Spatial pattern formation in non-equilibrium condensates

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Outline

- Introduction: Exciton–polariton condensates
- Gross-Pitaevskii equation with loss and gain
  - Radially symmetric stationary states
  - Spiral vortex states
  - Vortex lattices
- Non-equilibrium spinor system: interplay between interconversion and detuning
  - Stability of cross–polarized vortices
  - Synchronisation/desynchronisation
- Controllable half-vortex lattices
- Turbulence in nonequilibrium condensates

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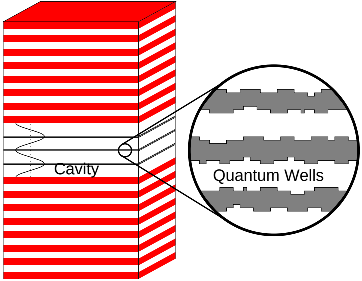
Nonequilibrium condensates: condensates made of light

Absorption of photon by semiconductor ⇒ exciton ⇒ emitting photon ⇒ mirrors ⇒ exciton photon superposition ⇒ polariton $m_{\mathrm{pol}} = 10^{-4} m_e$ ⇒

BEC expected at "high" temperature!



Nonequilibrium condensates: condensates made of light

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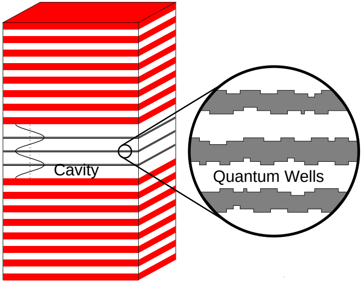

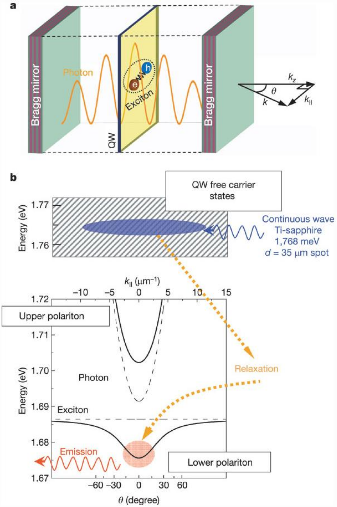

Lower and Upper polariton branches

Polariton frequency $\omega_{\vec{k}} = (c/n)\sqrt{k^2 + (2\pi N/L_w)^2}$
n is the refractive index, c the speed of light in vacuum, and N labels the transverse mode in a cavity of transverse size L_w.
For small k $\hbar\omega_{\vec{k}} = \hbar\omega_0 + \hbar^2 k^2/2m$ with photon mass $m = \hbar(n/c)(2\pi N/L_w)$.

$$i\hbar\partial_t \begin{pmatrix} \Psi_{\text{phot}} \\ \Psi_{\text{ex}} \end{pmatrix} = \begin{pmatrix} \hbar\omega_k & \frac{1}{2}g \\ \frac{1}{2}g & \varepsilon \end{pmatrix} \begin{pmatrix} \Psi_{\text{phot}} \\ \Psi_{\text{ex}} \end{pmatrix}$$

Eigenstates

$$E_{\vec{k}}^{\mathrm{LP,UP}} = \frac{1}{2}\left[\left(\hbar\omega_0 + \varepsilon + \frac{\hbar^2 k^2}{2m}\right) \mp \sqrt{\left(\hbar\omega_0 - \varepsilon + \frac{\hbar^2 k^2}{2m}\right)^2 + g^2}\right]$$

Direct observation of spectrum: transmission and reflection of the microcavity as a function of energy and incident angle.

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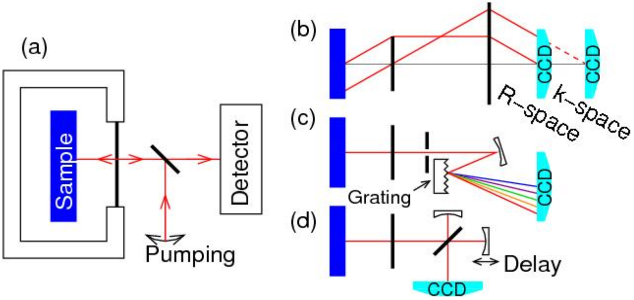
Properties of exciton-polaritons

- **polariton-polariton interactions:**
  interactions between charged particles, saturation of the exciton-photon interactions, electron-electron exchange;
  for low densities pseudo-potential $U(\mathbf{r}) \to U\delta(\mathbf{r})$;
  typical scale of U is 10^{-3} meVμm^2.
- **short lifetime (5-10 ps):**
  (i) non-equilibrium condensate (ii) helps image the properties.
  $ck = E_{\vec{k}}^{\mathrm{LP,UP}} \sin(\theta)$, therefore, refer to polariton momentum, wavevector or emission angle θ interchangeably.
- **two polarisation states:**
  left- and right-circularly polarised photon states;
- **coupling** between mechanical strain in the sample and the energy of electron and hole breaks symmetry and favours a particular linear polarisation.

Schematic of an experiment studying polaritons:



Detector measures:

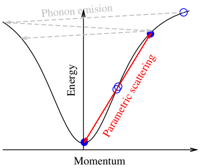
- (b) the real and momentum space images;
- (c) energy resolved images using a spectrometer;
- (d) first-order coherence using an interferometer.

From M.Richard, PhD Thesis, Universite Joseph Fourier, Grenoble, 2004.
http://tel.archives-ouvertes.fr/tel-00009088/fr

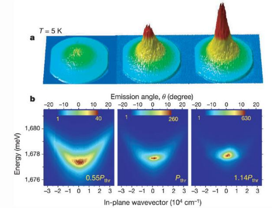
Materials: CdTe or GaAs

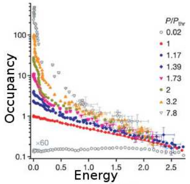
Polariton Injection

- directly creating zero momentum polaritons with a coherent pump laser;
- coherently creating polaritons at a 'magic angle';
- coherently creating polaritons at large angles;
- incoherent pump laser;
- injecting electrons and holes by electric currents.



Momentum distribution and thermalisation





[Kasprzak et al Nature (2006); Deng et al PRL (2006)]:

Non-equilibrium condensation vs lasing

Polariton condensates are non-equilibrium steady states emitting coherent light.

Should they be described as condensates or as lasers?

Criteria:

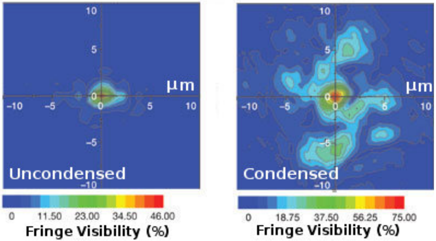
- (i)**Thermal distribution?** Polariton distribution is set by balance of pumping, decay and relaxation.
  Smooth cross-over between equilibrium BEC, polariton condensate and lasing.
- (ii) **Stimulated scattering into ground state.** Within polariton modes vs stimulated emission of photons in lasers.
- (iii) **Inversion of gain medium in lasers**. Polariton condensation occurs with a quasi-thermal distribution of polaritons. No need for inverted (negative temperature) distribution of gain medium in order for gain to exceed absorption.

