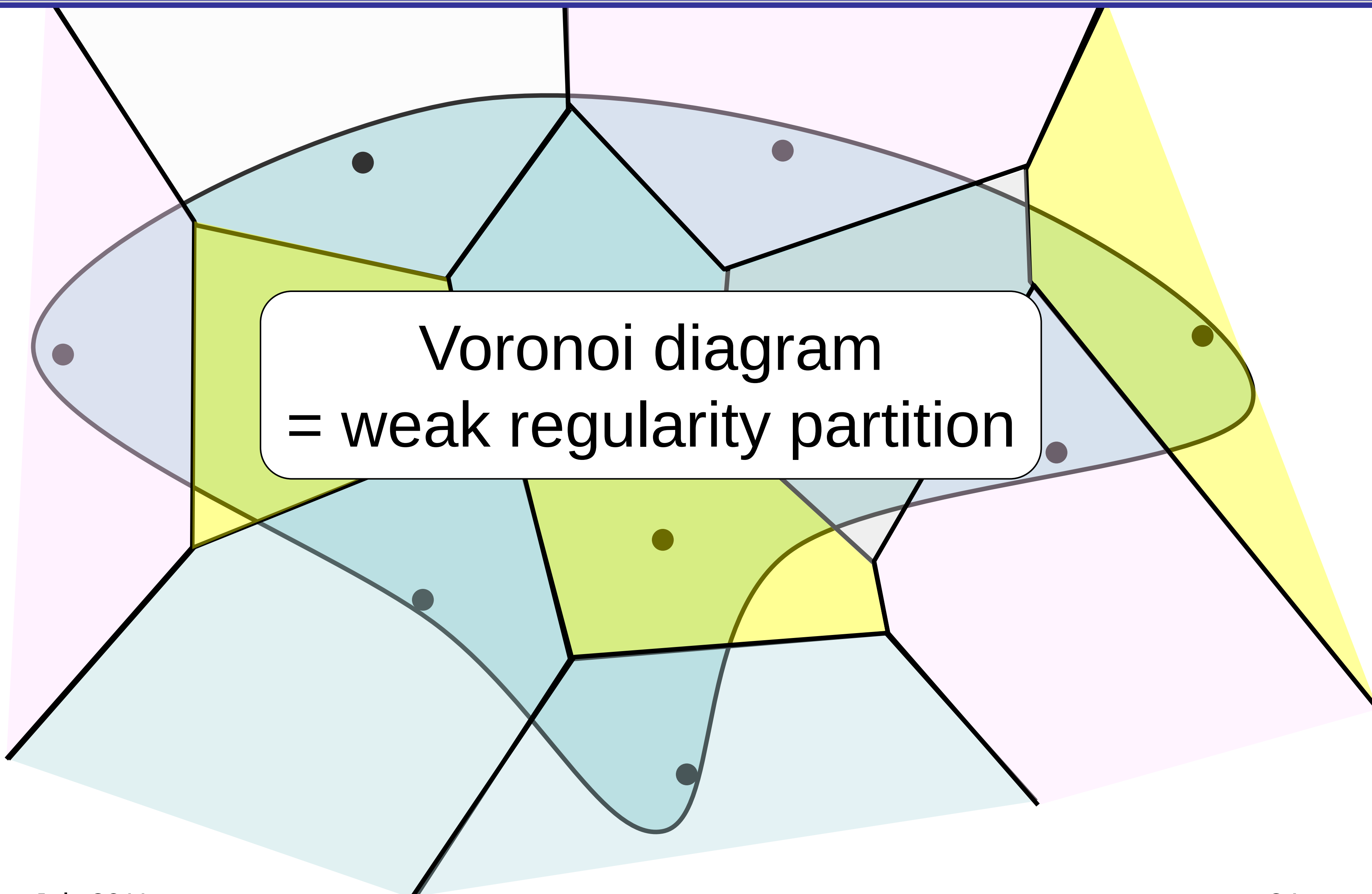
Representative set of nodes

(i) $u,v \in R \Rightarrow d_2(u,v) > \varepsilon$

(ii) $u \in V(G) \Rightarrow d_2(u,R) \leq \varepsilon$

Every graph contains an approximate representative set with at most $2^{2/\varepsilon^2}$ elements.

