Quantum liquids

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Most of the behaviour of a quantum liquid can be understood in the framework of 'extended' classical hydrodynamics.

Book chapter by Salman, NGB, Roberts "Extended hydrodynamics in the description of finite-temperature systems" in "Finite-temperature nonequilibrium systems" ed. by Proukakis et al 2011.

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Outline

Introduction:

- What are quantum liquids?
- Superfluid helium
- Recent experiments, novel systems
- Applications

Modelling of quantum liquids:

- Landau two-fluid model vs GPE
- Superfluid vs classical barotropic fluid
- Hills-Roberts theory
- GPE vs classical equations of vortex motion

Modelling in superfluid helium

- Turbulence
- Roton minimum as a ghost of a vanishing vortex ring?
- "Unindentified electron objects" UEO
- Turbulence induced by pressure variation

Modelling in exciton-polariton condensates

- Gross-Pitaevskii equation with loss/gain: vortices and lattices

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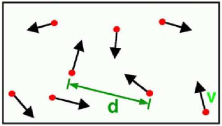
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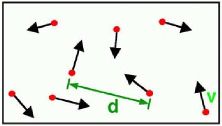
Introduction: "Giant matter wave"



Classical fluid mechanics

macroscopic

continuum approximation $d \ll L$

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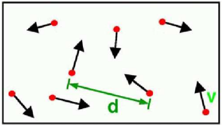

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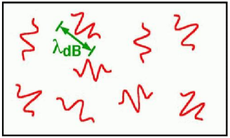
$\lambda_{dB} \sim T^{-1/2}$



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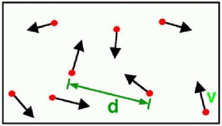

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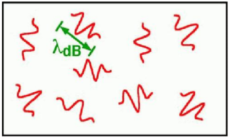
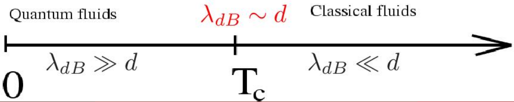


Quantum fluid mechanics – macroscopic dynamics & quantum effects



Bose-Einstein Condensation (BEC) - macroscopic occupation of the single quantum state.

Superfluidity - the ability to flow through narrow channels without friction; the existence of quantised vortices with the quantum of circulation h/m.

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Relationship between BEC and superfluidity

Existence of the classical field ψ (*order parameter, wave function*) associated with the macroscopic component of the field operator

$$\psi = \sqrt{n}\exp[\mathrm{i}S], \ \mathbf{v}_s = \frac{\hbar}{m}\nabla S,$$

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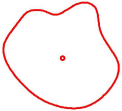

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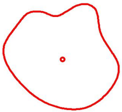
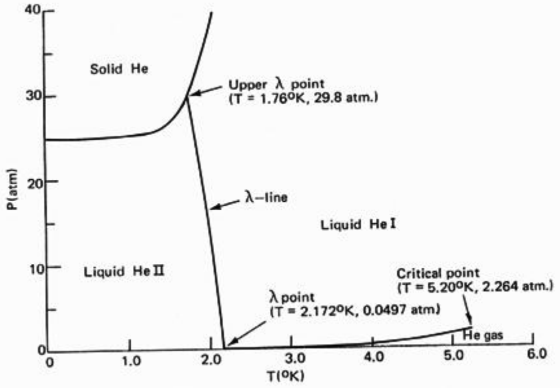


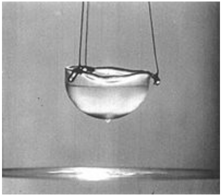
$\psi = 0$ represents quantised vortex line.



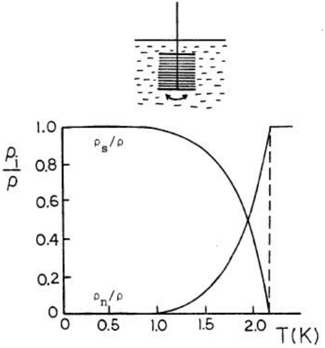
The phase diagram of He^4.

Discovered by Kapitza in 1938.

(a) Able to empty a beaker by flowing out of it via an absorbent film only about 100 atoms thick.



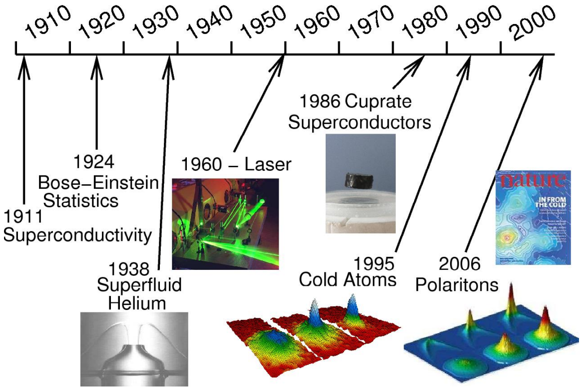
(b) Period and damping of torsional oscillations of a stack of closely spaced disks. Period is T-dependent.



Landau two-fluid model: $\rho = \rho_s + \rho_n$.

Milestones of superfluid discoveries

1908 Liquefaction of helium by Onnes

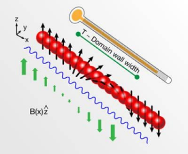


[Image courtesy Jonathan Keeling]

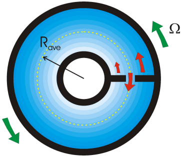
Nobel Prizes in Physics related to Superfluidity: 1962 Landau; 1978 Kapitza; 1996 Lee, Osheroff, Richardson; 1997 Chu, Cohen-Tannoudji, Phillips; 2001 Cornell, Ketterle, Wieman; 2003 Leggett;

Probing fundamental physics (eg. electrons in solid-state crystals), but also:

- "Atom lasers" – intense beams of coherent atoms (high precision atom-interferometric metrology)
- Spin gradient thermometry with resolution 50pK (Ketterle group, December 2009)



- Superfluid gyroscope (Packard)



Frictionless flow – property of the excitation spectrum $\epsilon(\mathbf{p})$.
Consider a heavy obtacle moving at a constant velocity v in a uniform fluid in its ground state.
Question:*At what velocity does it become possible for excitations to be created?*

	Original frame	Frame moving with obstacle
Ground state	E_0, $\mathbf{p} = \mathbf{0}$	$E(\mathbf{v}) = E_0 + \frac{1}{2}Nmv^2$
GS + Single excitation	$E = E_0 + \epsilon(\mathbf{p})$, $\mathbf{p}$	$E(\mathbf{v}) = E_0 + \epsilon(\mathbf{p}) - \mathbf{p}\cdot\mathbf{v} + \frac{1}{2}N$

No excitation can spontaneously grow in the fluid if

$$v < \min_{\mathbf{p}} \frac{\epsilon(\mathbf{p})}{p}, \qquad \text{Landau} \quad \text{criterion}$$

Mathematical description of superfluidity

- Phenomenological Landau two-fluid model: mixture of superfluid and normal fluid; Superfluid – ideal inviscid Euler fluid at $T = 0$K; Normal fluid – Navier-Stokes fluid; Validity: No vortices !!!
- Phenomenological Hall-Vinen-Bekharevich-Khalatnikov (HVBK) model: Landau + vortices. Extra forces: tension and friction with normal fluid. Validity: mean spacing between the vortex lines $\ll$ length scales of interest.
- Classical inviscid model of vortex motion
  Validity: Length scales $\gg$ vortex core, *ad hoc* vortex reconnections, no vortex formation
- Gross-Pitaevskii semi-classical model
  Validity: weakly interacting Bose gas

$$i\hbar\psi_t(\mathbf{r}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi + U|\psi|^2\psi$$

  U is the *effective interaction potential*, ψ is a classical complex valued matter field.

