Chapter 5

Contour Integration and Transform Theory

5.1 Path Integrals

For an integral $\int_a^b f(x)\,\mathrm{d}x$ on the real line, there is only one way of getting from a to b. For an integral $\int f(z)\,\mathrm{d}z$ between two *complex* points a and b we need to specify which *path* or *contour* C we will use. As an example, consider

$$I_1 = \int_{C_1} \frac{\mathrm{d}z}{z} \qquad \text{and} \qquad I_2 = \int_{C_2} \frac{\mathrm{d}z}{z}$$

where in both cases we integrate from $z = -1$ to $z = +1$ round a unit semicircle: C_1 above, C_2 below the real axis. Substitute $z = e^{i\theta}$, $\mathrm{d}z = ie^{i\theta}\,\mathrm{d}\theta$:

$$I_1 = \int_\pi^0 \frac{ie^{i\theta}\,\mathrm{d}\theta}{e^{i\theta}} = -i\pi$$

but

$$I_2 = \int_\pi^{2\pi} i\,\mathrm{d}\theta = +i\pi.$$

The result of a contour interaction *may* depend on the contour.

To formally define the integral, divide C into small intervals, separated at points z_k ($k = 0, \dots, N$) on C, where $z_0 = a$ and $z_N = b$. Let $\delta z_k = z_{k+1} - z_k$ and let $\Delta = \max\limits_{k=0,\dots,N-1} |\delta z_k|$. Then we define

$$\int_C f(z)\,\mathrm{d}z = \lim_{\Delta \to 0} \sum_{n=0}^{N-1} f(z_k)\,\delta z_k$$

where, as $\Delta \to 0$, $N \to \infty$. Note that if C lies along the real axis then this definition is exactly the normal definition of a real integral.

Elementary properties

If C_1 is a contour from w_1 to w_2 in $\mathbb{C}$, and C_2 a contour from w_2 to w_3, and C is the combined contour from w_1 to w_3 following first C_1 then C_2, we have that $\int_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z$. (Obvious from definition; compare with the equivalent result on the real line, $\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x$.)

If C^+ is a contour from w_1 to w_2, and C^- is exactly the same contour traversed backwards, then clearly $\int_{C^+} f(z)\,\mathrm{d}z = -\int_{C^-} f(z)\,\mathrm{d}z$. (Cf. $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$.)

Integration by substitution and by parts work in $\mathbb{C}$ also.

If C has length L, then

$$\boxed{\left|\int_C f(z)\,\mathrm{d}z\right| \le L \max_C |f(z)|}$$

because at each point on C, $|f(z)| \le \max\limits_C |f(z)|$.

Closed contours

If C is a closed curve, then it doesn't matter where we start from on C: $\oint_C f(z)\,\mathrm{d}z$ means the same thing in any case. (The notation $\oint$ denotes an integral round a closed curve.) Note that if we traverse C in a negative sense (clockwise) we get negative the result we would have obtained had we traversed it in a positive sense (anticlockwise).

5.2 Cauchy's Theorem

A *simply-connected domain* is a region R of the complex plane without any holes; formally, it is a region in which any closed curve encircles only points which are also in R. By a *simple closed curve* we mean one which is continuous, of finite length and does not intersect itself.

Cauchy's Theorem states simply that if $f(z)$ is analytic in a simply-connected domain R, then for any simple closed curve C in R,

$$\boxed{\oint_C f(z)\,\mathrm{d}z = 0.}$$

The proof is simple and follows from the Cauchy–Riemann equations and the Divergence Theorem in 2D:

$$\begin{aligned}
\oint_C f(z)\,\mathrm{d}z &= \oint_C (u+iv)(\mathrm{d}x + i\,\mathrm{d}y) \\
&= \oint_C (u\,\mathrm{d}x - v\,\mathrm{d}y) + i\oint_C (v\,\mathrm{d}x + u\,\mathrm{d}y) \\
&= \iint\limits_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\mathrm{d}y + i\iint\limits_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

by applying the Divergence Theorem, where S is the region enclosed by C. But the Cauchy–Riemann equations show that both brackets vanish, since f is analytic throughout S. The result follows.

This result is, of course, not true if C encircles a singularity (we could not then use the Cauchy–Riemann equations throughout S).

Changing the Contour

Suppose that C_1 and C_2 are two contours from a to b and that there are *no* singularities of f on *or between* the contours. Let C be the contour consisting of C_1 followed by the reverse of C_2. C is a simple closed contour, so

$$\oint_C f(z)\,\mathrm{d}z = 0$$

(no singularities are enclosed). Hence

$$\int_{C_1} f(z)\,\mathrm{d}z - \int_{C_2} f(z)\,\mathrm{d}z = 0,$$

i.e.

$$\int_{C_1} f(z)\,\mathrm{d}z = \int_{C_2} f(z)\,\mathrm{d}z.$$

So if we have one contour, we can move it around so long as we don't cross any singularities as we move it.

If f has no singularities anywhere, then $\int_a^b f(z)\,\mathrm{d}z$ does not depend at all on the path chosen.