The first and second order correlation functions of the electromagnetic field:

$$g_1(\vec{r},\vec{r}',t,t') = \frac{\langle E^*(\vec{r}',t')E(\vec{r},t)\rangle}{\sqrt{\langle E(\vec{r}',t')^2\rangle\langle E(\vec{r},t)^2\rangle}},$$

$$g_2(\vec{r},\vec{r}',t,t') = \frac{\langle E^*(\vec{r}',t')E^*(\vec{r},t)E(\vec{r},t)E(\vec{r}',t')\rangle}{\langle E(\vec{r}',t')^2\rangle\langle E(\vec{r},t)^2\rangle}.$$

- Temporal coherence $g_1(\tau) = g_1(\vec{r},\vec{r},t+\tau,t)$;
- Spatial coherence $g_1(|r|) = g_1(\mathbf{r_0}+\mathbf{r},\mathbf{r_0},t,t)$.



The complex Ginzburg-Landau equation:

$$i\partial_t \psi = c_1 \nabla^2 \psi + c_2 |\psi|^2 \psi + c_3 \psi.$$

Gross-Pitaevskii equation as a non-relativistic limit of the Klein-Gordon equation–the simplest equation consistent with special relativity.

$$\frac{\partial^2 \Psi}{\partial t^2} = c^2 \nabla^2 \Psi - \lambda^2 \Psi$$

Represent $\Psi = \psi \exp[\mp i \lambda t]$ for matter and anti-matter solutions.

$$-\lambda^2 \psi - 2i\lambda \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial t^2} = c^2 \nabla^2 \psi - \lambda^2 \psi$$

Non-relativistic limit $\left| \frac{\partial^2 \psi}{\partial t^2} \right| \ll \lambda \left| \frac{\partial \psi}{\partial t} \right|$

Gross-Pitaevskii equation

$$i \frac{\partial \psi}{\partial t} = -\frac{c^2}{2\lambda} \nabla^2 \psi$$

Modelling non-equilibrium condensates

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Gross-Pitaevskii equation

$$i \frac{\partial \psi}{\partial t} = -\frac{c^2}{2\lambda} \nabla^2 \psi + A |\psi|^2 \psi.$$

Equation for the macroscopically occupied polariton state $\Psi(\mathbf{r}, t)$:

$$i\hbar\partial_t\Psi = \left[E(i\nabla) + U|\Psi|^2 + V(\mathbf{r})\right]\Psi + i\left[P_{\mathrm{coh}}(\mathbf{r}, t) + \left(P_{\mathrm{inc}}(\mathbf{r}) - \kappa - \sigma|\Psi|^2\right)\Psi\right]$$

Polariton dispersion, $E(k)$ (eg. a quadratic dispersion $E(k) \simeq \hbar^2 k^2/2m_{\mathrm{pol}}$);
Strength of the $\delta-$function interaction (pseudo)potential U;
External potential $V(\mathbf{r})$;
Coherent pump field $P_{\mathrm{coh}}(\mathbf{r})e^{i\omega_p t}$;
Incoherent pump field $P_{\mathrm{inc}}(\mathbf{r})$;
κ and σ describe linear and nonlinear losses respectively.

cf. "generic laser model" of Wouters and Carusotto PRA (2007)

Bogoliubov spectrum comes from considering fluctuations of the form $\Psi(\vec{r},t) = e^{-i\mu t/\hbar}\left(\Psi_0 + \sum_k u_k e^{-i\xi_k t + i\vec{k}\cdot\vec{r}} + v_k e^{i\xi_k^* t - i\vec{k}\cdot\vec{r}}\right)$, and finding a self consistent set of equations for u_k, v_k and the frequency ξ_k.

Spectrum of non-equilibrium system $\hbar\xi_k \simeq -i\hbar\eta + \sqrt{\mu\hbar^2 k^2/m_{\mathrm{pol}} - \hbar^2\eta^2}$ for small k.

η is a characteristic size of the pump rate, e.g. $\eta \simeq P_{\mathrm{inc}} - \kappa$.

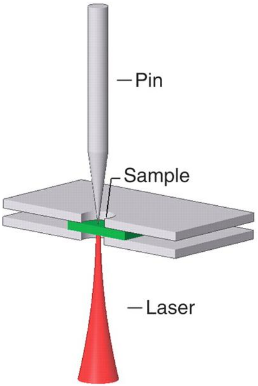
For small k, the real part of the spectrum is zero for $k < \eta\sqrt{m_{\mathrm{pol}}/\mu}$.

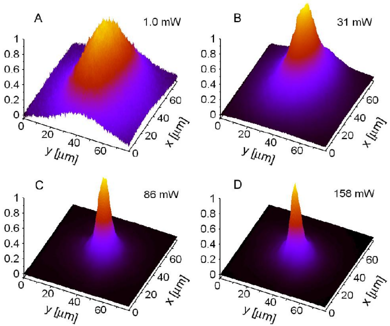
No superfluidity in non-equilibrium condensates?

Table 1 | Superfluidity checklist

	Quantized vortices	Landau critical velocity	Metastable persistent flow	Two-fluid hydro-dynamics	Local thermal equilibrium	Solitary waves
Superfluid ^{4}He/cold atom Bose-Einstein condensate	✓	✓	✓	✓	✓	✓
Non-interacting Bose-Einstein condensate	✓	X	X	X	✓	X
Classical irrotational fluid	X	✓	X	✓	✓	✓
Incoherently pumped polariton condensates	✓	X	?	?	X	?

[Balili et al Science **316**,(2007)]:
A harmonic trapping potential is created by squeezing the sample by a sharp pin.





Signatures of BEC:
spatial and spectral narrowing; coherence

Mean-field model of a non-equilibrium BEC of exciton-polaritons

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\mathrm{ext}} + U|\psi|^2 + i(\gamma_{\mathsf{net}} - \Gamma|\psi|^2)\right]\psi,$$

V_{ext} is an external trapping potential, $= \frac{1}{2}m\omega^2 r^2$, γ_{net}– net gain, Γ – effective loss, U – effective (pseudo-) interaction potential.