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Applicability of GP equation

Heisenberg representation of the field operator $\hat{\Psi}(\mathbf{r}, t)$:

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{r}, t) = \left[-\frac{\hbar^2 \nabla^2}{2m} + \int \hat{\Psi}^\dagger(\mathbf{r}, t) V(\mathbf{r}' - \mathbf{r}) \hat{\Psi}(\mathbf{r}', t)\, d\mathbf{r}' \right] \hat{\Psi}(\mathbf{r}, t).$$

$$\hat{\Psi}(\mathbf{r}, t) \sim \psi(\mathbf{r}, t)$$

- BEC (macroscopic occupation, large N)
- dilute gas at low temperature
  (range of interatomic forces $\ll$ average interparticle distance)

$$V \to V_{\text{eff}}, \qquad V_0 = \int V_{\text{eff}}(r)\, d\mathbf{r}, \qquad V_0 = \frac{4\pi\hbar^2 a}{m}$$

- interested in phenomena taking place over distances $\gg a$
  $\text{Trapped} \quad \text{BEC}: \quad i\hbar\psi_t(\mathbf{r}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi + V_0|\psi|^2\psi + V_{\text{ext}}(\mathbf{r})\psi$

$$i\hbar\psi_t(\mathbf{r}, t) = \delta E/\delta\psi^*, \quad \text{Energy} \quad E = \int \frac{\hbar^2}{2m}|\psi|^2 + V_{\text{ext}}(\mathbf{r})|\psi|^2 + \frac{V_0}{2}|\psi|^4\, d\mathbf{r}.$$

GPE as a phenomenological model

Gross-Pitaevskii equation as a non-relativistic limit of the Klein-Gordon equation–the simplest equation consistent with special relativity.

$$\frac{\partial^2 \Psi}{\partial t^2} = c^2 \nabla^2 \Psi - \lambda^2 \Psi$$

Represent $\Psi = \psi \exp[\mp i \lambda t]$ for matter and anti-matter solutions.

$$-\lambda^2 \psi - 2i\lambda \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial t^2} = c^2 \nabla^2 \psi - \lambda^2 \psi$$

Non-relativistic limit $\left|\frac{\partial^2 \psi}{\partial t^2}\right| \ll \lambda \left|\frac{\partial \psi}{\partial t}\right|$

Gross-Pitaevskii equation

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Gross-Pitaevskii Equation of BEC [1961]

Weakly–interacting dilute Bose gas

$$\mathrm{i}\hbar\psi_t(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\psi + U|\psi|^2\psi$$

Number density $n(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2$; Velocity $\mathbf{v}(\mathbf{r},t) = \frac{\hbar}{m}\nabla S(\mathbf{r},t)$.
Madelung transformation:

$$\psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}\exp[iS(\mathbf{r},t)]$$

Continuity equation:

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) = 0.$$

Integrated momentum equation

$$\hbar\frac{\partial S}{\partial t} + \left(\frac{1}{2}m\mathbf{v}^2 + Un - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}\right) = 0$$

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Landau two-fluid model vs GP theory

[Putterman & Roberts, Physica A (1983)]

Landau two-fluid theory can be obtained from the equations of conservation of mass and momentum for a one-component barotropic fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla \mu_0(\rho),$$

where $d\mu_0(\rho) = (1/\rho)dP_0(\rho)$, $P_0(\rho)$ is the pressure, $\mu_0(\rho)$ is the chemical potential.

GP equation is of this form with $\psi = \rho^{1/2}\exp(\mathrm{i}S)$, so that $\mathbf{v} = (\hbar/m)\nabla S$ and

$$\mu_0 = -\frac{\hbar^2}{2m^2}\frac{\nabla^2 \rho^{1/2}}{\rho^{1/2}} + \frac{V_0}{m}\rho.$$

Exact solution $(\rho, \mathbf{v})$ contains a long wavelength (background) contributions and short wavelength contributions determined by excitations. Nonlinearities couple these motions.

Key ideas:

Key nonlinear effects:

- Scattering of sound waves producing sum and difference frequencies and wave numbers;
- Refraction of short wavelengths sound waves by long wavelength background variations;
- Reaction of background so that total energy and momentum are conserved during the refraction.

Main idea: expand ρ (and $\mathbf{v}$) in terms of slowly varying background $\rho_0(\mathbf{r}, t)$ and high frequency waves with amplitudes ρ_n:

$$\rho = \rho_0 + \sum_{n=1}^{\infty} \int \rho_n(\mathbf{k}, \mathbf{r}, t) \exp[iS(\mathbf{k}, \mathbf{r}, t)] d\mathbf{k} + c.c,$$

where $\rho_n \sim \epsilon^n$, but the derivatives are of order ϵ^{n+2}, $dS = \mathbf{k} \cdot d\mathbf{r} - \omega dt$.

At ϵ^4 Boltzmann equation on occupation numbers

$$\left(\frac{\partial}{\partial t} + \frac{\partial\omega}{\partial k}_i \frac{\partial}{\partial x}_i - \frac{\partial\omega}{\partial x}_i \frac{\partial}{\partial k}_i\right) n(\mathbf{k}, t) = I_{\mathrm{coll}}(n),$$

where $\omega = \omega_0(k) + \mathbf{v_0} \cdot k$ with $\omega_0 = c_0 k$.
Local equilibrium $I_{\mathrm{coll}}(n) = 0$ yields $n = n_{eq}(\beta(\omega_0 - \mathbf{k} \cdot \mathbf{w}))$. Use general conservation laws for $I_{\mathrm{coll}}(n)$ and identify $\rho_0 \to \rho_s$, $\mathbf{v_0} \to \mathbf{v}_s$ and $\mathbf{w} \to \mathbf{v}_n - \mathbf{v}_s$.
Use Chapman– Engskog expansion to provide viscosities and thermal conductivity.

[Hill & Roberts (1977-1980)]
Theory that incorporates healing and relaxation – generalization of Landau theory with superfluid component described in a way similar to the GPE. Hills-Roberts theory rests on accepted macroscopic balance laws for mass, momentum and energy together with a postulate for entropy growth. Superfluid density is regarded as an independent thermodynamic variable. Boundary conditions on ρ_s need to be specified.
Successes:

- Predicted that the static healing length would increase with pressure – confirmed experimentally by Tam and Ahlers (1982)
- Dependence of the vortex core parameter on P and T agreed with experiments

Can Hills–Roberts theory be applied to superfluid turbulence? Evaluation of coefficients in HVBK theory?

Vortices in the GP equation

$$i\hbar\psi_t(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\psi + U|\psi|^2\psi$$

Stationary state: chemical potential μ via $i\hbar\psi_t = \mu\psi \Rightarrow \mu = U|\psi_\infty|^2$.

Wave function of a straight-line vortex in (r,θ,z) takes form

$$\psi = |\psi(r)|\exp[i\theta]$$

Fluid rotating around the z-axis with tangential velocity

$$\mathbf{v} = \frac{\hbar}{m}\nabla\theta = \frac{\hbar}{m}\frac{1}{r}\hat{\theta}$$

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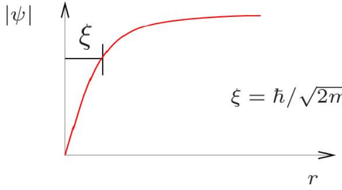
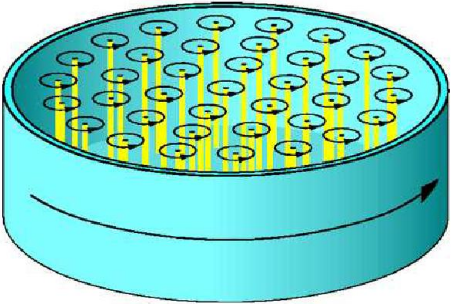
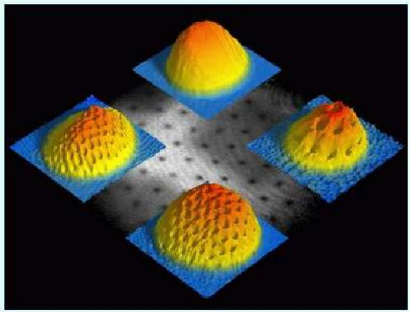


$\xi = \hbar/\sqrt{2m\mu}$ – healing length

[Pitaevskii, JETP, 1961]

Fo a sufficiently large angular velocity $\mathbf{\Omega}$, the state with superfluid at rest becomes energetically unfavourable.
In frame rotating with $\mathbf{\Omega}$,
the energy to minimize is $E_r = E - \mathbf{\Omega} \cdot \mathbf{L}$, where $\mathbf{L}$ is angular momentum $\Rightarrow$ vortices are created





^{4}He Vinen (1956),^{3}He-B Helsinki group 80s Rb^{87} Wolfgang Ketterle group (2001)

$$i\hbar\psi_t(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\psi + V_0|\psi|^2\psi + V_{ext}(\mathbf{r})\psi - \mu\psi$$

(1) Standard "magnetic" traps $V_{ext}(\mathbf{r}) = V_{tr} = \frac{1}{2}m(\omega_1^2x^2 + \omega_2^2y^2 + \omega_3^2z^2)$, where ω_i are trap frequencies.

(2) Other experimental potentials: quartic, periodic optical lattices, disordered potentials...