The same idea of "moving the contour" applies to closed contours; if C_1 and C_2 are closed contours as shown, then

$$\oint_{C_1} f(z)\,\mathrm{d}z = \oint_{C_2} f(z)\,\mathrm{d}z$$

so long as there are no singularities between C_1 and C_2. We prove this by considering the closed contour C shown: clearly

$$0 = \oint_C f(z)\,\mathrm{d}z = \oint_{C_1} f(z)\,\mathrm{d}z - \oint_{C_2} f(z)\,\mathrm{d}z$$

(the two integrals along the "joins" shown cancel).

5.3 The Integral of $f'(z)$

For a real function $f(x)$, $\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$. This result extends immediately to complex functions, so long as both f and f' are analytic in some simply-connected region R and the integration contour C lies entirely in R. Then

$$\boxed{\int_a^b f'(z)\,\mathrm{d}z = f(b) - f(a)}$$

for any complex points a, b in R.

Note that the specified conditions ensure that the integral on the LHS is independent of exactly which path in R is used from a to b, using the results of §5.2.

Examples:

(i) $\int_0^i z\,\mathrm{d}z = \frac{1}{2}(i^2 - 0^2) = -\frac{1}{2}$. ($f$ and f' are analytic in the whole of $\mathbb{C}$, so the LHS is path-independent.)

(ii) $\int_C e^z\,\mathrm{d}z$, where C is the semicircular contour joining -1 to $+1$ along $|z| = 1$ above the real axis, is equal to $e - e^{-1}$.

(iii) $\int_{1+i}^{-1+i} z^{-1}\,\mathrm{d}z$ via a straight contour. Note that z^{-1} is not analytic everywhere, so we *do* need to specify the contour; but we can define a simply-connected region R, given by $\operatorname{Im} z > \frac{1}{2}$ say, in which it *is* analytic, and C lies entirely in R. Let $f(z) = \log z$ with the standard branch cut, so that $f(z)$ is also analytic in R; then

$$\begin{aligned}\int_{1+i}^{-1+i} z^{-1}\,\mathrm{d}z &= \log(-1+i) - \log(1+i)\\ &= \log\sqrt{2} + \tfrac{3}{4}\pi i - (\log\sqrt{2} + \tfrac{1}{4}\pi i)\\ &= \tfrac{1}{2}\pi i.\end{aligned}$$

(iv) Now consider $\int_{1+i}^{-1+i} z^{-1}\,\mathrm{d}z$ via the contour shown. Define R as in the diagram; we cannot now choose the standard branch cut for $\log z$ (since C would cross it), so we choose a cut along the positive imaginary axis, and define $\log re^{i\theta} = \log r + i\theta$ where $-\frac{3\pi}{2} < \theta \le \frac{\pi}{2}$. Then

$$\begin{aligned}\int_C z^{-1}\,\mathrm{d}z &= \log(-1+i) - \log(1+i)\\ &= \log\sqrt{2} + (-\tfrac{5}{4}\pi)i - (\log\sqrt{2} + \tfrac{1}{4}\pi i)\\ &= -\tfrac{3}{2}\pi i.\end{aligned}$$

5.4 The Calculus of Residues

The Contour Integral of a Laurent Expansion

Consider a single term $a_n(z-z_0)^n$ of an expansion, integrated round a closed curve C which encircles z_0 in a positive sense (i.e., anticlockwise) once. For $n \ge 0$, we can use Cauchy's Theorem to obtain immediately

$$\oint_C a_n(z-z_0)^n\,\mathrm{d}z = 0.$$

For $n < 0$, first change the contour C to C_ε, a circle of radius ε about z_0, using the ideas of §5.2. On C_ε, $z = z_0 + \varepsilon e^{i\theta}$ and so

$$\begin{aligned}\oint_C a_n(z-z_0)^n\,\mathrm{d}z &= \int_0^{2\pi} a_n\varepsilon^n e^{in\theta}\, i\varepsilon e^{i\theta}\,\mathrm{d}\theta\\ &= ia_n\varepsilon^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\,\mathrm{d}\theta\\ &= \begin{cases} ia_n\varepsilon^{n+1}\left[\dfrac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} & n \ne -1\\ ia_n\varepsilon^{n+1}(2\pi) & n = -1\end{cases}\\ &= \begin{cases} 0 & n \ne -1\\ 2\pi i a_{-1} & n = -1\end{cases}\end{aligned}$$

We deduce that for a function $f(z)$ with a singularity at z_0, and a contour C encircling the singularity in a positive sense,

$$\oint_C f(z)\,\mathrm{d}z = \sum_{n=-\infty}^{\infty}\oint_C a_n(z-z_0)^n\,\mathrm{d}z = 2\pi i a_{-1} = 2\pi i \operatorname*{res}_{z=z_0} f(z).$$

We can also obtain the result as follows, using the method of §5.3:

$$\oint_C a_n (z-z_0)^n \, \mathrm{d}z = \begin{cases} \dfrac{a_n}{n+1} \big[(z-z_0)^{n+1} \big]_C & n \neq -1 \\ a_n \big[\log(z-z_0) \big]_C & n = -1 \end{cases}$$

$$= \begin{cases} 0 & n \neq -1 \quad \text{(because } (z-z_0)^{n+1} \text{ is single-valued)} \\ 2\pi i a_{-1} & n = -1 \quad \text{(because } \theta \text{ changes by } 2\pi) \end{cases}$$

The Residue Theorem

Suppose that $f(z)$ is analytic in a simply-connected region R except for a finite number of poles at $z_1, z_2, \ldots, z_n$; and that a simple closed curve C encircles the poles anticlockwise. Then

$$\boxed{\oint_C f(z) \, \mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname*{res}_{z=z_k} f(z).}$$

(We have just proved this in the case of a single pole.)