Length in units of oscillator length $\sqrt{\hbar/m\omega}$, energies in units of $\hbar\omega$, and $\psi \to \sqrt{\hbar\omega/2U}\psi$, yields:

$$i\partial_t\psi = \left[-\nabla^2 + r^2 + |\psi|^2 + i\left(\alpha - \sigma|\psi|^2\right)\right]\psi.$$

Two parameters: $\alpha = 2\gamma_{\mathsf{net}}/\hbar\omega$ (gain), and $\sigma = \Gamma/U$ (loss).
Estimate from experiments: $0 \leq \alpha \leq 10$ and $\sigma \sim 0.3$

Gross-Pitaevskii equation with loss and gain

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Radially symmetric stationary states

$$\mu\psi = \left[-\nabla^2 + r^2 + |\psi|^2 + i\left(\alpha - \sigma|\psi|^2\right) \right]\psi$$

α not too large, Thomas-Fermi solution $|\psi|^2 = (\mu - r^2)$ for $r < r_{TF} = \sqrt{\mu}$
$\int d^2r \left(\alpha - \sigma|\psi|^2\right)|\psi|^2 = 0 \Rightarrow \mu = 3\alpha/2\sigma$.
Madelung transformation, $\psi = \sqrt{\rho}e^{i\phi}$:

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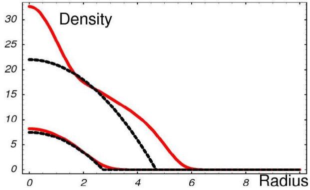

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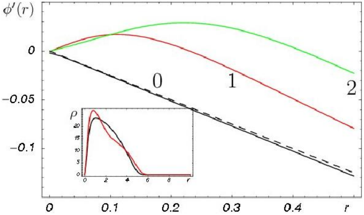

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Leading order
$\phi'(r) \sim \alpha/2(s+1)r$.

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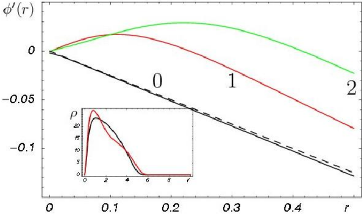
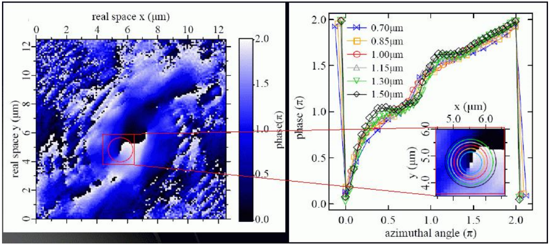


Experiment:

[Lagoudakis et al. Nature Physics (2008)]



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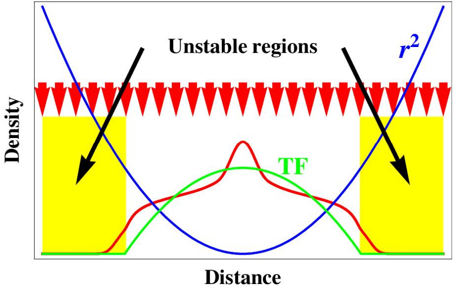
Instability of rotationally symmetric states



$$\frac{1}{2}\partial_t \rho + \nabla \cdot [\rho \mathbf{v}] = (\alpha - \sigma\rho)\,\rho, \qquad \partial_t \mathbf{v} + \nabla(\rho + r^2 + |\mathbf{v}|^2) = 0$$

If α, σ small, find normal modes in 2D trap: $\delta\rho_{n,m} = e^{im\theta} h_{n,m}(r) e^{i\omega_{n,m} t}$
$\omega_{n,m} = 2\sqrt{m(1+2n) + 2n(n+1)}$.
Add weak pumping and decay

$$\omega_{n,m} \to \omega_{n,m} + i\alpha\left[\frac{m(1+2n) + 2n(n+1) - m^2}{2m(1+2n) + 4n(n+1) + m^2}\right]$$

Finite Spot Size

In experiments: finite spot, of size comparable to observed cloud, is used.

Model this as $\alpha = \alpha(r) = \alpha\,\Theta(r_0 - r)$

For small r_0 ($r_0 < r_{TF} \sim \sqrt{3\alpha/2\sigma}$), this stabilises the radially symmetric modes and vortex modes:

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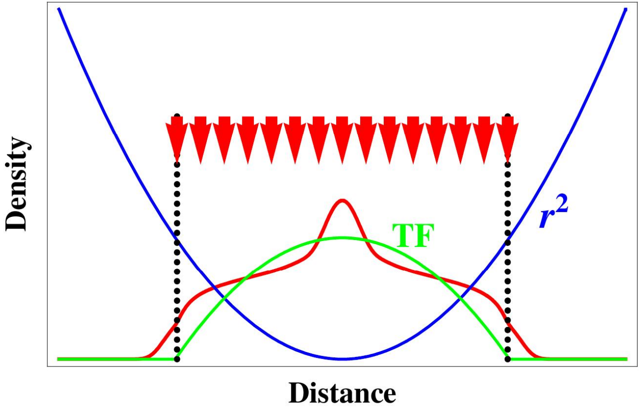
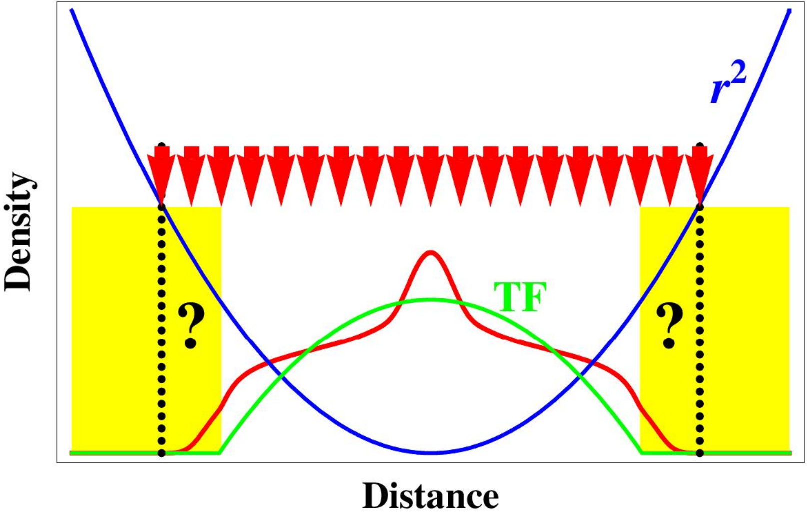
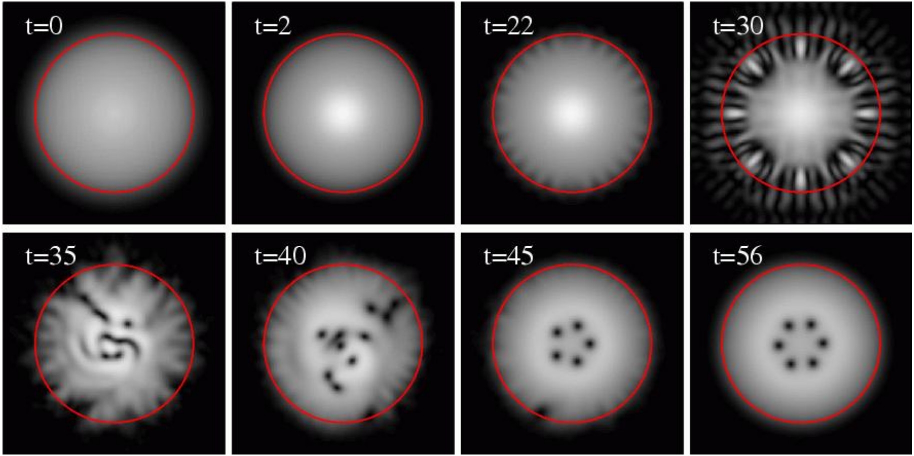


Development of instability?

