Critical layers

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1 Introduction

Theoretical models of waves in the atmosphere naturally require consideration of propagation on a background state that is a shear flow. One example is that of Rossby waves (or planetary waves) propagating from the extratropical troposphere into the stratosphere, The background state is here the longitudinally-averaged flow, which may include westerly winds increasing in strength with height (e.g. in the winter) or westerly winds at lower levels changing to easterly winds at upper levels (e.g. in the summer). Another example is that of small-scale internal-gravity waves excited by a mountain propagating upwards through a large-scale flow that changes strength (and perhaps direction) with height.

Suppose that the background flow (i.e. the flow in the absence of the waves) is in the x direction with speed U that is a function of a second space co-ordinate y and that the waves have a well-defined phase speed c in the x-direction. Then a location where $U(y) = c$, i.e. the flow speed matches the phase speed, is a line parallel to the x-axis and at a fixed value of y, called a *critical line*. Where the second space co-ordinate is height the equivalent term *critical level* is often used. If the speed U was a function of two space co-ordinates y and z then the location $U(y, z) = 0$ would define a *critical surface*.

Simple theories for the structure of waves are often based on the assumptions that the waves are steadily propagating, that dissipative or diabatic processes such as friction or radiative transfer may be neglected and that the waves are small-amplitude, so that terms in the equations of motion that are nonlinear in wave quantities may be neglected. These theories lead to a straightforward differential or partial differential equation that describes the spatial structure of the waves. The importance of the critical line is that it is a location where these differential equations are singular, in other words the solutions imply that some physical quantity becomes infinite.

As in many physical contexts, the appearance of singular behaviour in a mathematical model implies that the simplifications that lead to that model cannot be justified and that some physical process that was neglected must be retained. To remove the critical-line singularity one of the neglected processes mentioned above must therefore be included (however weak such processes might have been estimated to be). The neglected process will be essential only in a small, but finite, region around the critical line and may still be negligible elsewhere. This small but finite region is named the *critical layer*.

2 The Rossby-wave critical layer

2.1 A simple model

One of the simplest examples of the critical-line singularity and its resolution in a finite critical layer arises in a two-dimensional model of Rossby wave propagation on a β-plane (a mathematical device to include the effect of the variation of Coriolis parameter with latitude [[could be omitted if explained in another article]]). Two-dimensional cartesian co-ordinates (x, y) may be used, with x measured in the eastward direction and y measured in the northward direction. The corresponding velocity components are taken to be (u, v). The assumption of incompressibility implies that the velocity components may be expressed in terms of a streamfunction $\psi(x, y, t)$ (with t time) where $u = -\partial\psi/\partial y$ and $v = \partial\psi/\partial x$.

The governing equation is based on the fact that, in the absence of dissipation, the absolute vorticity, which is the sum of the relative vorticity, $\zeta = \partial v/\partial x - \partial u/\partial y = \nabla^2\psi$ and the planetary vorticity βy, is conserved following the fluid motion. β is a constant and in an Earth-like atmosphere is positive. It is convenient to include linear damping of vorticity in the model as a simple representation of a dissipative processes. The governing equation then becomes

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial t}\right)(\zeta + \beta y) = -\alpha\zeta \tag{1}$$

where where α is the damping rate. (Another possibility for a dissipative process would be diffusion of vorticity. Neither linear damping nor diffusion are likely to be realistic representations of dissipative processes in the real atmosphere, but either serves as a convenient example that captures the basic effect of dissipation in the critical layer.)

It is assumed that in the absence of waves the flow is in the x direction with speed $U(y)$. Waves are superimposed on this flow giving a contribution $\psi'(x, y, t)$ to the streamfunction. Then the equation (1) may be written in terms of ψ' as

$$\nabla^2\psi'_t + U(y)\nabla^2\psi'_x + (\beta - U''(y))\psi'_x = -\alpha\nabla^2\psi' - u'\frac{\partial\nabla^2\psi'}{\partial x} - v'\frac{\partial\nabla^2\psi'}{\partial y} \tag{2}$$

where $u' = -\partial\psi'/\partial y$ and $v' = \partial\psi'/\partial x$ are the wave velocity components

If the damping is weak then it is reasonable to neglect the first term on the right-hand side. If the waves are small-amplitude then it is reasonable to neglect the the second and third terms on the right-hand side, which are quadratic in wave quantities. Since the resulting equations are linear and contain no explicit x-dependence it is possible to consider waves with different wavelengths in the x-direction as independent.