Proof: Consider the curve $\widehat{C}$ shown. $\widehat{C}$ encircles no poles, so

$$\oint_{\widehat{C}} f(z) \, \mathrm{d}z = 0$$

by Cauchy's Theorem. But we can also work out the integral round $\widehat{C}$ by adding together several contributions: the large outer curve (which is the same as C), the small circles around each pole, and the contributions from the lines joining the outer curve to the inner circles. For each k, the contribution from the small circle round z_k is $-2\pi i \operatorname{res}_{z=z_k} f(z)$ because the small circle goes *clockwise* round z_k. Also, the contribution from the line joining the outer curve to the small circle cancels exactly with the contribution from the line going back. Hence

$$0 = \oint_{\widehat{C}} f(z) \, \mathrm{d}z = \oint_C f(z) \, \mathrm{d}z + \sum_{k=1}^{n} \Big(-2\pi i \operatorname*{res}_{z=z_k} f(z) \Big)$$

from which the result follows.

5.5 Cauchy's Formula for $f(z)$

Suppose that $f(z)$ is analytic in a region R and that z_0 lies in R. Then Cauchy's formula states that

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \, \mathrm{d}z}$$

where C is any closed contour in R encircling z_0 once anticlockwise.

Proof: $f(z)/(z - z_0)$ is analytic except for a simple pole at z_0, where it has residue $f(z_0)$. Using the Residue Theorem,

$$\oint_C \frac{f(z)}{z - z_0} \, \mathrm{d}z = 2\pi i f(z_0)$$

as required.

Note: Cauchy's formula says that if we know f on C then we know it at all points within C. We can see that this must be so by the uniqueness theorem of Chapter 2: u and v, the real and imaginary parts of f, are harmonic, so if they are specified on C (Dirichlet boundary conditions), then there is a *unique* solution for u and v inside C.

Exercise: show that if instead f is analytic except for a singularity at z_0, and has a Laurent expansion $\sum_{m=-\infty}^{\infty} a_m (z - z_0)^m$, then the coefficients of the expansion are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z.$$

If we differentiate Cauchy's formula with respect to z_0 (differentiating under the $\oint$ sign on the RHS), we see that

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} \, \mathrm{d}z.$$

So $f'(z_0)$ is known for all z_0 inside C. Continuing this process,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z,$$

and $f^{(n)}(z_0)$ is known. So at any point where f is analytic, i.e. differentiable once, *all* its derivatives exist; hence it is differentiable infinitely many times.

5.6 Applications of the Residue Calculus

Suppose we wish to evaluate

$$I = \int_0^\infty \frac{\mathrm{d}x}{1 + x^2}$$

(which we can already do using trigonometric substitutions).

Consider

$$\oint_C \frac{\mathrm{d}z}{1+z^2}$$

where C is the contour shown: from $-R$ to R along the real axis (C_0) then returning to $-R$ via a semicircle of radius R in the upper half-plane (C_R). Now $(1+z^2)^{-1} = (z+i)^{-1}(z-i)^{-1}$, so the only singularity enclosed by C is a simple pole at $z = i$, where the residue is $\lim_{z\to i}(z+i)^{-1} = 1/2i$. Hence

$$\int_{C_0} \frac{\mathrm{d}z}{1+z^2} + \int_{C_R} \frac{\mathrm{d}z}{1+z^2} = \oint_C \frac{\mathrm{d}z}{1+z^2} = 2\pi i \, \frac{1}{2i} = \pi.$$

Now

$$\int_{C_0} \frac{\mathrm{d}z}{1+z^2} = \int_{-R}^{R} \frac{\mathrm{d}x}{1+x^2} \to 2I \quad \text{as } R \to \infty.$$

Consider $\int_{C_R} \mathrm{d}z/(1+z^2)$: the integrand $(1+z^2)^{-1}$ is of order R^{-2} on the semicircle, but the length of the contour is πR. Hence

$$\left| \int_{C_R} \frac{\mathrm{d}z}{1+z^2} \right| \le \pi R \times O(R^{-2}) = O(R^{-1}) \to 0 \quad \text{as } R \to \infty.$$

Combining all these results and taking the limit as $R \to \infty$,

$$2I + 0 = \pi,$$

i.e. $I = \pi/2$.