Strong interactions, large number of particles, shallow trap: kinetic energy neglible compared to trap energy and interaction energy – Thomas-Fermi (TF) limit.

Ground state in Thomas-Fermi limit

$$V_0 n(\mathbf{r}) + V_{ext}(\mathbf{r}) = \mu$$

For example, if $V_{ext} = V_{tr}$

$$n(\mathbf{r}) \approx |\psi_{TF}(\mathbf{r})|^2 = \frac{\mu}{V_0}\left(1 - \sum_{j=x,y,z}\frac{x_j^2}{R_j^2}\right)$$

Assumptions: N vortices in (x, y)-plane, no external potential, vortices separated by distances far exceeding the healing length ξ.
Euler equation:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0$$

Derived from an assumption that the fluid can not be created or destroyed.

Vortex motion is according to classical irrotational ($\mathbf{v} = \nabla S$) incompressible (n =const) flow dynamics

$$\nabla^2 S = 0.$$

For a point vortex at origin $S = s\theta$, so

$$\mathbf{v} = s\nabla\theta = s\frac{\mathbf{e}_z \times \mathbf{x}}{|\mathbf{x}|^2} = s\mathcal{J}\frac{\mathbf{x}}{|\mathbf{x}|^2},$$

where $\mathcal{J}$ denotes rotation through $\pi/2$.

Vortex motion on uniform backgrounds

Assumptions: N vortices in (x, y)-plane, no external potential, vortices separated by distances far exceeding the healing length ξ.
Euler equation:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0$$

Derived from an assumption that the fluid can not be created or destroyed.
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Laplace's equation is linear, so we can linearly superpose a finite number N of point vortices with different strengths and positions $\mathbf{x}_i \quad (i = 1, ...N)$, thus

$$\mathbf{v} = \sum_i s_i \frac{\mathbf{e}_z \times (\mathbf{x} - \mathbf{x}_i)}{|\mathbf{x} - \mathbf{x}_i|^2}.$$

Each vortex is moved by the velocity field due to all the other vortices.
The dynamical system of vortex motion

$$\dot{\mathbf{x}}_i(t) = \sum_{j \neq i} s_j \frac{\mathbf{e}_z \times (\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^2} \qquad (i = 1, ...N) \ .$$

$$s_i \dot{\mathbf{x}}_i(t) = \mathcal{J} \frac{\partial H}{\partial \mathbf{x}_i}; \qquad H = \frac{1}{2} \sum_j \sum_{j \neq i} s_i s_j \ln |\mathbf{x}_i - \mathbf{x}_j|.$$

Two more invariants of motion: the dipole momentum $\mathbf{P}$ and the angular momentum Q

$$\mathbf{P} = \sum_i s_i \mathbf{x}_i, \qquad Q = \sum_i s_i \mathbf{x}_i \cdot \mathbf{x}_i.$$

- An unlike-charged pair with the same winding numbers separated by a distance d propagates normally to their common axis with speed $|s|/d$.
- Method of "images" to deal with boundaries. No mass flux across the boundary implies $\mathbf{v} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the normal to the boundary. This condition is satisfied by placing vortices and removing the boundary.
- Chaotic dynamics with the minimum of four vortices in the infinite plane; three in a half-plane or circle, and two in an arbitrary closed region.

Challenge to understand vortex motion in condensates: nonuniform background, interactions with sound, boundaries etc.

Methods

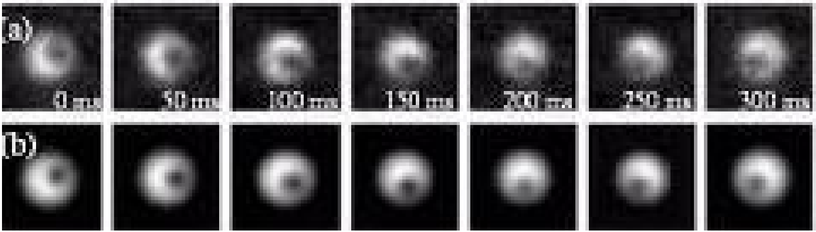
- boundary-layer theory: vortex problem solved exactly in the inner region (close to the core) and in the outer region (far away from the core); the asymptotics of two solutions are matched in the intermediate region.
- variational approach: trial function that depends on some parameters is used to evaluate the Lagrangian of the GP equation; the Euler-Lagrange equations are used to determine the dynamical evolution of parameters.
- Hamiltonian approach: approximation of a vortex solution is used to evaluate $U = \frac{\partial H}{\partial p}$.
- numerical simulations.

Vortex motion in condensates

$$i\hbar\psi_t(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\psi + V_0|\psi|^2\psi + V_{\rm ext}(\mathbf{r})\psi$$

Harmonic trapping potential $V_{\rm ext} = \frac{1}{2}m(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)$

Eric Cornell group (JILA) 2000



Question: how does a vortex move in trapped condensates?

Partial answer: [Rubinstein & Pismen (1994)]

Drift across the density gradient: $\mathbf{v} = \mathbf{v}_s - \mathcal{J}\mathbf{f}\ln\frac{\alpha}{|\mathbf{f}|}, \quad \mathbf{f} = \nabla\ln\rho_0.$

Vortex motion in condensates

- Depleted surface layer induces an effective shift in the position of the image proportional to the integral of the displaced density [Mason, Berloff and Fetter PRA (2006)];
- Global shape of the condensate has an effect on the vortex motion. Vortex velocity locally satisfies $U = \partial H/\partial p$. [Mason and Berloff, PRA (2007)]
- non-equilibrium condensates: vortex motion is subject of many additional effects.

Turbulence

Classical Turbulence

In 50th Batchelor wrote to his friend and close colleague, Alan Townsend, who remained in Australia:

You will come to Cambridge, study turbulence, and work with G. I. Taylor.

The answer came immediately: *I agree, but I have two questions:*
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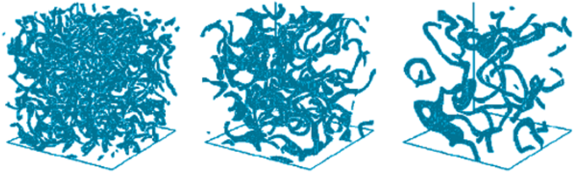
Classical turbulence – cascading vorticity;
Superfluid turbulence – quantisation of velocity circulation – differences with classical turbulence;
Strong turbulence– unstructured vortices (distance between vortices of the order of their core);
Weak turbulence regime – almost independent motion of weakly interacting dispersive waves.

Stages in condensate formation from a nonequilibrium state:
[Berloff & Svistunov Phys Rev A (2002)]
weak turbulence → strong turbulence → superfluid turbulence → condensate



Superfluid turbulence

Superfluid turbulence (tangle of quantised vortices in a superfluid) can be created in a number of ways:

- in the counterflow of normal and superfluid components [Vinen PRSL (1957), Schwarz PRB (1985,1986), Chagovets et al, PRE (2007)];
- by vibrating objects [Davis et al, Physica B (2000), Bradley et al PRL (2006), Hanninen et al PRB (2007), Blaukova et al, PRB (2009)];
- as a result of macroscopic motion of superfluid (quasi-classical turbulence) [Nore et al, PRL (1997), Walmsley et al PRL (2007, 2008), Stalp et al (1999, 2002)];
- by the recently developed technique of ion injection [Walmsley et al, PRL (2008)] ;
- in the process of strongly non-equilibrium Bose-Einstein condensation (Kibble-Zurek effect) [Berloff and Svistunov, PRB (2002), Weiler et al, Nature (2008)];

Counterflow set–up: Vinen's equations, extensive experimental studies and microscopic simulations of vortex line dynamics. Presence of the normal component leads to a simple relaxation mechanism.
Example: Vortex ring. Without normal component radius R of the ring is constant and velocity $U = U(R)$. With the drag force $\dot{R} \sim -\alpha U(R)$, where α is a friction coefficient.
Similarly, normal fluid causes the decay of Kelvin waves – precessing distortions on the vortex filaments – and α^{-1} gives the number of revolutions the distortion makes before its amplitude diminishes.