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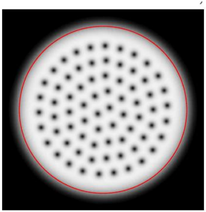
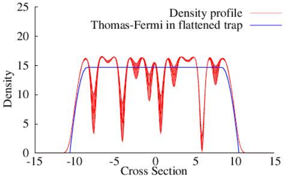
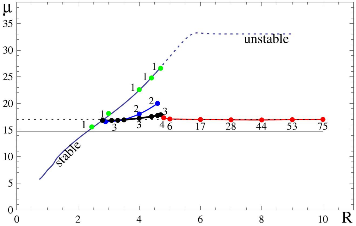
t=2
t=22
t=30
t=35
t=40
t=45
t=56





Stationary $\mu \sim 3\alpha/2\sigma$; Vortex lattice $\mu \sim \alpha/\sigma$



In rotating frame

$$\nabla \cdot [\rho(\nabla\phi - \Omega \times \mathbf{r})] = (\alpha \qquad - \sigma\rho)\,\rho,$$

$$\mu = |\nabla\phi - \Omega \times \mathbf{r}|^2 + r^2(1 - \Omega^2) + \rho \qquad .$$

In TF regime away from boundaries solution is $\nabla\phi = \Omega \times \mathbf{r}, \rho = \alpha/\sigma = \mu, \Omega^2 = 1.$

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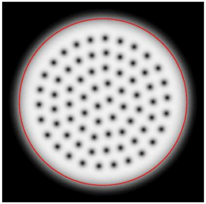

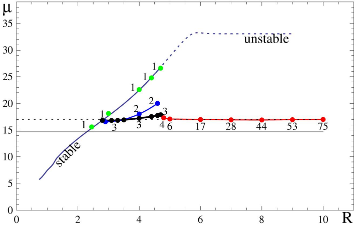

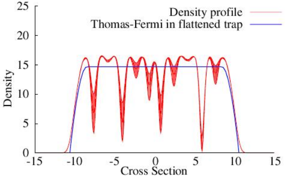

In rotating frame

$$\nabla \cdot [\rho(\nabla\phi - \Omega \times \mathbf{r})] = (\alpha\Theta(r_0 - r) - \sigma\rho)\,\rho,$$

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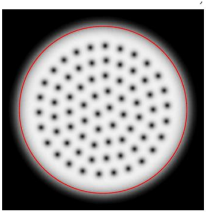

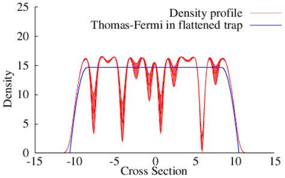

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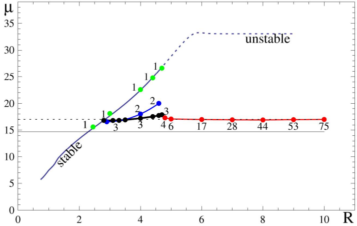

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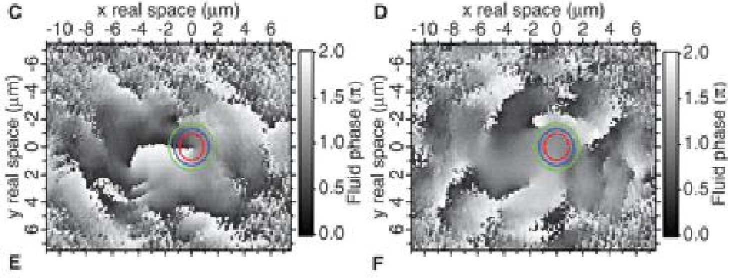
Results so far do not involve polariton spin:
[Lagoudakis et al, Science, November 2009]:

Phase maps of left- and right-circular polarized polariton states



Observed all possible $(\pm 1, \pm 1)$ vortex states.

Polariton spin degree of freedom

- Include spin degree of freedom: left- and right-circular polariton states ψ_L and ψ_R.
- For weakly-interacting dilute Bose gas model:

$$H = \frac{\hbar^2|\nabla\psi_L|^2}{2m} + \frac{\hbar^2|\nabla\psi_R|^2}{2m} + \frac{U_0}{2}\left(|\psi_L|^2 + |\psi_R|^2\right)^2$$

- Tendency to biexciton formation $\rightarrow U_1$. Magnetic field: Ω_B
- J_2 Circular symmetry broken – two equivalent axes.
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Non-equilibrium spinor system

Spinor Gross-Pitaevskii equation:

$$i\hbar\partial_t\psi_L = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\mathrm{ext}}(r) + \frac{\Omega_B}{2} + U_0|\psi_L|^2 + (U_0 - 2U_1)|\psi_R|^2 \right.$$
$$\left. + i\left(\gamma_{\mathrm{net}} - \Gamma|\psi_L|^2\right)\right]\psi_L + J_1\psi_R$$

Similarly for ψ_R with $\psi_L \leftrightarrow \psi_R$ and $\Omega_B \to -\Omega_B$.

Dimensionless cGPE:

$$i\partial_t\psi_L = \left[-\nabla^2 + v(r) + |\psi_L|^2 + (1-u_a)|\psi_R|^2 + \frac{\Delta}{2} + i\left(\alpha - \sigma|\psi_L|^2\right)\right]\psi_L + J\psi_R.$$

If $v(r) = r^2$ then take $\alpha \to \alpha\Theta(r_0 - r)$ as before.

Questions:

1. Normal modes of uniform model: diffusive, linear, gapped.
2. Effect of Δ and J on vortices?
3. How does interconversion J interact with currents?
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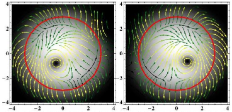

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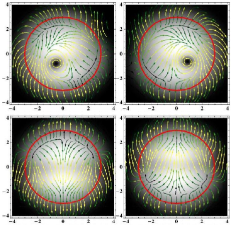

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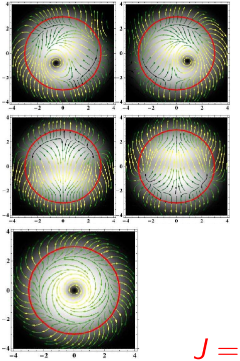

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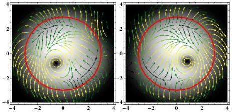

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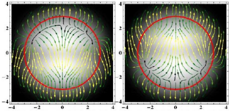
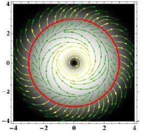
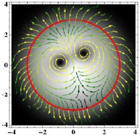


$J = 1$



$J = 1.5$



$J = 2$

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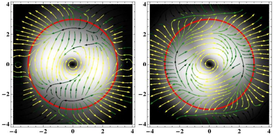
$J \neq 0, \Delta \neq 0$: For a given J, any sufficiently large Δ allows the vortex complexes $(+1, -1)$ and $(\pm 1, 0)$ to stabilize.