This example is not in itself impressive. But the power of the method is clear when we see how easily it adapts to other such integrals (for which it would not be easy, or would be impossible, to use substitutions). Examples:

(i) We wish to calculate

$$I = \int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}$$

where $a > 0$ is a real constant. We consider $\oint_C \mathrm{d}z/(z^2+a^2)^2$; most of the above analysis is unchanged. The poles now occur at $z = \pm ia$, and they both have order 2; only the pole at $+ia$ is enclosed by C. The residue there is

$$\lim_{z\to ia} \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{(z+ia)^2} = \lim_{z\to ia} \frac{-2}{(z+ia)^3} = \frac{-2}{-8ia^3} = -\tfrac{1}{4}ia^{-3}.$$

The integral round the semicircle still vanishes as $R \to \infty$, since now

$$\left| \int_{C_R} \frac{\mathrm{d}z}{(z^2+a^2)^2} \right| \le \pi R \times O(R^{-4}) = O(R^{-3}).$$

Therefore

$$2I = 2\pi i(-\tfrac{1}{4}ia^{-3}) = \pi/2a^3,$$

i.e., $I = \pi/4a^3$.

(ii) For $I = \int_0^\infty \mathrm{d}x/(1+x^4)$, the (simple) poles are at $e^{\pi i/4}$, $e^{3\pi i/4}$, $e^{-\pi i/4}$ and $e^{-3\pi i/4}$.

Only the first two poles are enclosed. The residue at $e^{\pi i/4}$ is

$$\lim_{z\to e^{\pi i/4}} \frac{z - e^{\pi i/4}}{1+z^4} = \lim_{z\to e^{\pi i/4}} \frac{1}{4z^3} = \tfrac{1}{4}e^{-3\pi i/4} = -\tfrac{1}{4}e^{\pi i/4}$$

using L'Hôpital's Rule, and at $e^{3\pi i/4}$ it is (similarly) $\frac{1}{4}e^{-\pi i/4}$. Hence

$$2I = 2\pi i(-\tfrac{1}{4}e^{\pi i/4} + \tfrac{1}{4}e^{-\pi i/4}) = 2\pi i(-\tfrac{1}{4})(2i\sin\tfrac{\pi}{4}) = \pi\sin\tfrac{\pi}{4},$$

i.e., $I = \pi/2\sqrt{2}$.

(iii) For $I = \int_0^\infty x^2\mathrm{d}x/(1+x^4)$, the poles are as in (ii) but with residues $+\frac{1}{4}e^{-\pi i/4}$ and $-\frac{1}{4}e^{\pi i/4}$ respectively (check for yourself). So the value of the integral is unchanged.

(iv) For $I = \int_0^\infty \mathrm{d}x/(1+x^4)$ again, an alternative to the method used in example (ii) above (and similarly in example (iii) above) is to use a contour which is just a quarter-circle, as shown.

Let C consist of the real axis from 0 to R (C_0); the arc of circle from R to iR (C_1); and the imaginary axis from iR to 0 (C_2). Now $\int_{C_0} \mathrm{d}z/(1+z^4) \to I$ as $R \to \infty$; and, along C_2, we substitute $z = iy$ to obtain

$$\int_{C_2} \frac{\mathrm{d}z}{1+z^4} = \int_R^0 \frac{i\,\mathrm{d}y}{1+(iy)^4} = -i \int_0^R \frac{\mathrm{d}y}{1+y^4} \to -iI \qquad \text{as } R \to \infty.$$

The integral along C_1 vanishes as $R \to \infty$, using the same argument as for C_R above, but this time we only enclose *one* pole, which makes the calculation easier. Hence

$$I - iI = 2\pi i(-\tfrac{1}{4}e^{\pi i/4}) = -\tfrac{1}{2}\pi e^{3\pi i/4} \qquad \Longrightarrow \qquad I = \pi/2\sqrt{2}$$

as before.

Jordan's Lemma

For many applications (in particular, ones involving Fourier transforms) we need to show that

$$\boxed{\int_{C_R} f(z)e^{i\lambda z}\,\mathrm{d}z \to 0}$$

as $R \to \infty$, where $\lambda > 0$ is some real constant and f is an analytic function (except possibly for a finite number of poles). Jordan's Lemma states that this is true so long as $f(z) \to 0$ as $|z| \to \infty$. For $\lambda < 0$, the same conclusion holds for the semicircular contour C'_R in the lower half-plane.

Note that this result is obvious if $f(z) = O(|z|^{-2})$ as $|z| \to \infty$ – i.e., if $f(z) = O(R^{-2})$ on C_R – by the following argument. First note that $e^{i\lambda z} = e^{i\lambda(x+iy)} = e^{-\lambda y}e^{i\lambda x}$, and $y \geq 0$ on C_R, so $|e^{i\lambda z}| = e^{-\lambda y} \leq 1$ on C_R. Hence

$$\begin{aligned} \left| \int_{C_R} f(z)e^{i\lambda z}\,\mathrm{d}z \right| &\leq \pi R \max_{C_R} |f(z)| \\ &= \pi R \times O(R^{-2}) \to 0 \quad \text{as } R \to \infty. \end{aligned}$$