Consider one such wave, with x-wavenumber k (i.e. wavelength $2\pi/k$ in the x-direction), assumed to be steadily propagating in the x-direction with phase speed c. It follows that the streamfunction for this wave may be written in the form $\psi'(x, y, t) = \mathrm{Re}[\hat{\psi}(y)e^{ik(x-ct)}]$, where $\hat{\psi}(y)$ is a (complex) function of y. Substituting into (2) and neglecting terms on the right-hand side gives the ordinary differential equation

$$\hat{\psi}'' + (\frac{\beta - U''(y)}{U(y) - c} - k^2)\hat{\psi} = 0. \tag{3}$$

This equation is known as the Rayleigh-Kuo equation and, depending on context, determines the stability of the shear flow $U(y)$ in a problem where c is an eigenvalue, or when c is determined by forcing, which may be included in the problem by an extra term on the right-hand side of (3) or by a boundary condition, describes through the function $\hat{\psi}(y)$ the structure of the forced disturbances. The focus here is on a steady forced problem, where c is given and real.

The appearance of the factor $(U(y)-c)^{-1}$ in part of the coefficient of $\hat{\psi}$ indicates that the equation has a singular point at values of y such that $U(y) = c$, i.e. where the phase speed matches the flow speed. These locations are the critical lines. If $U(y)$ is an increasing or decreasing function of y then there is at most one critical line. If $U(y)$ has a turning point (as would be the case for a jet-like flow, for example) then there may be more that one critical line.

Consider the solution near a critical line at $y = y_c$. The nature of (3) is such that whilst $\hat{\psi}(y)$ (proportional to the velocity in the y-direction) is finite and continous in the neighbourhood of the critical line, $\hat{\psi}''(y)$ (representing part of the vorticity) is proportional to $(y-y_c)^{-1}$ and $\psi'(y)$ (representing the velocity in the x direction) is proportional to $\log|y-y_c|$, i.e. both are singular. This is clearly unphysical, but what is more problematic, in a way, is that the singularity in $\hat{\psi}'(y)$ implies that there is no unique way to match solutions of (3) across $y = y_c$. In particular the jump in $\hat{\psi}'(y)$, corresponding to the jump in u', across $y = y_c$ remains unknown. It follows there is no unique solution in the whole flow domain for the function $\hat{\psi}(y)$. The critical line singularity must therefore be resolved not only to remove the local singular behaviour in certain physical quantities but also to determine the the structure of the waves over the whole flow.

2.2 Absorption-reflection

To note the implications of the critical layer for the waves elsewhere in the flow it is useful to focus on the following geometry, shown in Figure 1. Assume that the waves are forced at some large positive value of y, with phase speed $c = 0$, i.e. the waves are stationary. Assume also that the flow speed $U(y)$ is positive in $y > 0$ and negative in $y < 0$, so that the waves have a critical line at $y = 0$ and that the curvature term $U''(y)$ is not too large (so that $\beta - U''(y)$ is positive). The equation (3) predicts that the function $\hat{\psi}(y)$ is oscillatory in $y > 0$, implying that there are propagating waves (as expected from the basic properties of Rossby waves). In $y < 0$, on the other hand, the function $\hat{\psi}(y)$ is exponentially increasing or decreasing with y and physical considerations require that $\hat{\psi}(y)$ decreases as y decreases, representing evanescent waves. One feature of the solution that is naturally of interest is the relative amount of northward and southward propagating waves in the region between the wave forcing and the critical layer. This measures the absorption-reflection behaviour of the critical layer. If the critical layer acted as an absorber of waves then the region between the forcing and the critical line would contain only waves propagating southward. If it acted as a reflector of waves then there would be some contribution to the solution in this region from waves propagating northwards. The reflection could be partial or perfect. Indeed there could in principle be over-reflection, which would be associated with a greater proportion of northward-propagating than southward-propagating waves, implying that the critical layer was actively emitting waves.

