Part III - Lent Term 2005
Approximation Theory – Lecture 4

4 Existence and uniqueness of best approximation

4.1 Existence

Lemma 4.1 *Let $\mathcal{U}$ be a compact set in a metric space $\mathbb{X}$. Then, for every f in $\mathbb{X}$, there exists an element of best approximation.*

Proof. Let $d_* := \inf\{d(f,u) : u \in \mathcal{U}\}$, and let (u_i) be a minimizing sequence, i.e., $d(f,u_i) \to d_*$. By the compactness of $\mathcal{U}$, this sequence has at least one limit point $u_* \in \mathcal{U}$, and we may assume that $d(u_i,u_*) \to 0$. By the triangle inequality,

$$d(f,u_*) \le d(f,u_i) + d(u_i,u_*) \to d_*. \qquad \square$$

Theorem 4.2 *Let $\mathcal{U}$ be a finite–dimensional subspace of a normed linear space $\mathbb{X}$. Then, for every f in $\mathbb{X}$, there exists an element of best approximation.*

Proof. Let u_0 be any element of $\mathcal{U}$, e.g., $u_0 = 0$. The best approximant we seek lies in the set

$$\mathcal{U}_0 := \{u : u \in \mathcal{U},\ \|f-u\| \le \|f-u_0\|\}.$$

This set is compact because it is a closed and bounded subset of a finite–dimensional space. Therefore, by the previous theorem, there is an element u_* of best approximation from $\mathcal{U}_0$ to f. $\square$

4.2 Example of nonexistence

The finite-dimensionality hypothesis cannot be dropped as the following example shows.

Let c_0 be the Banach space of infinite sequences f such that

$$f = (\xi_1, \xi_2, \ldots), \quad \xi_k \to 0, \qquad \|f\| = \max|\xi_k|,$$

and let $\mathcal{U}_0 := \{u \in c_0 : \sum_{k=1}^\infty 2^{-k}\alpha_k = 0\}$.

Lemma 4.3 *For any $f \in c_0 \setminus \mathcal{U}_0$, the element of best approximation from $\mathcal{U}_0$ to f does not exist.*

Proof. Let $f \in c_0 \setminus \mathcal{U}_0$ and let $\lambda := \sum_{k=1}^\infty 2^{-k}\xi_k \ne 0$.

1) On the one hand, the following elements belong to $\mathcal{U}_0$:

$$u_1 = f - \tfrac{2}{1}\lambda\,(1,0,0,\ldots), \quad u_2 = f - \tfrac{4}{3}\lambda\,(1,1,0,0,\ldots), \quad u_3 = f - \tfrac{8}{7}\lambda\,(1,1,1,0,0,\ldots), \quad \text{etc.,}$$

and $\|f-u_n\| = (1-\frac{1}{2^n})^{-1}|\lambda| \searrow |\lambda|$. Hence, $d(f,\mathcal{U}_0) \le |\lambda|$.

2) On the other hand, for any $u \in \mathcal{U}_0$, we have $\|f-u\| > |\lambda|$ because

$$|\lambda| = |\sum_{k=1}^\infty 2^{-k}\xi_k| = |\sum_{k=1}^\infty 2^{-k}(\xi_k-\alpha_k)| \le \sum_{k=1}^\infty 2^{-k}|\xi_k-\alpha_k| < \|f-u\|\sum_{k=1}^\infty 2^{-k} = \|f-u\|,$$

the last inequality being strict since $\xi_k, \alpha_k \to 0$ implies $|\xi_k-\alpha_k| < \|f-u\|$ for $k > k_0$.

3) Thus, for any $f \in c_0 \setminus \mathcal{U}_0$, one has $d(f,\mathcal{U}_0) = |\lambda(f)|$, but the element of best approximation does not exist. $\square$

4.3 Uniqueness

Definition 4.4 A normed linear space $\mathbb{X}$ is called *strictly convex* if the unit sphere contains no line segments on its surface, i.e.,

$$\|x\| = \|y\| = 1, \quad x \neq y \quad \Rightarrow \quad \|\tfrac{1}{2}(x+y)\| < \tfrac{1}{2}\|x\| + \tfrac{1}{2}\|y\| = 1 \quad \forall\, x, y.$$

Lemma 4.5 *Let $\mathcal{U}$ be a subspace of a strictly convex normed linear space $\mathbb{X}$. Then, for each element $f \in \mathbb{X}$, there is at most one element of best approximation.*