Jordan's Lemma simply extends the result from functions satisfying $f(z) = O(|z|^{-2})$ to any function satisfying $f(z) \to 0$ as $|z| \to \infty$. Examples:

$$\int_{C_R} \frac{e^{2iz}}{z}\,\mathrm{d}z \to 0 \quad \text{as } R \to \infty; \qquad \int_{C'_R} \frac{e^{-iz}}{z^2}\,\mathrm{d}z \to 0 \quad \text{as } R \to \infty.$$

The proof of Jordan's Lemma stems from the fact that for $0 \le \theta \le \pi/2$, $\sin\theta \ge 2\theta/\pi$. Now

$$\begin{aligned}
\left|\int_{C_R} f(z)e^{i\lambda z}\,\mathrm{d}z\right| &\le \max_{C_R}|f(z)|\int_0^\pi |e^{i\lambda z}|\,|Re^{i\theta}|\,\mathrm{d}\theta \\
&= R\max|f(z)|\int_0^\pi e^{-\lambda R\sin\theta}\,\mathrm{d}\theta \\
&\qquad [\text{using } y = R\sin\theta] \\
&= 2R\max|f(z)|\int_0^{\pi/2} e^{-\lambda R\sin\theta}\,\mathrm{d}\theta \\
&\le 2R\max|f(z)|\int_0^{\pi/2} e^{-2\lambda R\theta/\pi}\,\mathrm{d}\theta \\
&= \frac{\pi}{\lambda}(1-e^{-\lambda R})\max|f(z)| \\
&\to 0 \quad \text{as } R\to\infty.
\end{aligned}$$

A similar proof holds on C'_R for $\lambda < 0$.

5.7 Laplace Transforms

The Fourier transform is a powerful technique for solving differential equations and for investigating many physical problems, but not all functions have a Fourier transform: the integral defining the transform does not converge unless the function tends to zero at infinity.

To get around this restriction, we can use another kind of transform known as the *Laplace transform*. The price we pay is a different restriction: it is only defined for functions which are zero for $t < 0$ (by convention). From now on, we shall make this assumption, so that if we refer to the function $f(t) = e^t$ for instance, we really mean the function $f(t) = \begin{cases} 0 & t < 0, \\ e^t & t \ge 0. \end{cases}$

The Laplace transform of a function $f(t)$ is defined by

$$\boxed{\bar{f}(p) = \int_0^\infty f(t)e^{-pt}\,\mathrm{d}t}$$

where p may be complex. The notation $\mathscr{L}[f]$ or $\mathscr{L}[f(t)]$ is also used for $\bar{f}(p)$; and the symbol s is often used instead of p. Many functions – for instance, t and e^t – which do not have Fourier transforms *do* have Laplace transforms; however, there are still exceptions (e.g., e^{t^2}). Laplace transforms are particularly useful in initial value problems, where we are given the state of a system at $t = 0$ and desire to find its state for $t > 0$.

Examples:

(i) $\mathscr{L}[1] = \displaystyle\int_0^\infty e^{-pt}\,\mathrm{d}t = \frac{1}{p}$.

(ii) $\mathscr{L}[t] = \displaystyle\int_0^\infty te^{-pt}\,\mathrm{d}t = \left[-\frac{1}{p}te^{-pt}\right]_0^\infty + \frac{1}{p}\int_0^\infty e^{-pt}\,\mathrm{d}t = \frac{1}{p^2}$.

(iii) $\mathscr{L}[e^{\lambda t}] = \displaystyle\int_0^\infty e^{(\lambda-p)t}\,\mathrm{d}t = \frac{1}{p-\lambda}$.

(iv) $\mathscr{L}[\sin t] = \mathscr{L}\left[\dfrac{1}{2i}(e^{it} - e^{-it})\right] = \dfrac{1}{2i}\left(\dfrac{1}{p-i} - \dfrac{1}{p+i}\right) = \dfrac{1}{p^2+1}$.

Note that, strictly speaking, in example (iii), the integral only converges for $\operatorname{Re} p > \operatorname{Re}\lambda$ (otherwise the integrand, $e^{(\lambda-p)t}$, diverges as $t \to \infty$). However, once we have calculated the integral for $\operatorname{Re} p > \operatorname{Re}\lambda$ we can consider $\bar{f}(p)$ to exist everywhere in the complex p-plane (except for singularities such as at $p = \lambda$ in this example). This process of extending a complex function which is initially only defined in some part of the complex plane to the whole of the plane is known as *analytic continuation*.

It is useful to have a "library" of Laplace transforms to hand; some common ones are listed below.

$f(t)$	$\bar{f}(p)$	$f(t)$	$\bar{f}(p)$
1	$\dfrac{1}{p}$	t^n	$\dfrac{n!}{p^{n+1}}$
$e^{\lambda t}$	$\dfrac{1}{p-\lambda}$	$t^n e^{\lambda t}$	$\dfrac{n!}{(p-\lambda)^{n+1}}$
$\sin\omega t$	$\dfrac{\omega}{p^2+\omega^2}$	$\cos\omega t$	$\dfrac{p}{p^2+\omega^2}$
$\sinh\lambda t$	$\dfrac{\lambda}{p^2-\lambda^2}$	$\cosh\lambda t$	$\dfrac{p}{p^2-\lambda^2}$
$e^{\lambda t}\sin\omega t$	$\dfrac{\omega}{(p-\lambda)^2+\omega^2}$	$e^{\lambda t}\cos\omega t$	$\dfrac{p-\lambda}{(p-\lambda)^2+\omega^2}$
$\delta(t)$	1	$\delta(t-t_0)$	e^{-pt_0}

Elementary Properties of the Laplace Transform

(i) Linearity: $\mathscr{L}[\alpha f(t) + \beta g(t)] = \alpha\bar{f}(p) + \beta\bar{g}(p)$.