$J = 1, \Delta = 8$

Neglect $v(r)$ and spatial variations, write

$$\psi_{L,R} = \sqrt{\rho_{L,R}}e^{i(\phi\pm\theta/2)}, \quad R = \frac{\rho_L + \rho_R}{2}, \quad z = \frac{\rho_L - \rho_R}{2},$$

$$\dot{\theta} = -\Delta - 2u_a z + \frac{2Jz\cos(\theta)}{\sqrt{R^2 - z^2}}$$

$$\dot{z} = 2(\alpha - 2\sigma R)z - 2J\sqrt{R^2 - z^2}\sin(\theta)$$

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Two-mode system

Neglect $v(r)$ and spatial variations, write

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Josephson regime $J \ll u_a R$ & $z \ll R$; $R = R_0 = \alpha/\sigma$

Equation for a driven, damped pendulum

$$\ddot{\theta} + 2\alpha\dot{\theta} = -2\alpha\Delta + 4u_a J\frac{\alpha}{\sigma}\sin(\theta).$$

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$$\psi_{L,R} = \sqrt{\rho_{L,R}}e^{i(\phi\pm\theta/2)}, \quad R = \frac{\rho_L+\rho_R}{2}, \quad z = \frac{\rho_L-\rho_R}{2},$$

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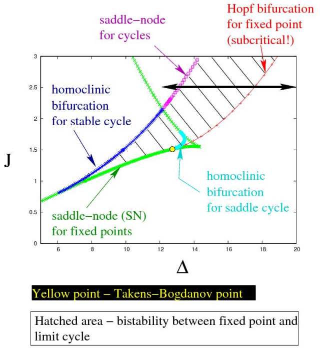
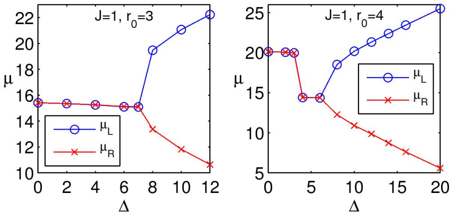
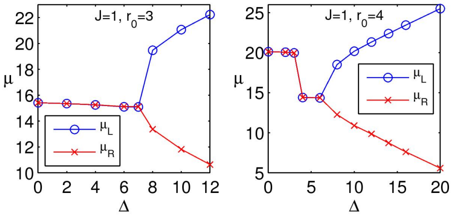
$$\dot R = 2\sigma\left(\frac{\alpha}{\sigma}R - R^2 - z^2\right).$$



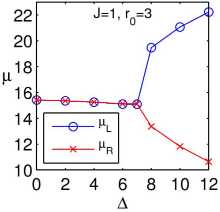
[with Balanov and Janson]

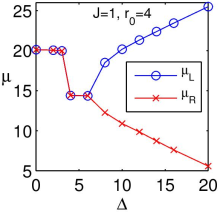

J=1, r_0=3
J=1, r_0=4
μ_L
μ_R
μ
Δ

Trapped spinor system: $\mu_{L,R} = \mathrm{i}\partial_t \langle \ln \psi_{L,R} \rangle$ vs Δ.



Simple case no vortices; $r_0 < r_{TF}$.





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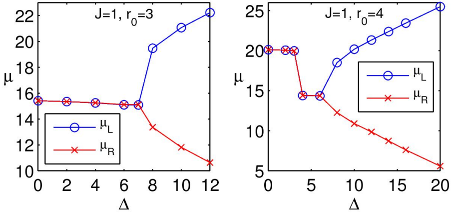
Marginal case $r_0 \sim r_{TF}$.
Δ causes $R(L)$ to grow (shrink).



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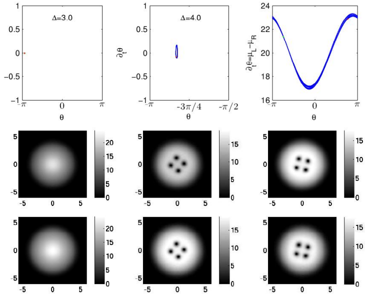

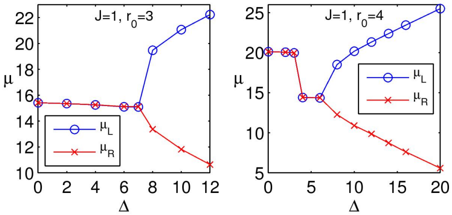

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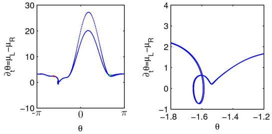
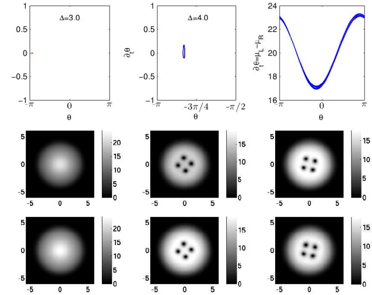
"Simple case" not so simple:

retrograde loop





Full two-component model

- Full model with a trap confirms the predictions of two-mode model, but has richer behaviour:

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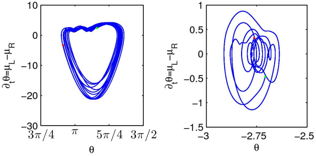
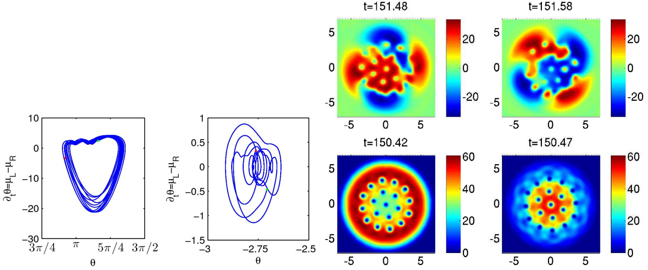
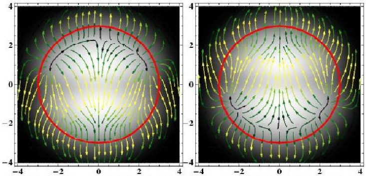
  - 

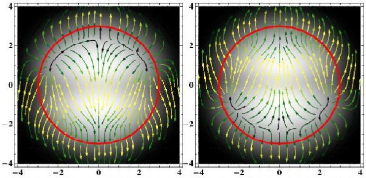
- Full model with a trap confirms the predictions of two-mode model, but has richer behaviour:
  - Phase portraits: fixed points, limit cycles with winding 0, 1, 2; retrograde loops, quasi-periodic and chaotic behaviours
  - Counter-rotating lattices; spatially non-uniform interconversions...
  - 

Stationary density depletion for intermediate J and small Δ



$\Delta = 0$

Stationary density depletion for intermediate J and small Δ



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Density depletions appear in trapped and uniform equilibrium condensates: dark/black/grey solitons; rarefaction waves;
Travelling hole solutions of the complex Ginzburg–Landau equations: e.g. Nozaki–Bekki solutions

Are these relevant?