A useful quantitative measure of the wave propagation is the momentum flux, $\overline{u'v'}$, where $\overline{(.)}$ indicates an average in the x-direction (keeping y and t constant). $\overline{u'v'}$ indicates the correlation between the two velocity components and the basic properties of Rossby waves imply $\overline{u'v'} > 0$ for southward propagating waves and $\overline{u'v'} < 0$ for northward propagating waves. It follows from (3) that $\overline{u'v'}$ is constant in $y > 0$, except at the critical line at

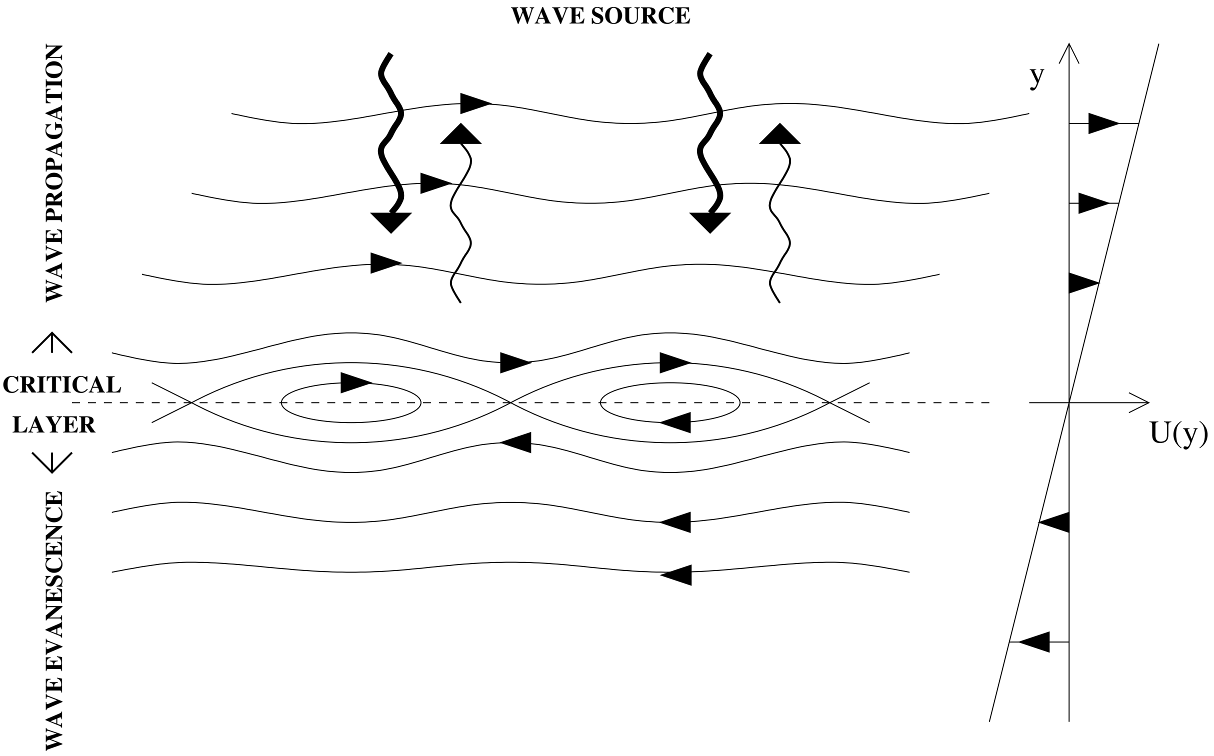

Figure 1: Schematic diagram of Rossby-wave propagation on a shear flow $U(y)$ with a critical line. The flow is positive (i.e. eastward) in $y > 0$ (upper portion of the diagram) and negative (i.e. westward) in $y < 0$ (lower portion of the diagram). The waves are forced, with zero phase speed in x-direction, in $y > 0$ and propagate towards $y = 0$. In $y < 0$ the waves are evanescent (i.e. non-propagating and decaying as y becomes more negative). The critical line is at $y = 0$, where $U(y) = 0$. In the neighbourhood of $y = 0$ the streamlines are closed and form a Kelvin's cat's eye pattern. The width of the closed streamline region, which increases as the wave amplitude increases, defines the width of the nonlinear critical layer. If dissipation were strong enough then dissipative effects would dominate over a relatively broad region near $y = 0$ and the closed streamlines would essentially be irrelevant to the dynamics. (The critical layer would then be linear and dissipative, rather than nonlinear.) There may be some reflected wave in $y > 0$, but the amount of reflection can be determined only by considering the detailed dynamics of the critical layer.