(ii) Change of scale: using the substitution $t' = \lambda t$,

$$\mathscr{L}[f(\lambda t)] = \int_0^\infty f(\lambda t)e^{-pt}\,\mathrm{d}t = \frac{1}{\lambda}\int_0^\infty f(t')e^{-(p/\lambda)t'}\,\mathrm{d}t' = \frac{1}{\lambda}\bar{f}\left(\frac{p}{\lambda}\right).$$

(iii) Shifting theorem: $\mathscr{L}[e^{\lambda t}f(t)] = \bar{f}(p-\lambda)$. (Easy to check.)

(iv) Derivative of a Laplace transform:

$$\boxed{\mathscr{L}[tf(t)] = -\frac{\mathrm{d}}{\mathrm{d}p}\bar{f}(p).}$$

Proof:

$$\bar{f}(p) = \int_0^\infty f(t)e^{-pt}\,\mathrm{d}t \qquad \Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}p}\bar{f}(p) = -\int_0^\infty tf(t)e^{-pt}\,\mathrm{d}t.$$

By repeating this trick n times, we see that the Laplace transform of $t^n f(t)$ is $(-1)^n\bar{f}^{(n)}(p)$.

Examples:

$$\mathscr{L}[t\sin t] = -\frac{\mathrm{d}}{\mathrm{d}p}\frac{1}{p^2+1} = \frac{2p}{(p^2+1)^2}; \qquad \mathscr{L}[t^n] = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}p^n}\frac{1}{p} = \frac{n!}{p^{n+1}}.$$

(v) Laplace transform of a derivative:

$$\boxed{\mathscr{L}\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] = p\bar{f}(p) - f(0).}$$

Proof:

$$\int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}t}e^{-pt}\,\mathrm{d}t = [f(t)e^{-pt}]_0^\infty + p\int_0^\infty f(t)e^{-pt}\,\mathrm{d}t = p\bar{f}(p) - f(0).$$

We can deduce that

$$\mathscr{L}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\right] = p\mathscr{L}\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] - \dot{f}(0) = p^2\bar{f}(p) - pf(0) - \dot{f}(0)$$

and so on.

(vi) Asymptotic limits: $p\bar{f}(p) \to f(0)$ as $p\to\infty$, and $p\bar{f}(p) \to \lim_{t\to\infty} f(t)$ as $p \to 0$. Proofs: from (v) above,

$$p\bar{f}(p) = f(0) + \int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}t}e^{-pt}\,\mathrm{d}t,$$

so as $p\to\infty$ (and therefore $e^{-pt}\to 0$ for all $t>0$), $p\bar{f}(p)\to f(0)$. Similarly, as $p\to 0$, $e^{-pt}\to 1$ so that

$$p\bar{f}(p) \to f(0) + \int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t = f(0) + [f(t)]_0^\infty = \lim_{t\to\infty} f(t).$$

Solving Differential Equations using Laplace Transforms

The Laplace transform is particularly suited to the solution of initial value problems. Example: solve

$$\ddot{y} + 5\dot{y} + 6y = 0$$

for $y(t)$ subject to $y(0) = 1$, $\dot{y}(0) = -4$. Taking Laplace transforms, and using the results for the Laplace transform of a derivative, we see that

$$(p^2\bar{y}(p) - p + 4) + 5(p\bar{y}(p) - 1) + 6\bar{y}(p) = 0,$$

which we may solve for $\bar{y}(p)$:

$$\bar{y}(p) = \frac{p+1}{p^2+5p+6} = \frac{p+1}{(p+2)(p+3)} = \frac{2}{p+3} - \frac{1}{p+2}$$

using partial fractions. We now need to invert $\bar{y}(p)$ to find $y(t)$; in general we must use the inversion formula described below, but in many cases (such as this one) it is possible to "spot" the answer using the "library" of transforms given above (and taking advantage of the fact that inverse Laplace transforms are unique). Here, we know that $\mathscr{L}[e^{\lambda t}] = 1/(p - \lambda)$; hence

$$y(t) = 2e^{-3t} - e^{-2t}.$$

The Convolution Theorem for Laplace Transforms

The convolution of two functions $f(t)$ and $g(t)$ is

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t - t')g(t')\,\mathrm{d}t'.$$