- $\alpha \sim 1$ Vortex lines reconnect producing Kelvin waves that decay due to drag force rendering the vortex tangle more and more dilute.
- $\alpha \ll 1$ ($\alpha \sim T^5$ as $T \to 0$ [Iordanskii (1966)]) relaxation of turbulence involves number of cascades [Kozik and Svistunov, (2004-2009)]

Superfluid turbulence: normal fluid

Problem: *How to model the normal fluid component?*

Nonlinear Schrödinger (NLS) equation describes evolution of all highly occupied modes $n_{\mathbf{k}} \gg 1$! [Levich and Yakhot, JPA (1978); Kagan and Svistunov, PRL (1997)]

Therefore, NLS equation gives an accurate microscopic description of

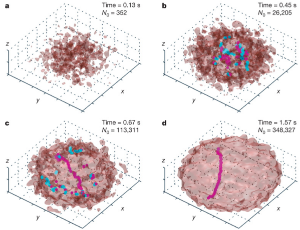
- formation of a BEC from a strongly degenerate gas of weakly interacting bosons;
- interactions of vortex tangle with normal fluid (above-the-condensate modes)

Two theoretical limits of mathematical analysis of NLS:

- "weak turbulence" [Zakharov *et al* (1985), Svistunov (1991)]
  NLS $\to$ Boltzmann kinetic equation $\to$ self-similar solution for motion of quasi-particles from high energies to low energies
- superfluid turbulence [Nore *et al* (1997); Kobayashi & Tsubota (2005)]
  start with *ad hoc* tangle of vortices $\to$ tangle decays $\to$ energy spectrum

NLS can be used to unify these results! [Berloff & Svistunov PRA (2002); Berloff and Youd PRL (2007)]
(1) NLS with initial state as random field: $\psi(\mathbf{x}, t = 0) = \sum_{\mathbf{k}} a_{\mathbf{k}} \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x})$,
(2) Initial evolution according to self-similar solution of Boltzmann kinetic equation on occupation numbers $n_{\mathbf{k}} = |a_{\mathbf{k}}|^2$;
(3) Characteristic time and characteristic wave vector at the beginning of the coherent regime;
(4) Criteria for the number of particles in quasi-condensate;
(5) Formation of vortex tangle;
(6) Decay of superfluid turbulence via interactions with normal fluid; Drag coefficients.

Vortex formed during nonequilibrium kinetics of BEC
[Weiler et al. Nature (2008)]



Reverse the process going from condensate to weak turbulent state?
[Henn at el PRL (2009)]: applied an external oscillatory perturbation to produce vortices.

Shortcomings of the GP equation (when applied to superfluid helium):

- Interactions

GP: only two-body interactions;

superfluid ^{4}He: many-body interactions;

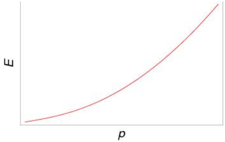
- Compressibility

GP: $c \propto \sqrt{\rho}$;

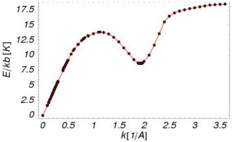
superfluid ^{4}He: $c \propto \rho^{2.8}$;

- Dispersion curves; critical velocities

GP $\omega^2 = c^2 k^2 + \left(\frac{\hbar}{2m}\right)^2 k^4$

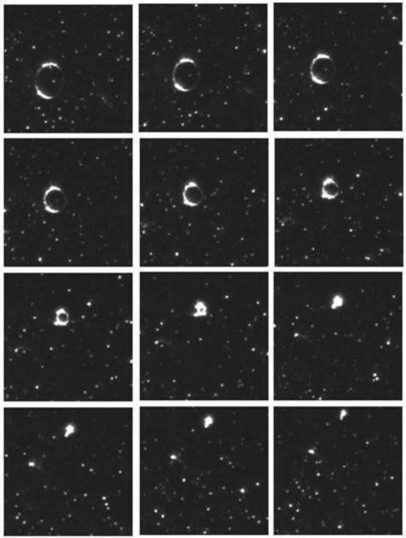


superfluid ^{4}He



- scaling problem; strictly positive pressure in GP

Recent experiments in Yale:



Nonlocal model

Take

$$W_c = \frac{1}{m^2} \int \left[\frac{1}{2} \int \rho(\mathbf{r}') V(\mathbf{r}' - \mathbf{r}) \rho(\mathbf{r})\, d\mathbf{r}' \right]$$

We consider a potential of the form

$$V(|\mathbf{x} - \mathbf{x}'|) = V(r) = (\alpha + \beta A^2 r^2 + \delta A^4 r^4) \exp(-A^2 r^2),$$

where A, B, α, β, and δ are parameters that can be chosen to give excellent agreement with the experimentally determined dispersion curve. The equation replacing the GP equation becomes
[Berloff JLTP (1999), Berloff & Roberts JPA (2000)]

$$i\hbar \frac{\partial \psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + \int V(|\mathbf{r} - \mathbf{r}'|) |\psi(\mathbf{r}', t)|^2 \, d\mathbf{r}' \right] \psi$$

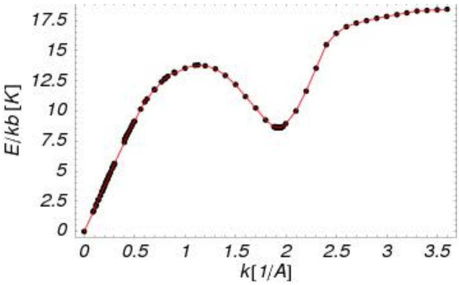
In dimensionless form

$$-2i \frac{\partial \psi}{\partial t} = \nabla^2 \psi + \psi \left[1 - \int |\psi(\mathbf{x}')|^2 V(|\mathbf{x} - \mathbf{x}'|) \, d\mathbf{x}' \right].$$

Dispersion relation

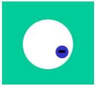
$$\omega^2 = \frac{1}{4}k^4 + 2\pi k \int \sin kr V(r) r \, dr.$$

The parameters α, β and δ of the nonlocal potential are chosen so that the bulk normalization condition is satisfied and the dispersion relation has the position of the roton minimum close to that experimentally observed.



- Roton minimum is not a vanishing vortex ring.
- Vortex rings dynamics.
- Structure of vortex line in agreement with MC simulations.
- Roton emission vs vortex nucleation.

Superfluid helium and Electron bubbles



Electron bubbles — useful experimental probes.

- Rayfield and Reif (1964): Above some critical velocity moving ions produce vortex rings;
- Packard and Sanders (1972): Vortex lines trap electrons;
- Walmsley and Golov (2008): Turbulence by injecting ions;
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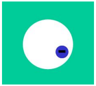

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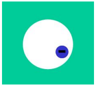

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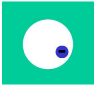

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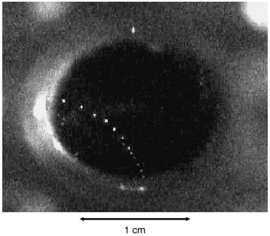

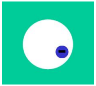

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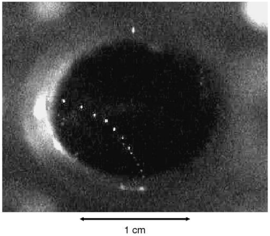

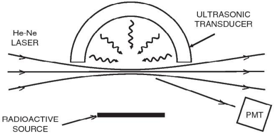

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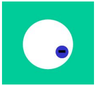

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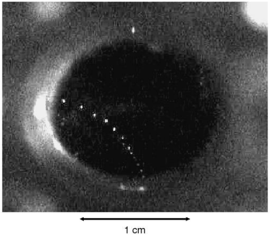
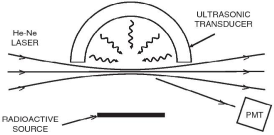
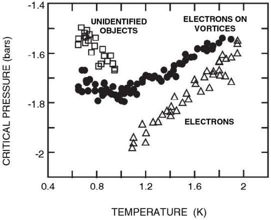






[Berloff and Roberts, JPA (2001); Berloff, FDR (2009)]

Use correct equation of state $H_0/N = -\frac{1}{2}V_0 n - \frac{1}{3}Q_0 n^2 + \frac{1}{4}W_0 n^3$.

Equations on wave function of the condensate, ψ, and the wave function of the impurity, ϕ:

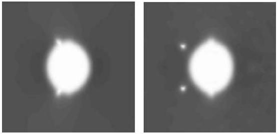
$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + (U_0|\phi|^2 - V_0|\psi|^2 - Q_0|\psi|^4 + W_0|\psi|^6 - E)\psi, \int|\psi|^2 dV = N$$

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2\mu}\nabla^2\phi + (U_0|\psi|^2 - E_e)\phi, \qquad \int|\phi|^2 dV = 1.$$

m – mass of boson; E – single particle energy of boson; μ – mass of electron; E_e – energy of electron; $U_0 = 2\pi l\hbar^2/\mu$ – effective interaction potentials between boson and electron; l – boson-impurity scattering length.