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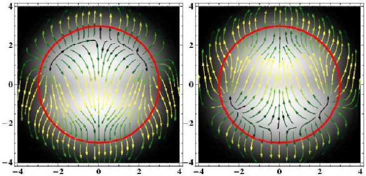

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Are these relevant?

From simulations $\psi_L(x,y) = \psi_R(x,-y)$, so this stationary state satisfies

$$i\partial_t\psi = [-\nabla^2 + r^2 + |\psi|^2 + \mathrm{i}(\alpha\Theta(r_0 - r) - \sigma|\psi|^2)]\psi + J\psi(x,-y).$$

One-dimensional modified GL equation

Consider solutions of a modified GL equation without trap

$$i\partial_t\psi = -\psi_{xx} + |\psi|^2\psi + \mathrm{i}(\alpha - \sigma|\psi|^2)\psi + J\psi(-x).$$

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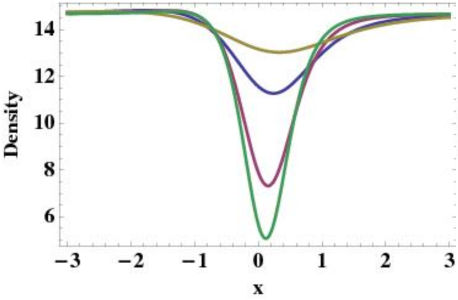
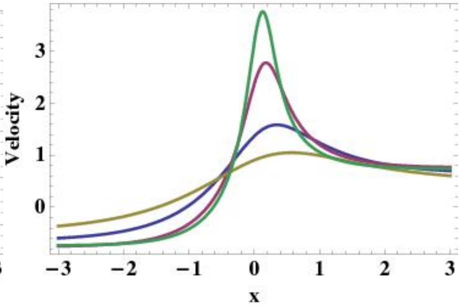
Stationary solutions exist for $0 < J < J_{cr}$. Black soliton evolves into these states.





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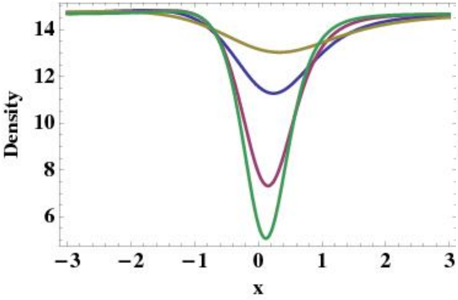
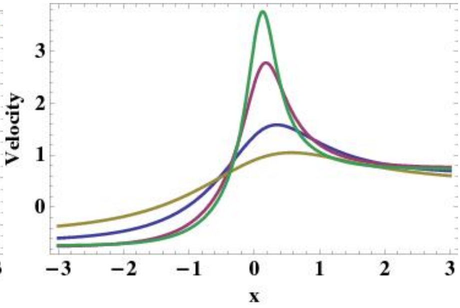
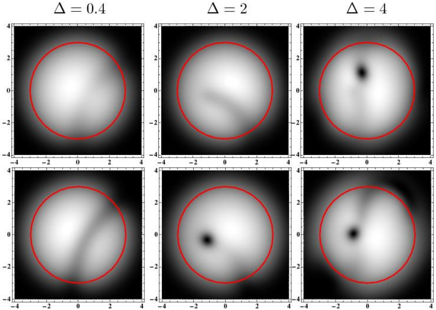




Note: For Nozaki–Bekki holes $J = 0$ but one needs diffusion $i\psi_{xx}$ (spectral filtering to stabilize the central frequency of the pulse)

Densities of L and R components for $J = 1$



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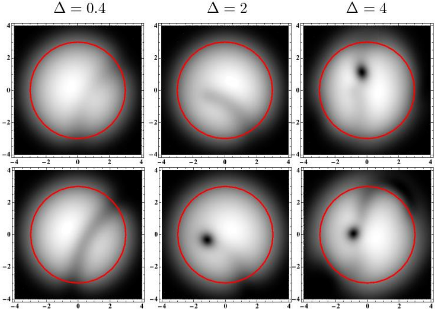
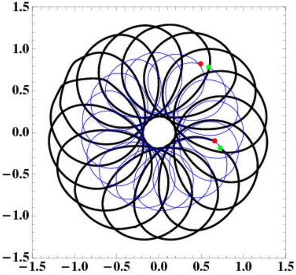


Trajectories for $\Delta = 4$



Spirographs (epitrochoids/hypotrochoid)

Similarly complicated cycloid trajectories of vortices are known for two-layer fluids with one vortex in each layer — e.g. in models of tropical vortices. Reaction difusion equations may lead to spiral wave dynamics.

Taking into account forces: Magnus force, radial advection, vortex interactions can explain stationary vortex pairs.

Variational technique and ansatz

$\psi_L = A(t)(z - z_L(t)) \exp(-|z|^2)$,

$\psi_R = B(t)(z^* - z_R^*(t)) \exp(-|z|^2)$

yield equations of motion.

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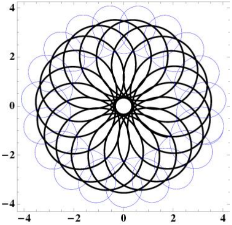

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$$
\begin{aligned}
z_L(t) &= x_L(t) + iy_L(t), \\
z_R(t) &= x_R(t) + iy_R(t), \\
\dot{z}_L &= a(\Delta - \delta)iz_L + 2Jbiz_R^*, \\
\dot{z}_R &= a(\Delta + \delta)iz_R - 2Jbiz_L^*.
\end{aligned}
$$

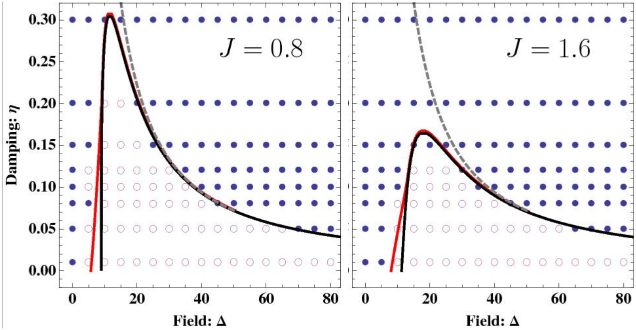
Vortex patterns generated by superposition of fluxes.
Spinor complex Ginzburg-Landau equation:

$$2i\partial_t\psi_{l,r} = \left[\pm\frac{\Delta}{2} - \nabla^2 + v(r) + |\psi_{l,r}|^2 + (1-u_a)|\psi_{r,l}|^2 \right.$$
$$\left. + i\left(\alpha - 2i\eta\partial_t - \sigma|\psi_{l,r}|^2 - \tau|\psi_{r,l}|^2\right)\right]\psi_{l,r} + J\psi_{r,l}.$$