$y = 0$. The evanescence of the waves in $y < 0$ implies that $\overline{u'v'} = 0$ there. However the value of $\overline{u'v'}$ cannot be determined from (3) alone. Instead the critical line must be resolved into a finite critical layer to allow the jump in $\overline{u'v'}$ across the layer, denoted by $[\overline{u'v'}]_-^+$, and hence the value of $\overline{u'v'}$ in $y > 0$, to be evaluated.

The continuity of $\hat{\psi}(y)$ across the critical line singularity suggests that, when the critical line is resolved into a thin critical layer, ψ' and hence $v' = \partial\psi'/\partial x$ will vary only weakly across the critical layer. In addition, y-derivatives within the critical layer will generally be much larger than x-derivatives (because the layer is thin), implying that $\zeta' \simeq -\partial u'/\partial y$. Putting these pieces of information together, it follows that

$$[u']_-^+ = \int \zeta' dy \text{ and } \overline{u'v'} \text{ (in } y > 0) \; = [\overline{u'v'}]_-^+ = \int \overline{\zeta' v'} dy \tag{4}$$

where $[u']_-^+$ denotes the jump in u' across the critical layer, the integrals are taken across the critical layer and v' may be taken as constant within the second integral. The first equality is the missing matching condition across the critical layer. The second shows that the critical layer acts as a net absorber of waves when there is (in a y-integrated sense) negative correlation between ζ' and v' in the critical layer, as a perfect reflector when there is zero correlation and as a net emitter (i.e. an over-reflector) when there is positive correlation.

To summarise, the non-uniqueness in the solution of (3) left by the critical-line singularity leaves the absorption-reflection behaviour of the critical layer uncertain. Only by determining the correlation between v' and ζ' in the critical layer is it possible to determine the precise absorption-reflection properties.

2.3 The dynamics of the critical layer

The dynamical balance in the critical layer depends on the parameters of the problem. Consider in turn the processes that have been neglected in arriving at (3). Firstly it has been assumed that the waves are steadily propagating, i.e. that their amplitude is not changing with time. It is possible to analyse the non-dissipative, linearised equations [(2) with the right-hand side set to zero] without making this assumption and show that the singular behaviour predicted by (3) develops with time. For example, both the vorticity ζ' and the x-component of velocity u' are predicted to increase without bound. The time-dependent analysis shows that the terms neglected in going from (2) to (3) inevitably become important at large times, however small they might have first appeared.

Secondly, consider the dissipative term $-\alpha\nabla^2\psi'$ in (2). This may be compared with the advection term $U(y)\partial\nabla^2\psi'/\partial x$. The relative sizes of these terms, near to $y = 0$, may be estimated as $\alpha/(kU'(0)y)$ and it follows that the dissipative term cannot be neglected in a region of size $\delta_\alpha = \alpha/(kU'(0))$. This is (potentially) the thickness of the dissipative critical layer.

Finally, consider the nonlinear term. It turns out that the most important part is $v'\partial\nabla^2\psi'/\partial y$. If this is to balance $U(y)\partial\nabla^2\psi'/\partial x \simeq U'(0)y\partial\nabla^2\psi'/\partial x$ in a thin region of thickness δ_{NL}, then $v'/\delta_{NL} \sim kU'(0)\delta_{NL}$, i.e. $\delta_{NL} \sim (v'/kU'(0))^{1/2}$. This is (potentially) the thickness of the nonlinear critical layer.

Whether nonlinearity or dissipation is dominant in the critical layer depends on the relative size of $\delta_{NL}/\delta_\alpha = (kv'U'(0))^{1/2}/\alpha$. If $\delta_{NL}/\delta_\alpha \ll 1$ then the critical-layer dynamics are dominated by dissipation and the critical layer thickness is δ_α. If $\delta_{NL}/\delta_\alpha \gg 1$ then the critical-layer dynamics are dominated by nonlinear processes and the critical layer thickness is δ_{NL}. In the case of Rossby waves in the real atmosphere, wave amplitudes are relatively large and dissipation is relatively weak, so that the nonlinear dynamics are the most relevant.