Results on electron bubbles:

- Moving electron experiences no drag below critical velocity; vortex nucleation above critical velocity;



- Electron bubble gets trapped in the vortex core;
- Bubble collapses below critical -2 bar (agrees with experiment);
- Shrinking bubble leads to formation of vortex rings; (compare with collapse of cavity NGB & Barenghi, PRL (2004));
- Transfer of energy between vortex rings allows them to grow and slow down; (compare with NGB, PRA (2004))
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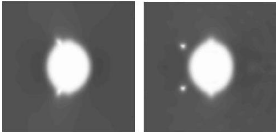

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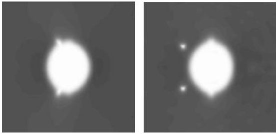

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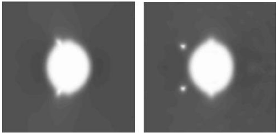

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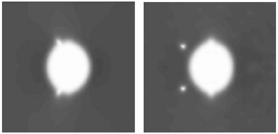

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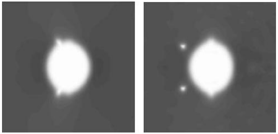

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Proposal: generation of turbulence

Vortex lines $\psi = f(r)\exp[is\theta]$. Near the origin $f(r) \sim a_s r^{|s|}$.

Instability occurs when $a_1 = 0$ and the negative pressure forces can no longer be balanced by the centrifugal energy of the fluid flow.
Criticality corresponds to a critical pressure of -6 bar (agrees with experiments).

Vortex rings and rarefaction pulses.
Energy vs impulse

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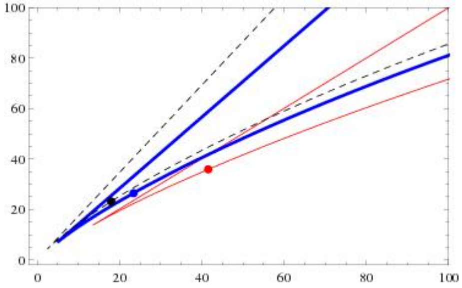

Increasing density of vortex lines

Periodically varying pressure may lead to multiplication of vortices.

Variations of the velocity field around the core $\rightarrow$ vortex core breaks into odd number of vortices (vortex rings) to preserve the total unit of circulation of ± 1 around the initial vortex.

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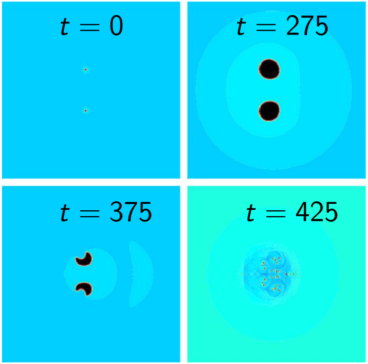
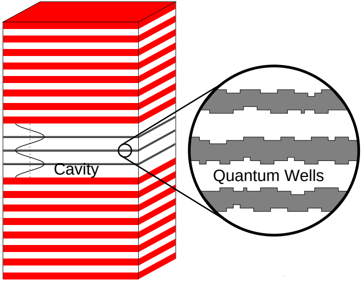


Absorption of photon by semiconductor $\Rightarrow$ exciton $\Rightarrow$ emitting photon $\Rightarrow$ mirrors $\Rightarrow$ exciton photon superposition $\Rightarrow$ polariton $m_{\mathrm{pol}} = 10^{-4} m_e$ $\Rightarrow$

BEC expected at "high" temperature!



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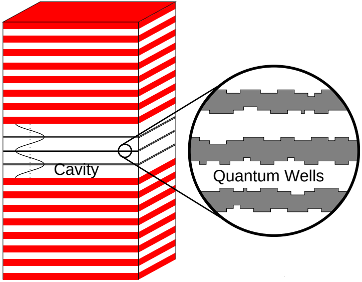
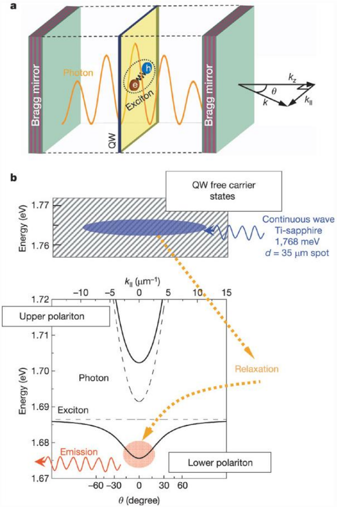




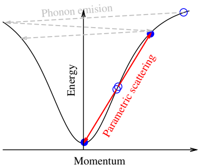
Properties of exciton-polaritons

- **polariton-polariton interactions:**
  interactions between charged particles, saturation of the exciton-photon interactions, electron-electron exchange;
  for low densities pseudo-potential $U(\mathbf{r}) \to U\delta(\mathbf{r})$;
  typical scale of U is 10^{-3} meVμm^2.
- **short lifetime (5-10 ps):**
  (i) non-equilibrium condensate (ii) helps image the properties.
  $ck = E_{\vec{k}}^{\mathrm{LP,UP}} \sin(\theta)$, therefore, refer to polariton momentum, wavevector or emission angle θ interchangeably.
- **two polarisation states:**
  left- and right-circularly polarised photon states;
- **coupling** between mechanical strain in the sample and the energy of electron and hole breaks symmetry and favours a particular linear polarisation.

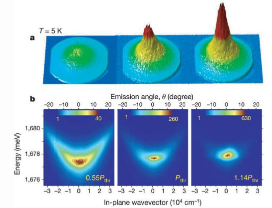
Materials: CdTe or GaAs

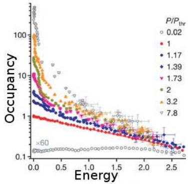
Polariton Injection

- directly creating zero momentum polaritons with a coherent pump laser;
- coherently creating polaritons at a 'magic angle';
- coherently creating polaritons at large angles;
- incoherent pump laser;
- injecting electrons and holes by electric currents.



Momentum distribution and thermalisation





[Kasprzak et al Nature (2006); Deng et al PRL (2006)]:

Non-equilibrium condensation vs lasing

Polariton condensates are non-equilibrium steady states emitting coherent light.

Should they be described as condensates or as lasers?

Criteria:

- (i)**Thermal distribution?** Polariton distribution is set by balance of pumping, decay and relaxation.
  Smooth cross-over between equilibrium BEC, polariton condensate and lasing.
- (ii) **Stimulated scattering into ground state.** Within polariton modes vs stimulated emission of photons in lasers.
- (iii) **Inversion of gain medium in lasers**. Polariton condensation occurs with a quasi-thermal distribution of polaritons. No need for inverted (negative temperature) distribution of gain medium in order for gain to exceed absorption.

[Keeling & NB, PRL, **100**, 250401 (2008)]
Equation for the macroscopically occupied polariton state $\Psi(\mathbf{r}, t)$:

$$i\hbar\partial_t\Psi = \left[E(i\nabla) + U|\Psi|^2 + V(\mathbf{r})\right]\Psi + i\left[P_{\mathrm{coh}}(\mathbf{r}, t) + \left(P_{\mathrm{inc}}(\mathbf{r}) - \kappa - \sigma|\Psi|^2\right)\Psi\right.$$

Polariton dispersion, $E(k)$ (eg. a quadratic dispersion $E(k) \simeq \hbar^2 k^2/2m_{\mathrm{pol}}$);
Strength of the $\delta-$function interaction (pseudo)potential U;
External potential $V(\mathbf{r})$;
Coherent pump field $P_{\mathrm{coh}}(\mathbf{r})e^{i\omega_p t}$;
Incoherent pump field $P_{\mathrm{inc}}(\mathbf{r})$;
κ and σ describe linear and nonlinear losses respectively.

cf. "generic laser model" of Wouters and Carusotto PRA (2007)

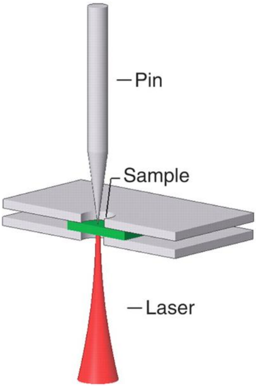
Superfluidity checklist

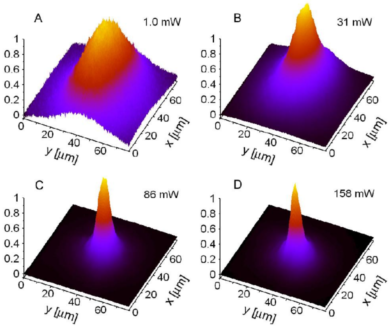
Table 1 | Superfluidity checklist

	Quantized vortices	Landau critical velocity	Metastable persistent flow	Two-fluid hydro-dynamics	Local thermal equilibrium	Solitary waves
Superfluid ^{4}He/cold atom Bose-Einstein condensate	✓	✓	✓	✓	✓	✓
Non-interacting Bose-Einstein condensate	✓	X	X	X	✓	X
Classical irrotational fluid	X	✓	X	✓	✓	✓
Incoherently pumped polariton condensates	✓	X	?	?	X	?