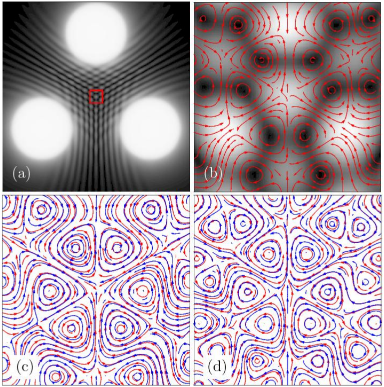
η – energy relaxation [Wouters and Savona arXiv:1007.5431 (2010)];
τ – cross-spin nonlinear dissipation;
Δ – effect of the magnetic fied (in Hamiltonian $\sim \Delta(|\psi_r|^2 - |\psi_l|^2)$);
J – electric field, stress or due to asymmetry of quantum well interfaces;

Magnetic field, Δ, drives the transition from synchronized to desyncronized regimes for $\eta = \tau = 0$.

For nonzero η there is a second transition at Δ_{c2} back to synchronized state, $\Delta_{c2} \simeq (2\alpha/\eta)(\sigma - \tau + \eta u_a)/(\sigma + \tau + \eta(2 - u_a))$ (dashed line)

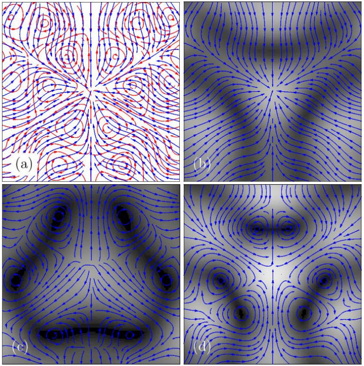


• –synchronized states (vortex-free states or synchronized vortices);
○– desynchronized states (vortices of opposite sign for l and r).
Conclude: homogeneous model gives good prediction of spatially varying system.



(a) $\Delta = 0$ showing geometry of pumping;
(b) Desynchronized $\Delta = 20$ steady majority density with streamlines;
(c) Lower synchronized $\Delta = 5$ steamlines of both polarizations;
(d) Upper synchronized $\Delta = 40$ steamlines of both polarizations.

"Half-vortices" have been seen in experiments:
[Lagoudakis et al Nature Phys. (2008)]
Are "half-vortices" pinned and stabilized by disorder?



(a) Desyncronized $\Delta = 20$ half-vortex lattice;
(b) -(c) -(d) evolution of minority component in desyncronized regime $\Delta = 20$.

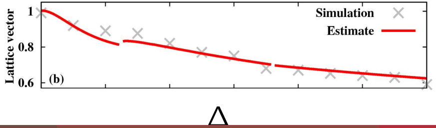
Majority component is stationary in both regimes;
Minority component is stationary in syncronized regime only.
In desyncronized regime averages to vortex-free state.

Vortex Lattice Spacing

Currents are neglible at the pumping centre, $\mu(\rho_{l,r})$;
away from pumping spot – densities are neglible.

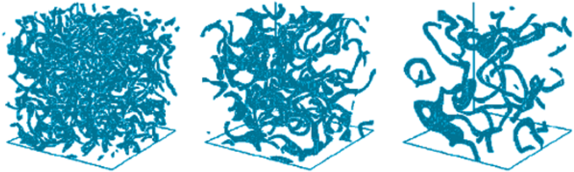
Synchronized regime: away from the pump
$\mu - |\vec{u}|^2 \mp \Delta/2 = J\,(\rho_l/\rho_r)^{\mp 1/2}\cos(\theta)$ and
$\nabla\cdot(\rho_{l,r}\vec{u}) + \alpha_1\rho_{l,r} = \mp J\sqrt{\rho_l\rho_r}\sin(\theta)$.
These are solved by $\sin(\theta) = 0$ and $\nabla(\rho_l/\rho_r) = 0$,
so $|\vec{u}|^2 = \mu + \sqrt{J^2 + \Delta^2/4}$.

Desynchronized regime: θ and ρ_l/ρ_r are not time independent, so one calculates averages. If $\rho_r \gg \rho_l$, then for majority component
$\langle|\vec{u}_r|^2\rangle = \langle\mu_r\rangle + \Delta/2$.
Superposition of such currents results in hexagonal vortex lattice with spacing $l = (2\pi/|\vec{u}|)\times 2/3\sqrt{3}$.

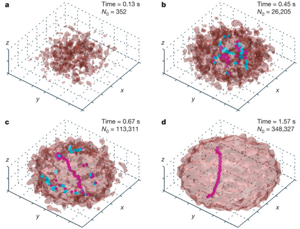


Classical turbulence – cascading vorticity;
Superfluid turbulence – quantisation of velocity circulation – differences with classical turbulence;
Strong turbulence– unstructured vortices (distance between vortices of the order of their core);
Weak turbulence regime – almost independent motion of weakly interacting dispersive waves.

Stages in condensate formation from a nonequilibrium state:
[Berloff & Svistunov Phys Rev A (2002)]
weak turbulence $\rightarrow$ **strong turbulence** $\rightarrow$ **superfluid turbulence** $\rightarrow$ **condensate**



Vortex formed during nonequilibrium kinetics of BEC
[Weiler et al. Nature (2008)]



Reverse the process going from condensate to weak turbulent state?
[Henn at el PRL (2009)]: applied an external oscillatory perturbation to produce vortices.

Complex Ginzburg–Landau equation

Modelling Exciton-polariton condensates:
the complex Ginzburg-Landau equation

$$2i\partial_t\psi = \left[-\nabla^2 + v(\mathbf{r}) + |\psi|^2 + i\left(\alpha(\mathbf{r}) - i2\eta\partial_t\psi - \sigma|\psi|^2\right)\right]\psi,$$

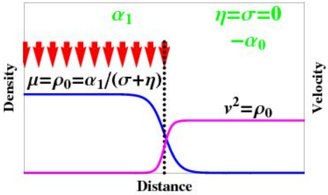
$v(\mathbf{r})$ – external disorder potential (ex. $v(r) = m\omega^2 r^2/2$);
α – an effective gain (intensity of the pumping field);
σ – nonlinear losses.
Energy and length rescaled using harmonic oscillator energy and length.
From experiments, $0 \leq \alpha \leq 10, \sigma \sim 0.3$.
η – energy relaxation [Wouters and Savona arXiv:1007.5431 (2010)] – interactions with normal fluid [Pitaevskii, JETP (1959)].

Vortex formation in equilibrium condensates:

- interactions of finite amplitude sound waves;
- existence of critical velocities of the flow;
- modulational instabilities.