The fully nonlinear equations state that $\zeta + \beta y$ is conserved following the flow (which is in turn determined by the ζ field). If the critical layer is thin, i.e. δ_{NL} is small, there is a simplification because the flow may be approximated by the superposition of the basic flow $U(y) \simeq U'(0)y$ and the y-component of the disturbance velocity field, which is simply a function of x, because of the continuity of ψ' across the critical layer. This superposition gives a flow whose streamlines form a pattern known as Kelvin's cat's eyes, with closed streamlines near $y = 0$ (see Figure 1). The dynamics of the nonlinear critical layer is therefore that fluid particles are advected around these streamlines, conserving their values of $\zeta + \beta y$. The rearrangement of the $\zeta + \beta y$ field changes the ζ' field, thereby changing $[u']^+_-$ and hence the structure of the waves outside the critical layer. Furthermore there is a corresponding change in the correlation between v' and ζ', which determines the absorption-reflection properties.

A schematic diagram of the evolution of the vorticity field in a simple nonlinear critical layer is presented in Figure 2 and the absorption-reflection properties deduced. At early times v' and ζ' are anti-correlated and the critical layer acts as an absorber. If there is strong dissipation then the vorticity in the centre of the critical layer is essentially frozen in its early-time configuration and the critical layer continues to act as an absorber at later times. (Detailed calculation shows that in this early-time/dissipative regime,

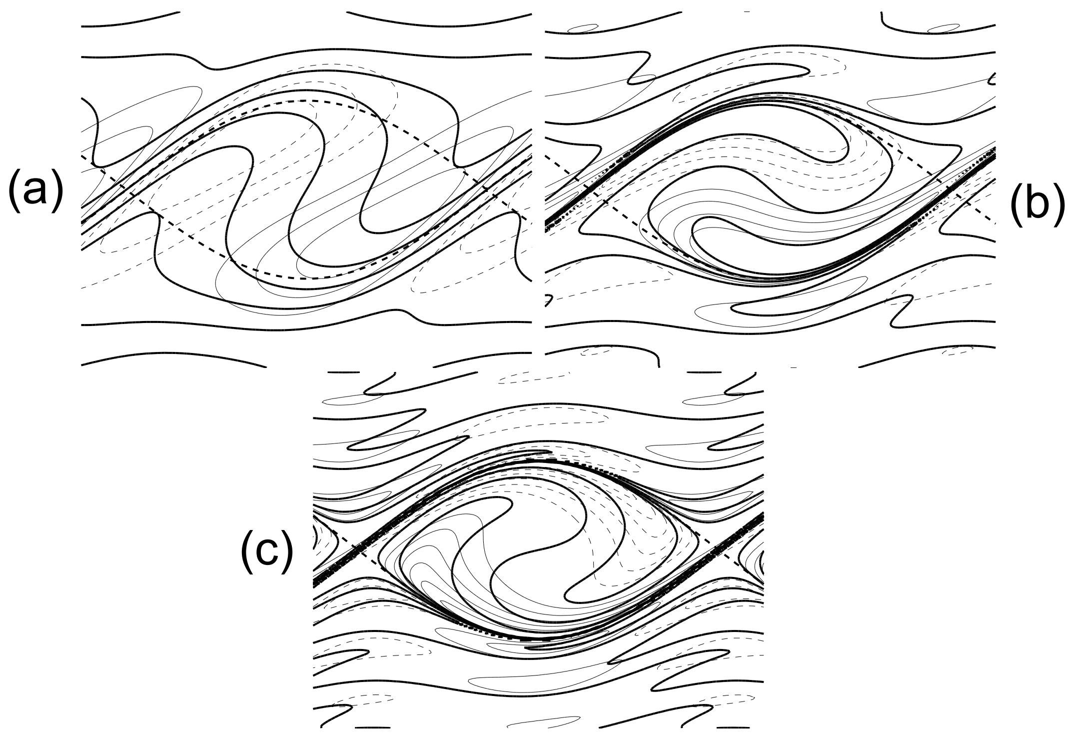

Figure 2: Evolution of vorticity field in the nonlinear critical layer. The panels show an expanded view of the vorticity field in the closed streamline region indicated in Figure 1. Note that this region may be very thin in the y-direction if the wave amplitude is small. Thick dotted curves are the bounding closed streamlines. Thick solid curves are contours of absolute vorticity $\zeta + \beta y$. Thin curves are contours of wave relative vorticity ζ', with solid curves indicating positive values and dashed curves indicating negative values. Three panels show (a) absorbing stage, where y-velocity v' is negatively correlated with ζ', (b) reflecting stage, where correlation between v' and ζ' is close to zero and (c) over-reflecting stage, where v' is positively correlated with ζ'.