[J. Keeling and NB, Nature (2009)]

[Balili et al Science **316**,(2007)]:
A harmonic trapping potential is created by squeezing the sample by a sharp pin.





Signatures of BEC:
spatial and spectral narrowing; coherence

Gross-Pitaevskii equation with loss and gain

Mean-field model of a non-equilibrium BEC of exciton-polaritons

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\rm ext} + U|\psi|^2 + i(\gamma_{\sf net} - \Gamma|\psi|^2)\right]\psi,$$

$V_{\rm ext}$ is an external trapping potential, $= \frac{1}{2}m\omega^2 r^2$, $\gamma_{\rm net}$– net gain, Γ – effective loss, U – effective (pseudo-) interaction potential.

Length in units of oscillator length $\sqrt{\hbar/m\omega}$, energies in units of $\hbar\omega$, and $\psi \to \sqrt{\hbar\omega/2U}\psi$, yields:

$$i\partial_t\psi = \left[-\nabla^2 + r^2 + |\psi|^2 + i\left(\alpha - \sigma|\psi|^2\right)\right]\psi.$$

Two parameters: $\alpha = 2\gamma_{\sf net}/\hbar\omega$ (gain), and $\sigma = \Gamma/U$ (loss).
Estimate from experiments: $0 \le \alpha \le 10$ and $\sigma \sim 0.3$

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Radially symmetric stationary states

$$\mu\psi = \left[\, -\nabla^2 + r^2 + |\psi|^2 + i\left(\alpha - \sigma|\psi|^2\right) \,\right]\psi$$

α not too large, Thomas-Fermi solution $|\psi|^2 = (\mu - r^2)$ for $r < r_{TF} = \sqrt{\mu}$

$\int d^2 r\, (\alpha - \sigma|\psi|^2)\, |\psi|^2 = 0 \Rightarrow \mu = 3\alpha/2\sigma.$

Madelung transformation, $\psi = \sqrt{\rho}e^{i\varphi}$:

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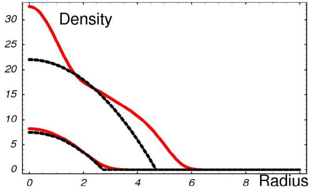

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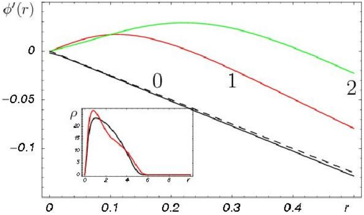

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Leading order
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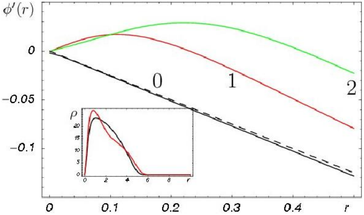

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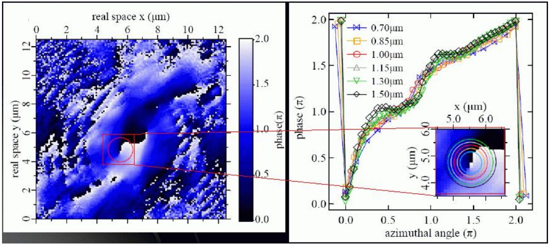
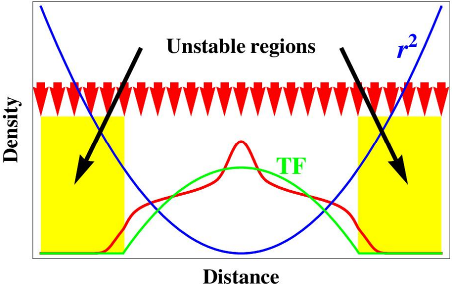
Experiment:

[Lagoudakis et al. Nature Physics (2008)]





$$\frac{1}{2}\partial_t\rho + \nabla\cdot[\rho\mathbf{v}] = (\alpha - \sigma\rho)\,\rho, \qquad \partial_t\mathbf{v} + \nabla(\rho + r^2 + |\mathbf{v}|^2) = 0$$

If α, σ small, find normal modes in 2D trap: $\delta\rho_{n,m} = e^{im\theta}h_{n,m}(r)e^{i\omega_{n,m}t}$
$\omega_{n,m} = 2\sqrt{m(1+2n)+2n(n+1)}$.
Add weak pumping and decay

$$\omega_{n,m} \to \omega_{n,m} + i\alpha\left[\frac{m(1+2n)+2n(n+1)-m^2}{2m(1+2n)+4n(n+1)+m^2}\right]$$

Finite Spot Size

In experiments: finite spot, of size comparable to observed cloud, is used.

Model this as $\alpha = \alpha(r) = \alpha\,\Theta(r_0 - r)$

For small r_0 ($r_0 < r_{TF} \sim \sqrt{3\alpha/2\sigma}$), this stabilises the radially symmetric modes and vortex modes:

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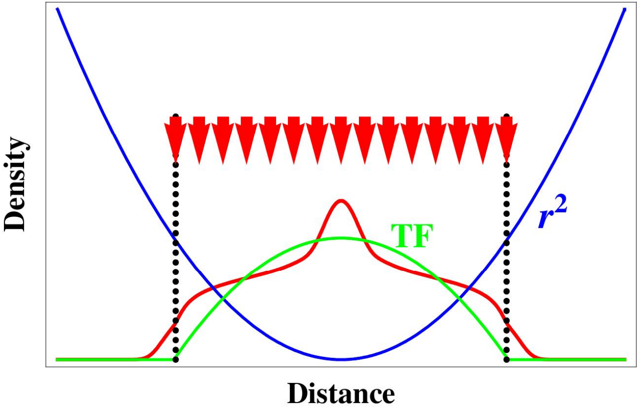
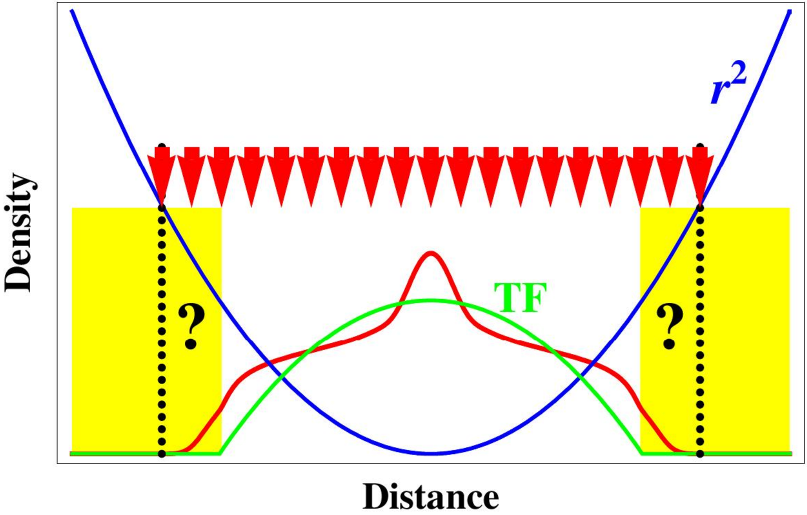
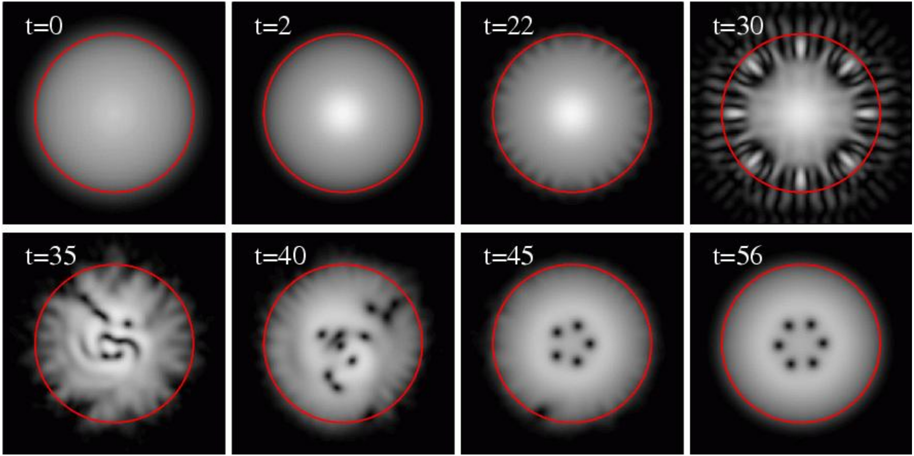
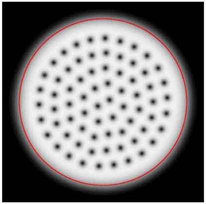


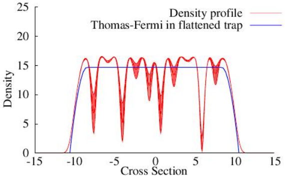
Development of instability?