We are dealing here with functions which vanish for $t < 0$, so this reduces to

$$(f * g)(t) = \int_0^t f(t - t')g(t')\,\mathrm{d}t'$$

since $g(t') = 0$ for $t' < 0$ and $f(t-t') = 0$ for $t' > t$. The convolution theorem for Laplace transforms then states that

$$\boxed{\mathscr{L}[f * g] = \bar{f}(p)\bar{g}(p).}$$

Proof:

$$\begin{aligned}\mathscr{L}[f * g] &= \int_0^\infty \left\{ \int_0^t f(t - t')g(t')\,\mathrm{d}t' \right\} e^{-pt}\,\mathrm{d}t \\ &= \int_0^\infty \left\{ \int_0^t f(t - t')g(t')e^{-pt}\,\mathrm{d}t' \right\} \mathrm{d}t.\end{aligned}$$

From the diagram, we see that we can change the order of integration in the (t, t')-plane, giving

$$
\begin{aligned}
\mathscr{L}[f * g] &= \int_0^\infty \left\{ \int_{t'}^\infty f(t - t') g(t') e^{-pt} \, \mathrm{d}t \right\} \mathrm{d}t' \\
&= \int_0^\infty \left\{ \int_0^\infty f(t'') e^{-pt''} e^{-pt'} \, \mathrm{d}t'' \right\} g(t') \, \mathrm{d}t' \\
&\qquad\qquad [\text{substituting } t'' = t - t'] \\
&= \int_0^\infty \{ \bar{f}(p) e^{-pt'} \} g(t') \, \mathrm{d}t' \\
&= \bar{f}(p) \int_0^\infty g(t') e^{-pt'} \, \mathrm{d}t' \\
&= \bar{f}(p) \bar{g}(p)
\end{aligned}
$$

as required.

The Inverse Laplace Transform

Inverting Laplace transforms is more difficult than inverting Fourier transforms because it is always necessary to perform a contour integration. Given $\bar{f}(p)$, we can calculate $f(t)$ using the *Bromwich inversion formula*

$$
\boxed{f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(p) e^{pt} \, \mathrm{d}p.}
$$

Here γ is a real constant, and the Bromwich inversion contour Γ runs from $\gamma - i\infty$ to $\gamma + i\infty$ along a straight line. Γ *must* lie to the right of all the singularities of $\bar{f}(p)$.

Note that it is possible to derive the Bromwich inversion formula from the inverse Fourier transform by substituting $p = ik$ and noting that $\bar{f}(p) = \widetilde{f}(-ip)$ where $\widetilde{f}(k)$ is the Fourier transform of $f(t)$. The only difference is in the detail of the inversion contour.

Suppose that $\bar{f}(p)$ has only poles, and no other singularities; all these poles lie to the left of Γ. When $t < 0$, consider the integral round the contour C shown consisting of C_0 followed by C'_R. C encloses no poles, so

$$\oint_C \bar{f}(p)e^{pt}\,\mathrm{d}p = 0.$$

Now on C'_R, $\operatorname{Re} p \geq \gamma$, so $\operatorname{Re}(pt) \leq \gamma t$ (since $t < 0$) and hence $|e^{pt}| \leq e^{\gamma t}$. Therefore if $\bar{f}(p) = O(|p|^{-2})$ as $|p| \to \infty$ – i.e., if $\bar{f}(p) = O(R^{-2})$ on C'_R – then

$$\left|\int_{C'_R} \bar{f}(p)e^{pt}\,\mathrm{d}p\right| \leq \pi R e^{\gamma t} \times O(R^{-2}) \to 0$$

as $R \to \infty$. In fact the same is true even if we only have $\bar{f}(p) \to 0$ as $|p| \to \infty$, by a slight modification of Jordan's Lemma. So in either case,

$$\begin{aligned}
\int_\Gamma \bar{f}(p)e^{pt}\,\mathrm{d}p &= \lim_{R\to\infty} \int_{C_0} \bar{f}(p)e^{pt}\,\mathrm{d}p \\
&= \lim_{R\to\infty} \left(\oint_C \bar{f}(p)e^{pt}\,\mathrm{d}p - \int_{C'_R} \bar{f}(p)e^{pt}\,\mathrm{d}p \right) \\
&= 0 - 0 = 0,
\end{aligned}$$

and therefore for $t < 0$ the inversion formula gives

$$f(t) = \frac{1}{2\pi i} \int_\Gamma \bar{f}(p)e^{pt}\,\mathrm{d}p = 0$$

(as it must do, since $f(t) = 0$ for $t < 0$ by our initial assumption).

When $t > 0$, we close the contour to the left instead, and once again we can show that

$$\int_{C_R} \bar{f}(p)e^{pt}\,\mathrm{d}p \to 0$$

as $R \to \infty$, so long as $\bar{f}(p) \to 0$ as $|p| \to \infty$. Hence in the limit $R \to \infty$ we obtain

$$\int_\Gamma \bar{f}(p)e^{pt}\,\mathrm{d}p = 2\pi i \sum_{k=1}^{n} \operatorname*{res}_{p=p_k} \left(\bar{f}(p)e^{pt}\right)$$

by the Residue Theorem, where $p_1, \dots, p_n$ are the poles of $\bar{f}(p)$. We deduce that

$$\boxed{f(t) = \sum_{k=1}^{n} \operatorname*{res}_{p=p_k} \left(\bar{f}(p)e^{pt}\right)}$$

for $t > 0$, *so long as* $\bar{f}(p) \to 0$ as $|p| \to \infty$.