the absorption is effectively perfect.) However if dissipation is weak then the advective rearrangement continues and, after about half a turn-round time for the closed-streamline flow, the ζ' field in the centre of the critical layer (which gives the major contribution to the integral) is such that there is no y-integrated correlation with the v' field, i.e. the critical layer acts as a perfect reflector. According to this particular model the advective rearrangement continues, to give a positive correlation between v' and ζ' and hence over-reflection and the critical layer subsequently oscillates between a weakly absorbing and weakly over-reflecting state, converging to a state of perfect reflection at large times.

The precise details of the evolution depend on the particular flow configuration. However a general description of absorption-reflection behaviour can be formulated by considering -$\overline{u'v'}$ as the flux (in the y-direction) of wave activity (i.e. a quantity that is positive when waves are present and zero when they are not). In the early-time absorbing stage wave activity builds up in the critical layer. As the reflecting stage approaches the rate of buildup decreases to zero and in the over-reflecting stage the critical layer re-emits some of its wave activity. If there is dissipation then the flux of wave activity into the critical layer may be balanced by local dissipation of wave activity and an absorbing state may persist. However, for the critical layer to continue to act as an absorber without dissipation then, the amount of wave activity in the critical layer must continue to increase. The total amount of wave activity in the critical layer may be shown to depend on the thickness of the region over which the vorticity field has been rearranged, i.e. the thickness of the critical layer. If this thickness is finite then there is an upper bound to the total amount of wave activity that can be stored there and it is therefore not possible to sustain absorption. In such a case the long-time average of the flux of wave activity must approach zero and one can therefore say that the long-time average behaviour is perfect reflection. The only way that absorption could be sustained in the long-term would be if the thickness of the critical layer systematically increased in size.

A complementary viewpoint comes from considering $\overline{u'v'}$ as a momentum flux. In the absorbing stage $[\overline{u'v'}]_-^+$ is positive, implying that there is a negative force exerted on the x-averaged flow in the critical layer. The time-averaged perfect reflection in the case where dissipation is zero translates into no time-averaged x-average force exerted on the flow in the critical layer. (If there was such a force then the critical line, and hence the critical layer, would move closer and closer to the wave source.) Sustained absorption where there is dissipation translates into a non-zero time-averaged x-average force exerted on the flow in the critical layer, with this force being balanced by forces provided by dissipative processes (i.e. by the linear vorticity damping in the model described above).

The critical layer theory makes clear the nature of the two-way interaction between the wave propagation region outside the critical layer, and the flow in the critical layer itself. The waves outside the critical layer directly determine the flow pattern inside it (because of the continuity of v' across the critical layer). However inside the critical layer the flow changes the vorticity field and hence the jump in u' across the critical layer, thereby changing the waves outside it. It is important to note that there is no wave propagation within the critical layer itself. The dynamics is simply that of vorticity advection by a simple cat's eye flow whose structure is determined by the waves outside the critical layer. It is not the case that waves can be said to propagate into the critical layer and are reflected by the structure of the flow profile that they encounter there.

2.4 Wave breaking

The behaviour seen in the nonlinear critical layer for δ_{NL} small may be interpreted as an example of the breaking of Rossby waves. By 'breaking' it is meant that the material contours or surfaces that would, in wave propagation, be reversibly undulated, are strongly and irreversibly deformed. The most familiar example of wave breaking occurs for surface waves. There the wave dynamics is associated with the undulation of the ocean surface. Waves might be forced in one region, e.g. by a storm, and propagate through large distances. The presence of the waves in this propagation stage is associated with distortion of the ocean surface, but the distortion is weak and reversible. As the waves enter shallow water in a coastal region the distortion of the ocean surface becomes stronger and, ultimately, complex and irreversible and the flow will become three-dimensionally turbulent.