Representative set – Voronoi diagram



Representative set – Voronoi diagram

$S \subseteq V(G): \quad \bar{d}(S) = \mathrm{E}_x d_2(x,S)$ average ε-net

$\mathcal{P}$ partition: $r(\mathcal{P}) = \delta_{\square}(G, G_{\mathcal{P}})$ regular partition

Voronoi cells of S form a partition with

$$r(\mathcal{P}) < 8\sqrt{\bar{d}(S)}$$

$\forall$ partition $\mathcal{P}=\{V_1,\dots,V_k\}$ of [0,1] $\exists\, v_i \in V_i$ with

$$\bar{d}(\{v_1,\dots,v_k\}) < 8r(\mathcal{P})$$

Representative set – algorithm

- Begin with $U=\varnothing$.
- Select random nodes v_1, v_2, ...
- Add v_i to U iff $d_2(v_i,u)>\varepsilon$ for all $u\in U$.
- Stop if for more than $1/\varepsilon^2$ trials, U did not grow.

size bounded by $2^{2/\varepsilon^2}$

Representative set – algorithm

In which class does node v belong?

Let $U=\{u_1,...,u_k\}$.

Put node v in V_i iff i is the first index

with $d_2(u_i,v)\leq\varepsilon$.

Max cut – algorithm

Constructing representation of cut:

- Construct representative set U

- Compute p_{ij} = density between classes V_i and V_j (use sampling)

- Compute max cut (U_1,U_2) in complete graph on U with edge-weights p_{ij}

Max cut – algorithm

On which side of the cut does v belong?

Put node v of left side of cut iff

$$d_2(U_1,v) \leq d_2(U_2,v).$$

(Different algorithm implicit by Frieze-Kannan.)

Representative set of nodes

(i) $u,v \in R \Rightarrow d_2(u,v) > \varepsilon$

(ii) $u \in V(G) \Rightarrow d_2(u,R) \leq \varepsilon$

Looks like dimension.

Every graph contains a representative set with at most $2^{2/\varepsilon^2}$ elements.

Typically there is a much smaller one.

Convergence and limit objects

$t(F,G)$: Probability that random map $V(F) \rightarrow V(G)$ preserves edges

$(G_1, G_2, \ldots)$ convergent: $\forall F \;\; t(F, G_n)$ is convergent

distribution of k-samples is convergent for all k

Parameter estimation (dense case)

A graph parameter *f* can be estimated from samples if and only if

$$(G_n) \text{ convergent} \Rightarrow f(G_n) \text{ convergent}$$

Borgs, Chayes, L, Sós, Vesztergombi

Convergence and limit objects

$\mathcal{W}_0$ = {W: $[0,1]^2 \to [0,1]$, symmetric, measurable}

(graphons)

$$t(F,W) = \int_{[0,1]^{V(F)}} \prod_{ij \in E(F)} W(x_i, x_j)\, dx$$

$G_n \to W$: $\forall F$: $t(F,G_n) \to t(F,W)$

Convergence and limit objects

For every convergent graph sequence (G_n)

there is a $W \in \mathcal{W}_0$ such that $G_n \to W$.

Conversely, $\forall W \ \exists (G_n)$ such that $G_n \to W$.

L – B. Szegedy

W is essentially unique

(up to measure-preserving transform).

Borgs – Chayes - L

Limit objects

The distance δ between graphons,

the distance d_2 between points,

representative sets, regularity partitions,….

can be defined for graphons

$(\mathcal{W}_0, \delta)$ is a compact metric space.

The completion of $([0,1],d_2)$ is a compact metric space for every graphon.

Limit objects

The distance δ between graphons,

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The completion of ([0,1

compact metric spac

Equivalent to all
versions of the
Regularity lemma

Dimension of limit objects

If $([0,1],d_2)$ has finite dimension for some graphon W, then $\forall\varepsilon$ it has a representative set/ weak regularity partition with $(1/\varepsilon)^{\mathrm{const}}$ elements.

If G is a graph that does not contain F as a bipartite-induced subgraph (F bipartite), then $\forall\varepsilon$ it has a representative set/ weak regularity partition with $(1/\varepsilon)^{10|F|}$ elements.