Proof. Suppose that u_1 and u_2 are two different best approximations from $\mathcal{U}$ to f and $\|f - u_i\| = \lambda$. Then

$$\|f - \tfrac{1}{2}(u_1+u_2)\| = \|\tfrac{1}{2}(f-u_1) + \tfrac{1}{2}(f-u_2)\| < \tfrac{1}{2}\|f-u_1\| + \tfrac{1}{2}\|f-u_2\| = \lambda,$$

a contradiction to the definition of best approximation. □

Theorem 4.6 *Let $\mathcal{U}_n$ be a finite-dimensional subspace of a strictly convex normed linear space $\mathbb{X}$. Then, for each $f \in \mathbb{X}$, there exists a unique element of best approximation.*

Example 4.7 Any Hilbert space with a scalar product (f,g) and the norm $\|f\| := (f,f)^{1/2}$ is strictly convex. This follows from the identity

$$\|\tfrac{1}{2}(f+g)\|^2 + \|\tfrac{1}{2}(f-g)\|^2 = \tfrac{1}{2}\|f\|^2 + \tfrac{1}{2}\|g\|^2.$$

Example 4.8 The spaces $L_p[a,b]$ with the norm $\|f\|_p := \{\int_a^b |f(t)|^p dt\}^{1/p}$ are strictly convex if $1 < p < \infty$.

Example 4.9 The spaces $L_1[a,b]$ and $L_\infty[a,b] := C[a,b]$ are *not* strictly convex and examples of *some* subspaces which provide several b.a.'s to *some* functions can be easily given.

a) Let $\mathbb{X} = L_1[-1,1]$, $f(x) = \operatorname{sgn} x, \mathcal{U}_1 = \{\alpha\}$. Then any $\alpha \in [-1,1]$ is a b.a. to f.

b) Let $\mathbb{X} = C[0,1]$, $g \equiv 1, \mathcal{U}_1 = \{\alpha x\}$. Then any αx with $\alpha \in [0,2]$ is a b.a. to g.

However, the nonuniqueness is not guranteed. If in (a) one takes $f(x) = \operatorname{sgn}(x - x_0)$ with any $x_0 \neq 0$, then a b.a. from $\mathcal{U}_1$ is unique.

4.4 Example of nonuniqueness in L_1

Lemma 4.10 1) *Let $f \in L_1(\mathbb{T})$. Then, for $0 < k < n$, we have $\int_{-\pi}^{\pi} f(nx) {\cos kx \atop \sin kx} = 0$*

2) *If also $f \perp 1$, then $f(n\cdot) \perp \mathcal{T}_{n-1}$.*

Proof. Since ${\cos kx \atop \sin kx}$ are linear combination of e^{imx} with $0 < |m| < n$, it will be enough to prove that $I := \int_{-\pi}^{\pi} f(nx) e^{imx} dx = 0$. Setting $x = t + 2\pi/n$, and using the equality $\int_{\mathbb{T}} g = \int_{\mathbb{T}+\alpha} g$, we obtain

$$I = \int_{-\pi}^{\pi} f(nx)e^{imx}dx = \int_{-\pi-2\pi/n}^{\pi-2\pi/n} f(nt+2\pi)e^{imt+2im\pi/n}dt = e^{2im\pi/n}\int_{-\pi}^{\pi} f(nt)e^{imt}dt = e^{2im\pi/n} I.$$

Since $0 < |m| < n$, we have $e^{2i\pi m/n} \neq 1$, hence $I = 0$. □

Lemma 4.11 *For $F(x) = \operatorname{sgn}\sin nx$, any $s \in \mathcal{T}_{n-1}$ with $\|s\|_\infty < 1$ is a polynomial of b.a. to F in $L_1(\mathbb{T})$.*

Proof. By previous lemma, $F \perp \mathcal{T}_{n-1}$. Then, for any $s \in \mathcal{T}_{n-1}$,

$$\int_{\mathbb{T}} |F(x) - s(x)|\, dx \geq \Big| \int_{\mathbb{T}} [F(x) - s(x)] \operatorname{sgn}\sin nx\, dx \Big| = \Big| \int_{\mathbb{T}} F(x) \operatorname{sgn}\sin nx\, dx \Big| = 2\pi.$$

If $|s(x)| < 1 = |F(x)|$, then $\operatorname{sgn}[F(x) - s(x)] = \operatorname{sgn} F(x) = \operatorname{sgn}\sin nx$, and the first inequality becomes equality. □

This example has a remarkable generalization.