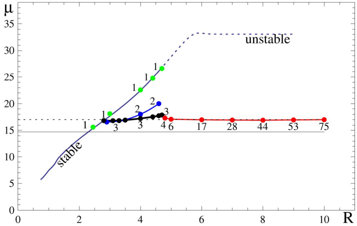
Vortex Lattices

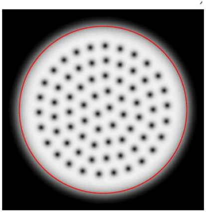


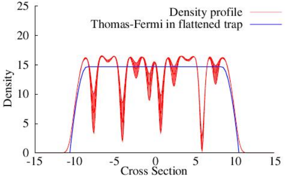




Stationary $\mu \sim 3\alpha/2\sigma$; Vortex lattice $\mu \sim \alpha/\sigma$







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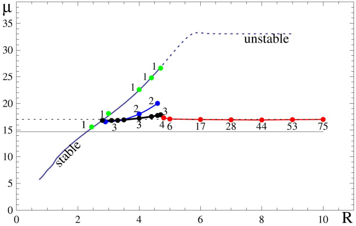

In rotating frame

$$\nabla \cdot [\rho(\nabla\phi - \Omega \times \mathbf{r})] = (\alpha\Theta(r_0 - r) - \sigma\rho)\,\rho,$$

$$\mu = |\nabla\phi - \Omega \times \mathbf{r}|^2 + r^2(1 - \Omega^2) + \rho - \frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}.$$

In TF regime away from boundaries solution is
$\nabla\phi = \Omega \times \mathbf{r} + v.c., \rho = \alpha/\sigma = \mu, \Omega^2 = 1.$

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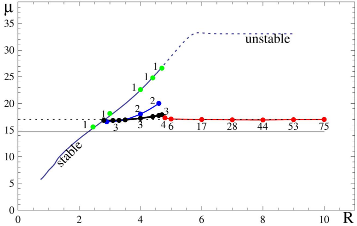

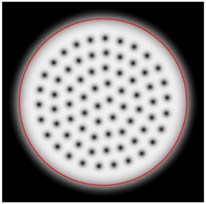

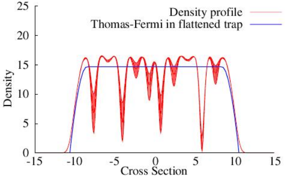

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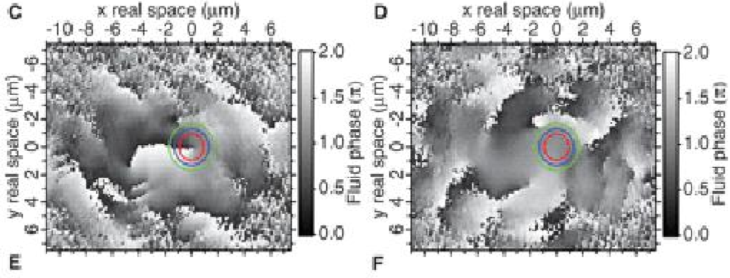
Results so far do not involve polariton spin:
[Lagoudakis et al, Science, November 2009]:

Phase maps of left- and right-circular polarized polariton states



Observed all possible $(\pm 1, \pm 1)$ vortex states.

Polariton spin degree of freedom

[Borgh, Keeling, NB, PRB, **81**, 235302 (2010)]

- Include spin degree of freedom: left- and right-circular polariton states ψ_L and ψ_R.
- For weakly-interacting dilute Bose gas model:

$$H = \frac{\hbar^2|\nabla\psi_L|^2}{2m} + \frac{\hbar^2|\nabla\psi_R|^2}{2m} + \frac{U_0}{2}\left(|\psi_L|^2 + |\psi_R|^2\right)^2$$

- Tendency to biexciton formation $\to U_1$. Magnetic field: Ω_B
- J_2 Circular symmetry broken – two equivalent axes.
  J_1 preferred direction – inequivalent axes.

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Non-equilibrium spinor system

Spinor Gross-Pitaevskii equation:

$$i\hbar\partial_t\psi_L = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(r) + \frac{\Omega_B}{2} + U_0|\psi_L|^2 + (U_0 - 2U_1)|\psi_R|^2 + i\left(\gamma_{\text{net}} - \Gamma|\psi_L|^2\right)\right]\psi_L + J_1\psi_R$$

Similarly for ψ_R with $\psi_L \leftrightarrow \psi_R$ and $\Omega_B \to -\Omega_B$.

Dimensionless cGPE:

$$i\partial_t\psi_L = \left[-\nabla^2 + v(r) + |\psi_L|^2 + (1-u_a)|\psi_R|^2 + \frac{\Delta}{2} + i\left(\alpha - \sigma|\psi_L|^2\right)\right]\psi_L + J\psi_R.$$

If $v(r) = r^2$ then take $\alpha \to \alpha\Theta(r_0 - r)$ as before.

Questions:

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[J. Keeling and NB, arXiv:1102.5302 (2011)]

Vortex patterns generated by superposition of fluxes.

Spinor complex Ginzburg-Landau equation:

$$2i\partial_t\psi_{l,r} = \left[\pm\frac{\Delta}{2} - \nabla^2 + v(r) + |\psi_{l,r}|^2 + (1-u_a)|\psi_{r,l}|^2 \right.$$
$$\left. + i\left(\alpha - 2i\eta\partial_t - \sigma|\psi_{l,r}|^2 - \tau|\psi_{r,l}|^2\right)\right]\psi_{l,r} + J\psi_{r,l}.$$

η – energy relaxation [Wouters and Savona arXiv:1007.5431 (2010)];

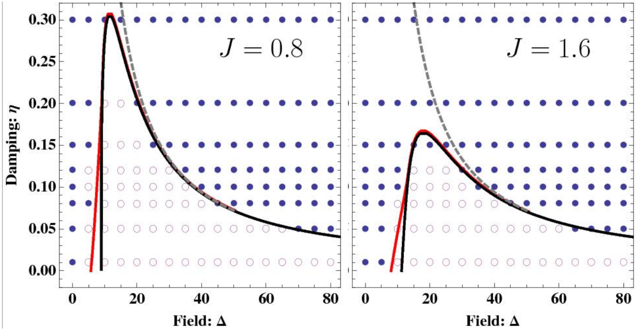
τ – cross-spin nonlinear dissipation;

Δ – effect of the magnetic fied;

J – electric field, stress or due to asymmetry of quantum well interfaces;

Parameters estimated from [Larionov et al, PRL, **105**, 256401 (2010)]

For nonzero η there is a second transition at Δ_{c2} back to synchronized state, $\Delta_{c2} \simeq (2\alpha/\eta)(\sigma - \tau + \eta u_a)/(\sigma + \tau + \eta(2 - u_a))$ (dashed line)



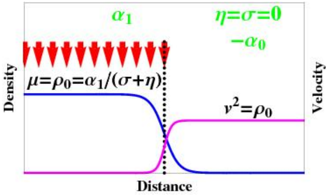
• –synchronized states (vortex-free states or synchronized vortices);
○– desynchronized states (vortices of opposite sign for l and r).
Conclude: homogeneous model gives good prediction of spatially varying system.

Vortex formation in equilibrium condensates:

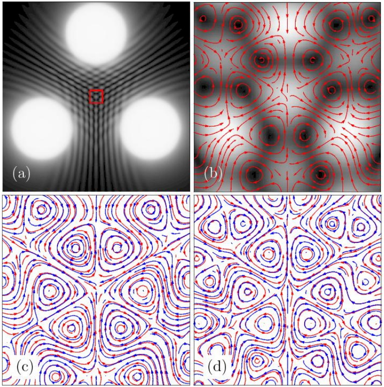
- interactions of finite amplitude sound waves;
- existence of critical velocities of the flow;
- modulational instabilities.