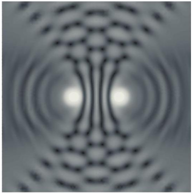
In addition in nonequilibrium condensates – pattern forming, interaction of fluxes with a disorder etc.

Vortex formation due to interference of supercurrents

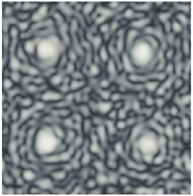


Analytical solution for the velocity $r(u)$ on $\infty < r < \infty$.

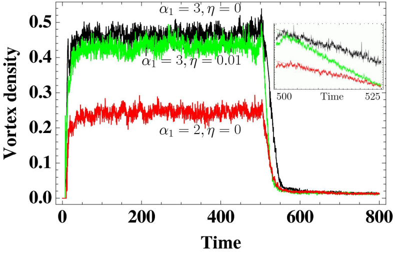
Regular emission of vortices



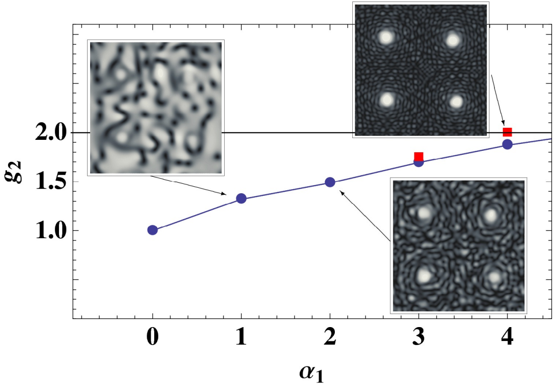
Many irregular spots: turbulence



Two regimes: forced turbulence and turbulence decay.



In forced turbulence it is possible to reach a weak turbulence state: $g_2 = \langle|\psi|^4\rangle / \langle|\psi|^2\rangle^2$. Weak turbulence implies $g_2 \sim 2$.



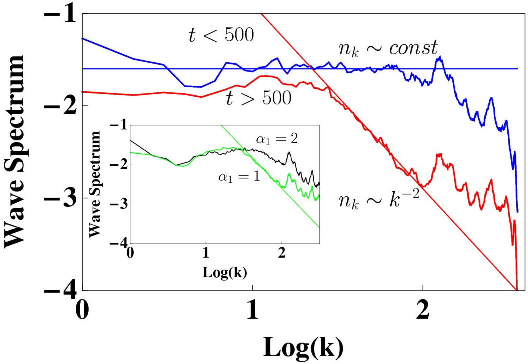
Red Squares – nonzero η facilitates the transition to weak turbulence.

Assume
(i) the existence of inertial range in the momentum space;
(ii) neglect pumping and dissipation there.

Weak turbulence theory
[Zhakharov et al (1992); Salman and Berloff, Physica D (2009)]:

Main idea:
Use random phase approximation to obtain evolution equation for the wave spectrum $\langle a_{\mathbf{k}_1} a^*_{\mathbf{k}_2} \rangle = n_{\mathbf{k}_1} \delta(\mathbf{k}_1 - \mathbf{k}_2)$,
$a_{\mathbf{k}}$ – the Fourier transform of ψ and $\mathbf{k}_i$ are discrete wave vectors.
$\partial_t n_{\mathbf{k}_1}(t) =$
$\int d^2 k_2 d^2 k_3 d^2 k_4 W_{k_1,k_2;k_3,k_4} \left(n_{\mathbf{k}_3} n_{\mathbf{k}_4} n_{\mathbf{k}_1} + n_{\mathbf{k}_3} n_{\mathbf{k}_4} n_{\mathbf{k}_2} - n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_4}\right)$,
where $W_{k_1,k_2;k_3,k_4} = \frac{4\pi}{(2\pi)^2} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \delta(k_1^2 + k_2^2 - k_3^2 - k_4^2)$

Two solutions:
(i) Equipartition of the total kinetic energy $E = \int k^2 n_k \, d\mathbf{k}$, so that $n_k \sim k^{-2}$;
(ii) Equipartition of the total number of particles $N = \int n_k \, d\mathbf{k}$, so that $n_k \sim \mathrm{const}$.



- Nonequilibrium condensates: condensates made of light
  - Gross-Pitaevskii equation with loss and gain

$$\mathrm{i}\partial_t \psi = \left[-\nabla^2 + r^2 + |\psi|^2 + i\left(\alpha\Theta(r_0 - r) - \sigma|\psi|^2\right)\right]\psi.$$

  - Radially symmetric stationary states: narrowing of density profile
  - Spiral vortex states
  - Vortex lattices

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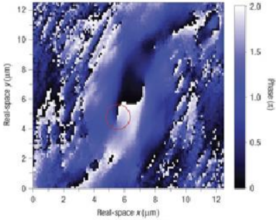

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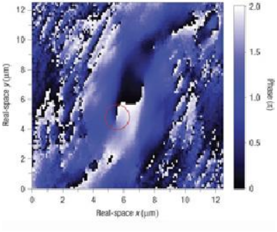

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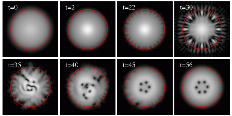
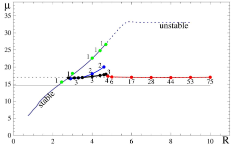


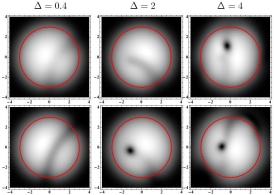


- Non-equilibrium spinor system

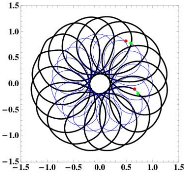
$$i\partial_t\psi_L = \Big[-\nabla^2 + V(r) + \frac{\Delta}{2} + |\psi_L|^2 + (1-u_a)|\psi_R|^2 + i\left(\alpha\Theta(r_0 - r) - \sigma|\psi_L|^2\right)\Big]\psi_L + J\psi_R$$

- Effect of Δ and J on vortices.

Densities of L and R components for $J = 1$



Trajectories for $\Delta = 4$



Spirographs (epitrochoids/hypotrochoid)

- Synchronization/desynchronization with the region of bistability.

Conclusions-3

- Turbulence in exciton-polariton condensates may lead to novel regimes of turbulence of classical matter field.
  - The regimes can be distinguished by finding second order correlation function; by looking at the wave spectrum.
  - What are the stages in transition from strong turbululence to weak turbulence and back?
- Spinor condensates: predictions of homogeneous model (syncronization/desynchronization) are not significantly modified by spatial inhomogeneity.
  - Observation of the experimental behaviour in an applied field can thus be used to distinguish the the loss nonlinearities σ, τ and η.
  - Vortices, vortex lattices and half-vortex latices in spinor condensates. Being stationary these textures can be studied experimentally.
- Turbulence in spinor condensates.

Scaling laws? Cross-overs of different regimes? Interplay between turbulent regimes and the effects of magnetic field?...