Theorem 4.12 (Hobby–Rice) *For any n-dimensional subspace $\mathcal{U}_n$ of $L_1[a,b]$ there exists a sign function h with n breakpoints such that $h \perp \mathcal{U}_n$.*

Theorem 4.13 (Krein[1][1938]) *No finite-dimensional subspace of $L_1[a,b]$ is a unicity space, i.e., for any $\mathcal{U}_n$ in $L_1[a,b]$ there exists a function $f \in L_1[a,b]$ that has several b.a.*

[1]Mark Krein (1907-1989), Russian mathematician, one of the recipients of prestigious Wolf Prize in mathematics "for his fundamental contributions to functional analysis and its applications".

4.5 Exercises

4.1. Prove that the equality $\|x\| = \|y\| = \|\frac{1}{2}x + \frac{1}{2}y\| = 1$ implies that $\|\alpha x + (1-\alpha)y\| = 1$ for any $\alpha \in [0,1]$, i.e., that two conditions in Definition 4.4 are really equivalent.

4.2. Does the unicity Theorem 4.6 have a converse? That is, can we infer strict convexity from a knowledge that for each $f \in \mathbb{X}$ and for each finite-dimensional $\mathcal{U} \subset \mathbb{X}$ there exists a *unique* element of best approximation?

Hint. Let $\mathbb{X} = \mathbb{R}^2$ with a norm which is not strictly convex, i.e., the set $\|x\| = 1$ contains a line segment. Find f and one-dimensional subspace $\mathcal{U}_1$ with several b.a. to f.

4.3. Fill in the details of the following proof of Theorem 4.2. It is based on the Weierstrass theorem: *A continuous function on a closed bounded set in $\mathbb{R}^n$ attains its minimum.*

a) Let $f \in \mathbb{X}$, an let $\mathcal{U}_n = \mathrm{span}\,(g_k)_{k=1}^n$ where g_k are linearly indepdendent.

b) For $x = (x_1, \ldots, x_n)$, the function $G(x) := \|\sum_{k=1}^n x_k g_k\|$ is a continuous function of n real variables x_k, hence, on the unit ball $\sum x_k^2 = 1$, it takes some minimal value $\rho > 0$.

c) The function $F(x) := \|f - \sum_{k=1}^n x_k g_k\|$ is continuous as well, and it follows from (b) that $F(x) \to \infty$ as $x \to \infty$.

d) Thus, looking for the infimum of $F(x)$ over $x \in \mathbb{R}^n$, we may restrict ourselves to some ball $B_r := \{x : \sum x_k^2 \le r\}$, this is a bounded closed set in $\mathbb{R}^n$, hence, on B_r, $F(x)$ attains its minimal value.

4.4. The essence of the example given in Lemma 4.3 is that the linear functional

$$\lambda : c_0 \to \mathbb{R}, \quad \lambda(f) := \sum_{k=1}^\infty 2^{-k}\xi_k, \quad f = (\xi_1, \xi_2, \ldots) \in c_0$$

does not take its norm

$$\|\lambda\| := \sup\{|\lambda(f)| : \|f\| = 1\},$$

i.e., there is no function $f \in c_0$ such that $|\lambda(f)| = \|\lambda\|\,\|f\|$.

a) Prove that, for any $\mathbb{X}$ and any λ in $\mathbb{X}^*$ (the space of all linear bounded functionals on $\mathbb{X}$), we have the equality $|\lambda(f)| = \|\lambda\|\,\mathrm{dist}(f, \ker\lambda)$ for any $f \in \mathbb{X}$.

b) Using (a), even if you have not proved it, prove that if $\lambda : \mathbb{X} \to \mathbb{R}$ does not take its norm, then the subspace $\mathcal{U}_0 := \ker\lambda$ is a non-existence set.

c) For ℓ_1, the Banach space of infinite sequences f such that

$$f = (\xi_1, \xi_2, \ldots), \quad \sum_{k=1}^\infty |\xi_k| < \infty, \quad \|f\| = \sum_k |\xi_k|,$$

construct a functional $\lambda(f) = \sum_{k=1}^\infty \lambda_k \xi_k$ which does not take its norm, thus find a subspace of ℓ_1 which is a non-existence set.

4.5. Derive Krein's Theorem 4.13 from Hobby–Rice Theorem 4.12.

Hint. For any $f \in \mathcal{U}_n$ consider the function $F = |f| \cdot h$, where h is a sign function from Hobby-Rice theorem.