In addition in nonequilibrium condensates – pattern forming, interaction of fluxes with a disorder etc.

Vortex formation due to interference of supercurrents



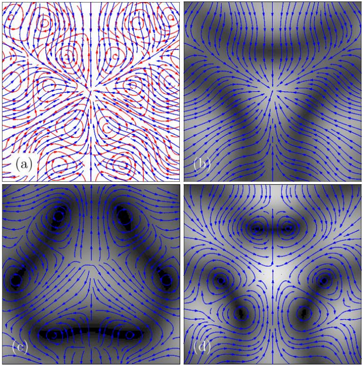
Analytical solution for the velocity $u(r)$ on $\infty < r < \infty$.



(a) $\Delta = 0$ showing geometry of pumping;
(b) Desynchronized $\Delta = 20$ steady majority density with streamlines;
(c) Lower synchronized $\Delta = 5$ steamlines of both polarizations;
(d) Upper synchronized $\Delta = 40$ steamlines of both polarizations.

Half-vortices

"Half-vortices" have been seen in experiments:
[Lagoudakis et al Nature Phys. (2008)]
Are "half-vortices" pinned and stabilized by disorder?



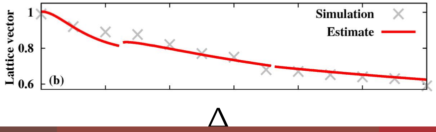
(a) Desyncronized $\Delta = 20$ half-vortex lattice;
(b) -(c) -(d) evolution of minority component in desyncronized regime $\Delta = 20$.

Majority component is stationary in both regimes;
Minority component is stationary in syncronized regime only.
In desyncronized regime averages to vortex-free state.

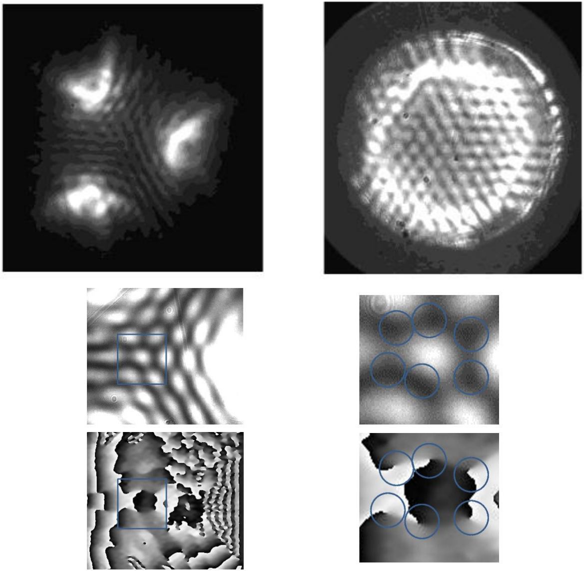
Currents are neglible at the pumping centre, $\mu(\rho_{l,r})$;
away from pumping spot – densities are neglible.

Synchronized regime: away from the pump
$\mu - |\vec{u}|^2 \mp \Delta/2 = J\left(\rho_l/\rho_r\right)^{\mp 1/2}\cos(\theta)$ and
$\nabla \cdot (\rho_{l,r}\vec{u}) + \alpha_1\rho_{l,r} = \mp J\sqrt{\rho_l\rho_r}\sin(\theta)$.
These are solved by $\sin(\theta) = 0$ and $\nabla(\rho_l/\rho_r) = 0$,
so $|\vec{u}|^2 = \mu + \sqrt{J^2 + \Delta^2/4}$.

Desynchronized regime: θ and ρ_l/ρ_r are not time independent, so one calculates averages. If $\rho_r \gg \rho_l$, then for majority component
$\langle|\vec{u}_r|^2\rangle = \langle\mu_r\rangle + \Delta/2$.
Superposition of such currents results in hexagonal vortex lattice with spacing $l = (2\pi/|\vec{u}|) \times 2/3\sqrt{3}$.



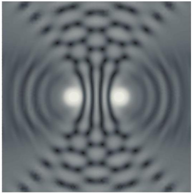
Experimental realization: Cavendish, UK

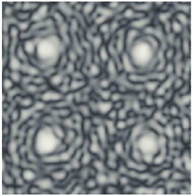


[N.G.Berloff, arXiv:1010.5225 (2010)]

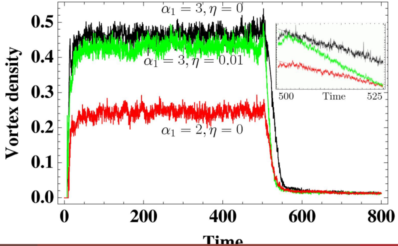
Regular emission of vortices

Many irregular spots: turbulence

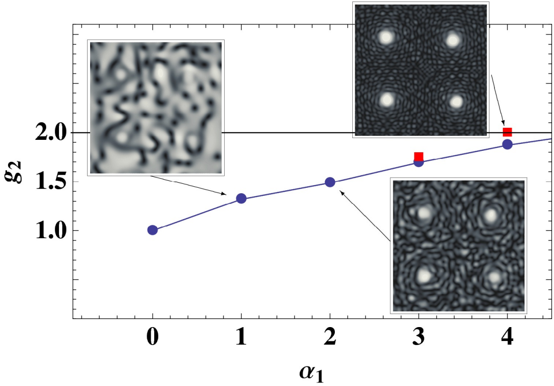




Two regimes: forced turbulence and turbulence decay.



In forced turbulence it is possible to reach a weak turbulence state: $g_2 = \langle|\psi|^4\rangle/\langle|\psi|^2\rangle^2$. Weak turbulence implies $g_2 \sim 2$.



Red Squares – nonzero η facilitates the transition to weak turbulence.

- Modelling superfluid helium using GP-like equations with a correct equation of state

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + (U_0|\phi|^2 - V_0|\psi|^2 - Q_0|\psi|^4 + W_0|\psi|^6 - E)\psi,$$

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2\mu}\nabla^2\phi + (U_0|\psi|^2 - E_e)\phi, \qquad \int|\phi|^2 dV = 1.$$

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- Nonequilibrium condensates: condensates made of light
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$$\mathrm{i}\partial_t\psi = \left[-\nabla^2 + r^2 + |\psi|^2 + i\left(\alpha\Theta(r_0 - r) - \sigma|\psi|^2\right)\right]\psi.$$

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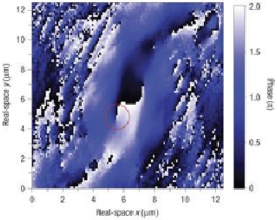

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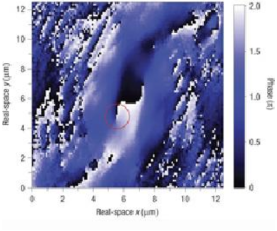

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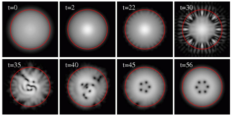

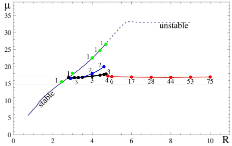

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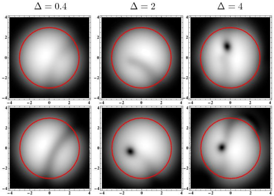
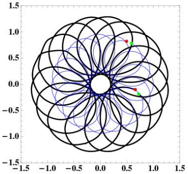
Densities of L and R components for $J = 1$



Trajectories for $\Delta = 4$



Spirographs (epitrochoids/hypotrochoid)

- Synchronization/desynchronization with the region of bistability.

Conclusions

- Turbulence in exciton-polariton condensates may lead to novel regimes of turbulence of classical matter field.
  - The regimes can be distinguished by finding second order correlation function; by looking at the wave spectrum.
  - What are the stages in transition from strong turbululence to weak turbulence and back?
- Spinor condensates: predictions of homogeneous model (syncronization/desynchronization) are not significantly modified by spatial inhomogeneity.
  - Observation of the experimental behaviour in an applied field can thus be used to distinguish the loss nonlinearities σ, τ and η.
  - Vortices, vortex lattices and half-vortex latices in spinor condensates. Being stationary these textures can be studied experimentally.
- Turbulence in spinor condensates.

Scaling laws? Cross-overs of different regimes? Interplay between turbulent regimes and the effects of magnetic field?...