Mathematical Tripos Part IB: Lent 2010

Numerical Analysis – Lecture 2[1]

1.2 Divided differences: a definition

Given pairwise-distinct points $x_0, x_1, \ldots, x_n \in [a, b]$, we let $p \in \mathbb{P}_n[x]$ interpolate $f \in C[a, b]$ there. The coefficient of x^n in p is called the *divided difference* and denoted by $f[x_0, x_1, \ldots, x_n]$. We say that this divided difference is of *degree n.*
We can derive $f[x_0, \ldots, x_n]$ from the Lagrange formula,

$$f[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} f(x_k) \prod_{\substack{\ell=0 \\ \ell \neq k}}^{n} \frac{1}{x_k - x_\ell}. \tag{1.2}$$

Theorem Let $[\bar{a}, \bar{b}]$ be the shortest interval that contains $x_0, x_1, \ldots, x_n$ and let $f \in C^n[\bar{a}, \bar{b}]$. Then there exists $\xi \in [\bar{a}, \bar{b}]$ such that

$$f[x_0, x_1, \ldots, x_n] = \tfrac{1}{n!} f^{(n)}(\xi). \tag{1.3}$$

Proof. Let p be the interpolating polynomial. The error function $f - p$ has at least $n+1$ zeros in $[\bar{a}, \bar{b}]$ and, applying Rolle's theorem n times, it follows that $f^{(n)} - p^{(n)}$ vanishes at some $\xi \in [\bar{a}, \bar{b}]$. But $p(x) = \frac{1}{n!} p^{(n)}(\zeta) x^n +$ lower order terms (for any $\zeta \in \mathbb{R}$), therefore, letting $\zeta = \xi$,

$$f[x_0, x_1, \ldots, x_n] = \tfrac{1}{n!} p^{(n)}(\xi) = \tfrac{1}{n!} f^{(n)}(\xi)$$

and we deduce (1.3). □

Application It is a consequence of the theorem that divided differences can be used to approximate derivatives.

1.3 Recurrence relations for divided differences

Our next topic is a useful way to calculate divided differences (and, ultimately, to derive yet another means to construct an interpolating polynomial). We commence with the remark that $f[x_i]$ is the coefficient of x^0 in the polynomial of degree 0 (i.e., a constant) that interpolates $f(x_i)$, hence $f[x_i] = f(x_i)$.

Theorem Suppose that $x_0, x_1, \ldots, x_{k+1}$ are pairwise distinct, where $k \geq 0$. Then

$$f[x_0, x_1, \ldots, x_{k+1}] = \frac{f[x_1, x_2, \ldots, x_{k+1}] - f[x_0, x_1, \ldots, x_k]}{x_{k+1} - x_0}. \tag{1.4}$$

Proof. Let $p, q \in \mathbb{P}_k[x]$ be the polynomials that interpolate f at

$$\{x_0, x_1, \ldots, x_k\} \qquad \text{and} \qquad \{x_1, x_2, \ldots, x_{k+1}\}$$

respectively and define

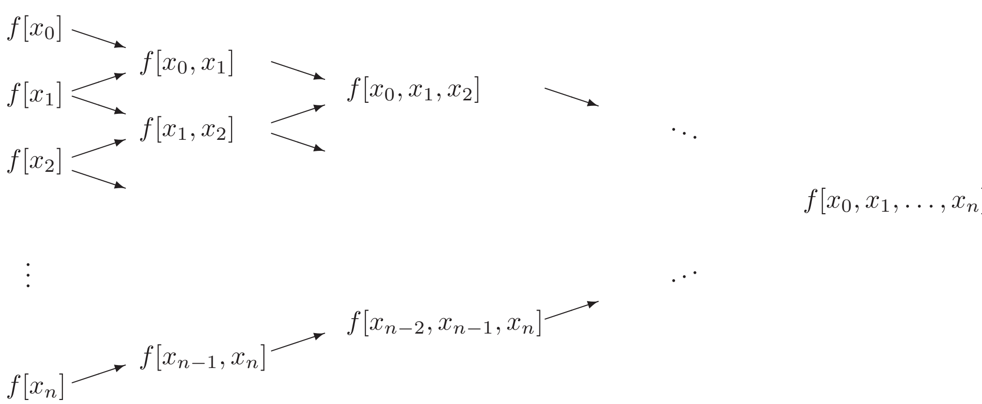
$$r(x) := \frac{(x - x_0)q(x) + (x_{k+1} - x)p(x)}{x_{k+1} - x_0} \in \mathbb{P}_{k+1}[x].$$

[1] Corrections and suggestions to these notes should be emailed to `A.Iserles@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartIB/`.

We readily verify that $r(x_i) = f(x_i)$, $i = 0, 1, \ldots, k+1$. Hence r is the $(k+1)$-degree interpolating polynomial and $f[x_0, \ldots, x_{k+1}]$ is the coefficient of x^{k+1} therein. The recurrence (1.4) follows from the definition of divided differences. $\square$

1.4 The Newton interpolation formula

Recalling that $f[x_i] = f(x_i)$, the recursive formula allows for rapid evaluation of the *divided difference table,* in the following manner:



This can be done in $\mathcal{O}(n^2)$ operations and the outcome are the numbers $\{f[x_0, x_1, \ldots, x_l]\}_{l=0}^{k}$. We now provide an alternative representation of the interpolating polynomial. Again, $f(x_i)$, $i = 0, 1, \ldots, k$, are given and we seek $p \in \mathbb{P}_k[x]$ such that $p(x_i) = f(x_i)$, $i = 0, \ldots, k$.

Theorem Suppose that $x_0, x_1, \ldots, x_k$ are pairwise distinct. The polynomial

$$p_k(x) := f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \ldots, x_k] \prod_{i=0}^{k-1}(x - x_i) \in \mathbb{P}_k[x]$$

obeys $p_k(x_i) = f(x_i)$, $i = 0, 1, \ldots, k$.

Proof. By induction on k. The statement is obvious for $k = 0$ and we suppose that it is true for k. We now prove that $p_{k+1}(x) - p_k(x) = f[x_0, x_1, \ldots, x_{k+1}] \prod_{i=0}^{k}(x - x_i)$. Clearly, $p_{k+1} - p_k \in \mathbb{P}_{k+1}[x]$ and the coefficient of x^{k+1} therein is, by definition, $f[x_0, \ldots, x_{k+1}]$. Moreover, $p_{k+1}(x_i) - p_k(x_i) = 0$, $i = 0, 1, \ldots, k$, hence it is a multiple of $\prod_{i=0}^{k}(x - x_i)$, and this proves the asserted form of $p_{k+1} - p_k$. The explicit form of p_{k+1} follows by adding $p_{k+1} - p_k$ to p_k. $\square$

We have derived the *Newton interpolation formula,* which requires only the top row of the divided difference table. It has several advantages over Lagrange's. In particular, its evaluation at a given point x (provided that divided differences are known) requires just $\mathcal{O}(k)$ operations, as long as we do it by the *Horner scheme*

$$\begin{aligned} p_k(x) = \{\{\{f[x_0, \ldots, x_k](x - x_{k-1}) + f[x_0, \ldots, x_{k-1}]\} \times (x - x_{k-2}) + f[x_0, \ldots, x_{k-2}]\} \\ \times (x - x_3) + \cdots\} + f[x_0]. \end{aligned}$$

On the other hand, the Lagrange formula is often better when we wish to manipulate the interpolation polynomial as part of a larger mathematical expression. We'll see an example in the section on *Gaussian quadrature.*

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