Examples:

(i) $\bar{f}(p) = 1/(p-1)$. This has a pole at $p = 1$, so we must use $\gamma > 1$. We have $\bar{f}(p) \to 0$ as $|p| \to \infty$, so Jordan's Lemma applies as above. For $t < 0$, therefore, $f(t) = 0$, and for $t > 0$,

$$f(t) = \operatorname*{res}_{p=1} \left(\frac{e^{pt}}{p-1}\right) = e^t.$$

This agrees with our earlier result for the Laplace transform of $e^{\lambda t}$ when $\lambda = 1$.

(ii) $\bar{f}(p) = p^{-n}$. Here we need $\gamma > 0$, because there is a pole of order n at $p = 0$. For $t < 0$, $f(t) = 0$ as usual. For $t > 0$,

$$\begin{aligned} f(t) = \operatorname*{res}_{p=0} \left(\frac{e^{pt}}{p^n}\right) &= \lim_{p \to 0} \left\{ \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}p^{n-1}} e^{pt} \right\} \\ &= \lim_{p \to 0} \left\{ \frac{1}{(n-1)!} (t^{n-1} e^{pt}) \right\} \\ &= \frac{t^{n-1}}{(n-1)!}. \end{aligned}$$

(iii) What if $\bar{f}(p) \not\to 0$ as $|p| \to \infty$? Consider the example

$$\bar{f}(p) = \frac{e^{-p}}{p};$$

here, as $p \to -\infty$ on the real axis, $\bar{f}(p) \to \infty$. We need to calculate

$$f(t) = \frac{1}{2\pi i} \int_\Gamma \frac{e^{-p}}{p} e^{pt} \, \mathrm{d}p,$$

but Jordan's Lemma does not immediately apply. Note, however, that $e^{-p}e^{pt} = e^{p(t-1)} = e^{pt'}$ where $t' = t - 1$; so

$$f(t) = \frac{1}{2\pi i} \int_\Gamma \frac{e^{pt'}}{p} \, \mathrm{d}p.$$

Now we *can* use Jordan's Lemma: when $t' < 0$, close to the right, and when $t' > 0$, close to the left, picking up the residue from the pole at $p = 0$. Hence

$$f(t) = \begin{cases} 0 & t' < 0, \\ 1 & t' > 0 \end{cases}$$

$$= \begin{cases} 0 & t < 1, \\ 1 & t > 1. \end{cases}$$

What function $f(t)$ has Laplace transform $\bar{f}(p) = p^{-1/2}$? We need to find

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} p^{-1/2} e^{pt} \, \mathrm{d}p.$$

For $t > 0$ we can close the contour to the right as usual and obtain $f(t) = 0$. For $t < 0$, however, the branch cut gets in the way.

Use a contour as shown, with a small circle of radius ε round the origin and two large quarter-circles of radius R. Substituting $p = \varepsilon e^{i\theta}$ on the small circle gives a contribution of

$$\int_{\pi}^{-\pi} \varepsilon^{-1/2} e^{-i\theta/2} e^{\varepsilon e^{i\theta} t} \, i\varepsilon e^{i\theta} \, \mathrm{d}\theta = O(\varepsilon^{1/2}) \to 0 \quad \text{as } \varepsilon \to 0.$$

Similarly, the integrals round the two large quarter-circles vanish as $R \to \infty$, using the method used to prove Jordan's Lemma. Hence the required integral is equal to the sum of the integrals on either side of the branch cut: i.e., for $t > 0$,

$$f(t) = \frac{1}{2\pi i} \left\{ - \int_{\infty}^{0} r^{-1/2} e^{-i\pi/2} e^{-rt} (-\mathrm{d}r) - \int_{0}^{\infty} r^{-1/2} e^{i\pi/2} e^{-rt} (-\mathrm{d}r) \right\}$$

[substituting $p = re^{i\pi}$ and $p = re^{-i\pi}$ respectively]

$$= \frac{1}{2\pi i} \left\{ 2i \int_{0}^{\infty} r^{-1/2} e^{-rt} \, \mathrm{d}r \right\}$$

$$= \frac{2}{\pi} \int_{0}^{\infty} e^{-s^2 t} \, \mathrm{d}s$$

[substituting $r = s^2$]

$$
\begin{aligned}
&= \frac{1}{\pi}\sqrt{\frac{\pi}{t}} \\
&= \frac{1}{\sqrt{\pi t}}.
\end{aligned}
$$

So $\mathscr{L}[t^{-1/2}] = \sqrt{\pi}\, p^{-1/2}$. This is a generalisation of the result that $\mathscr{L}[t^n] = n!/p^{n+1}$ to $\mathscr{L}[t^\alpha] = \Gamma(\alpha + 1)/p^{\alpha+1}$ where the *Gamma function* is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,\mathrm{d}x$$

and can easily be shown to be equal to $(\alpha - 1)!$ when α is a positive integer.