Rossby-wave propagation involves the reversible distortion of contours of potential vorticity (absolute vorticity in the simple two-dimensional context discussed above). In the critical layer region the distortion of these contours is strong and irreversible and the waves may be said to be breaking. Indeed in many cases the flow in the critical layer may be shown to involve a sort of turbulence (quasi-geostrophic or two-dimensional), but this is not essential for the behaviour to be described as breaking. As in the surface-wave case, where the breaking may allow the waves to drive systematic long-shore currents, the breaking of Rossby waves allows a systematic force to be exerted by the waves.

2.5 Implications for the atmosphere

In the nonlinear Rossby wave critical layer described above there is a clear division (described by simplified mathematical equations that may be derived by a formal approximation procedure) between the broad region outside the critical layer where the dynamics is wave-like and the thin critical layer itself where the dynamics is strong advective rearrangement of the potential vorticity or absolute vorticity field, that might be called wave breaking. In the real atmosphere the wave amplitudes are very large and the formal estimate δ_{NL} of the nonlinear critical layer thickness is generally as large as the other length scales in the problem. The same simplified mathematical equations do not hold precisely. Nonetheless observations and numerical models show clearly that there are regions of wave propagation and regions of wave breaking and that these exist side-by-side. There are at least two important examples. One is in the winter stratosphere, where planetaryscale Rossby waves propagate up from the troposphere, distort and shift the polar-night vortex and break in what is now called the stratospheric 'surf zone' (which covers a large region of the midlatitudes and subtropics). The stirring of potential vorticity in the surf zone leads to weak large-scale gradients in the interior of the surf zone and corresponding strong gradients at its poleward edge, which is the boundary of the polar vortex. These strong potential vorticity gradients act as a kind of wave guide for upward propagation of Rossby waves. A second is in the upper troposphere and lower stratosphere, where synoptic-scale baroclinic eddies lead to a wave-like distortion of the subtropical jet and to wave-breaking regions on the poleward and equatorward sides of the jet. Again the effect on PV gradients is to lead to a kind of wave guide for tropospheric Rossby waves.

In these examples nonlinear critical layer theory provides quantitive guidance as to how the different regions interact. For example, it indicates that the wave-breaking regions may be considered to absorb, reflect or emit wave activity and that the waves may have a systematic effect on the flow in the wave breaking region.

In the last ten years or so Rossby-wave critical layer ideas have also been applied to hurricane dynamics. For example the role of non-axisymmetric Rossby-wave disturbances to a hurricane in promoting hurricane intensification have been described in terms of the propagation of such disturbances and their subsequent dissipation in a critical layer. In a separate line of argument, the nonlinear critical layer of a subtropical Rossby wave has been interpreted as a preferred location for hurricane vortex development, with one stage of the development being a period of co-propagation (in longitude) of the pre-existing wave and the growing vortex.

3 Internal gravity wave critical layers

3.1 Description

Critical lines and critical layers arise generically in any problem of wave propagation in a fluid. Another example that is particularly important for the atmosphere is that of internal gravity waves. This has some important differences from the Rossby wave case.

Consider the propagation of internal gravity waves on a background state that has stable stratification with buoyancy frequency $N(z)$ and flow in the x direction with speed $U(z)$, where z is height. Assuming that the flow is incompressible (which is not necessarily defensible for many atmospheric gravity waves, but the model serves to illustrate important points that continue over to the compressible case) it may be shown that the analogue of (3) is

$$\hat{\psi}'' + \{\frac{N^2(z)}{(U(z)-c)^2} - \frac{U''(z)}{U(z)-c} - k^2\}\hat{\psi} = 0. \tag{5}$$

This equation is known as the Taylor-Goldstein equation and, as is the case for (3) for Rossby waves, depending on context determines the stability of density-stratified shear flows or the structure of waves propagating on such flows. Again the critical line singularity at $U(z) = c$ is manifested by the inverse powers of $U(z) - c$ appearing in the expression multiplying $\hat{\psi}$. There is an important difference from (3) in that the one of the expressions contains the factor of $(U(z) - c)^2$. This means that the behaviour of solutions near the critical-line singularity is quite different from the Rossby-wave case. In fact, provided that the local Richardson number $N(z)^2/U'(z)^2 > \frac{1}{4}$ (which is precisely the condition required for the flow to be stable) the function $\hat{\psi}(z)$ oscillates rapidly in z near to the critical line and the oscillations become infinitely rapid as the critical line is approached. Indeed there are infinitely many oscillations before the critical lines is reached. These oscillations are a manifestation of the rapid shrinking of the vertical wavelength of the wave as the critical line is approached, due to the tilting of the wave by the shear. An analogous shrinking of the wavelength occurs in the Rossby-wave case, but there are only a finite number of oscillations before the critical line is reached – a subtle and important difference between this and the internal-gravity wave case. The reason for the difference is that in the internal-gravity wave case decrease in wavelength gives a stronger decrease in the group velocity (i.e. the propagation velocity for wave packets). Indeed in the Rossby-wave case the idea of group velocity is simply not at all useful in the neighbourhood of the critical line, whereas in the internal-gravity wave case it is.

Again the critical line singularity can be resolved either by dissipation or nonlinearity, depending on the relative strengths of the two. One possibility is that the wave will eventually dissipate. This is possible however weak dissipative processes might seem, since the decrease in group velocity as the critical line is approached means that there is infinite time for the dissipation to act. Indeed the wave will dissipate before the critical line is reached. In this case the thickness of the dissipative critical layer may be defined as the distance to the critical line at which the dissipation occurs, and the critical layer may be regarded as a wave absorber. If wave amplitudes are sufficiently large compared to dissipative processes, however, then nonlinear terms in the equations may become important before dissipation occurs and, again, before the critical line is reached. The distance to the critical line defines the thickness of the nonlinear critical layer. Here the situation is more complicated than in the Rossby-wave case. For example, it is not possible to argue that the velocity component in the z-direction (analogous to v' in the Rossby-wave case) is continuous across the nonlinear critical layer (and therefore independent of z within the critical layer). If governing equations for the nonlinear critical layer are derived, they are essentially the full nonlinear governing dynamical equations, with a slight simplification because the structure is very thin in the z direction. The critical-layer dynamics is therefore a complex juxtaposition of wave propagation and nonlinearity. Furthermore if nonlinearity is important it is also almost inevitable that there will be the potential for gravitational instability and therefore, in reality, breakdown of the flow into complex three-dimensional turbulence. For this reason, while there have been some formal asymptotic studies of nonlinear internal-gravity wave critical layers, the behaviour that is most likely to be relevant in the real atmosphere is better investigated using full three-dimensional numerical simulation. Evidence from such simulations is that some nonlinear reflection effect is possible, particularly when the local Richardson number (see above) is not too large.

3.2 Implications for the atmosphere

Dissipation and breaking of internal gravity waves as they approach critical lines is potentially an important process in the atmosphere, since it implies the possibility of wave-induced forces. Breaking may also be caused by the decrease of density with height, which leads to a corresponding increase in wave amplitudes. However there is little doubt that breaking at (or more strictly near) critical lines also plays a major role. For example the mechanism for the equatorial quasi-biennial oscillation in the stratosphere requires selective filtering, breaking and dissipation of waves (depending on their horizontal phase speed) by the background flow. Such waves are believed to arise primarily from convection in the tropical troposphere (on a whole range of different scales). Observations confirm the expected relation between the phase speed of the waves observed at a particular height and the background flow at lower levels, through which they would have propagated.

Critical-line/critical-layer behaviour is an important ingredient of gravity-wave parametrizations, that seek to represent the effects (primarily the wave-induced forces) of small-scale gravity waves in global-scale numerical models. Such parametrization is essential for useful simulation of the stratosphere and mesosphere. One very simple parametrization would be that, for a spectrum of upward-propagating gravity waves, each component of the spectrum dissipates at its critical line and therefore gives rise to a force at that location. In practice, some kind of breaking criterion is applied so that waves break before the critical line is reached. Almost all current parametrizations assume the equivalent of critical-layer absorption. If critical-layer reflection had to be taken into account then it would greatly increase the complexity of the parametrization problem.

See also: Dynamic Meteorology (Potential Vorticity, Waves), Quasi-biennial Oscillation, Middle Atmosphere (Gravity Waves, Planetary Waves), Rossby Waves, Wave Mean-Flow Interaction

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