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Statistical Field Theory

Examples 1

Thermodynamics

(1) In standard notation, the first law of thermodynamics for a magnetic system is

$$dU \;=\; TdS \;-\; MdH\ .$$

where U is the internal energy, M is the magnetization and H the applied magnetic field. Define the other functions of state by appropriate Legendre transformations, for example $F = U - TS$ and hence obtain the Maxwell relations:

$$\left(\frac{\partial T}{\partial H}\right)_S \;=\; -\left(\frac{\partial M}{\partial S}\right)_H \qquad\qquad \left(\frac{\partial S}{\partial H}\right)_T \;=\; \left(\frac{\partial M}{\partial T}\right)_H$$

$$\left(\frac{\partial S}{\partial M}\right)_T \;=\; -\left(\frac{\partial H}{\partial T}\right)_M \qquad\qquad \left(\frac{\partial T}{\partial M}\right)_S \;=\; \left(\frac{\partial H}{\partial S}\right)_M$$

(2) The specific heats at constant magnetic field and at constant magnetization for the magnetic system are

$$C_H \;=\; T\left(\frac{\partial S}{\partial T}\right)_H\ , \qquad C_M \;=\; T\left(\frac{\partial S}{\partial T}\right)_M\ .$$

The isothermal and adiabatic susceptibilities are

$$\chi_T \;=\; \left(\frac{\partial M}{\partial H}\right)_T\ , \qquad \chi_S \;=\; \left(\frac{\partial M}{\partial H}\right)_S\ .$$

Also define

$$\alpha_H \;=\; \left(\frac{\partial M}{\partial T}\right)_H\ .$$

From the identity

$$\left(\frac{\partial S}{\partial T}\right)_M \;=\; \left(\frac{\partial S}{\partial T}\right)_H \;+\; \left(\frac{\partial S}{\partial H}\right)_T\left(\frac{\partial H}{\partial T}\right)_M\ ,$$

deduce that

$$\chi_T\,(C_H - C_M) \;=\; -\,T\left(\frac{\partial M}{\partial H}\right)_T\left(\frac{\partial M}{\partial T}\right)_H\left(\frac{\partial H}{\partial T}\right)_M\ ,$$

and hence show that

$$\chi_T\,(C_H - C_M) \;=\; T\alpha_H^2 \quad (\dagger)\ .$$

By similar means show that

$$C_H\,(\chi_T - \chi_S) \;=\; T\alpha_H^2\ .$$

Hence show that

$$\chi_S C_H = \chi_T C_M\ .$$

[*For this question you will need the identity*

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \;=\; -1\,. \qquad]$$

(3) For $T \to T_c$ for $T < T_c$ and $H = 0$ the dependence of the following observables on T is parametrized by

$$\begin{aligned} C_H &\sim (T_c - T)^{-\alpha} \\ M &\sim (T_c - T)^{\beta} \\ \chi_T &\sim (T_c - T)^{-\gamma}\,. \end{aligned}$$

Using that $C_M > 0$ and (†) above derive Rushbrooke's inequality:

$$\alpha + 2\beta + \gamma \geq 2.$$

What is the equivalent set of statements for a gaseous system?

Statistical Models and Landau Theory

(4) The Hamiltonian for a set of N spins $\{\mathbf{s}_n\}$ that are 3-dimensional vectors in the presence of a magnetic field $\mathbf{H}$ is

$$\mathcal{H} \;=\; \sum_n \mu H \cos\theta_n \;,$$

where θ_n is the angle between between $\mathbf{s}_n$ and $\mathbf{H}$. Show that the partition function is

$$\mathcal{Z} \;=\; \left[4\pi\left(\frac{\sinh\,\beta\mu H}{\beta\mu H}\right)\right]^N \;.$$

Compute the free energy F, the entropy S and the internal energy U. Find the equation of state and compute the susceptibility χ_T. Examine the behaviour of χ_T at low T.

(5) In a modification of the 1-dimensional Ising model the spins can take the values $\sigma_n = 1, 0, -1$. Show that the partition function is

$$\mathcal{Z} \;=\; \mathrm{Tr}\, W^n \;,$$

where W is the 3×3 matrix

$$\begin{pmatrix} z\mu^2 & \mu & z^{-1} \\ \mu & 1 & \mu^{-1} \\ z^{-1} & \mu^{-1} & z\mu^{-2} \end{pmatrix}$$

with $z = e^{\beta J}$ and $\mu = e^{\beta h/2}$.

For the case $h = 0$ show that this matrix can be expressed in the form $W = P\Lambda P^{-1}$ where

$$\Lambda \;=\; \begin{pmatrix} 2\cosh\beta J & \sqrt{2} & 0 \\ \sqrt{2} & 1 & 0 \\ 0 & 0 & 2\,\mathrm{sinh}\beta J \end{pmatrix} \;, \qquad P \;=\; \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \;.$$

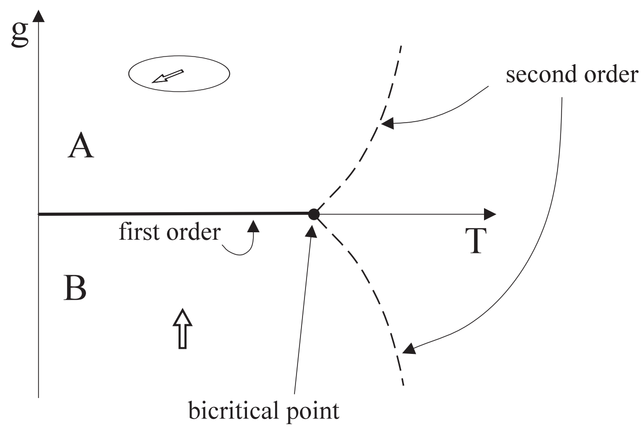
Hence find the eigenvalues of W and show that in the thermodynamic limit the free energy of the system is

$$F \;=\; -\,NkT\log\,\left\{\left(1+2\cosh\beta J \;+\; \sqrt{(2\cosh\beta J-1)^2+8}\right)/2\right\}\;.$$

(6) Give a plausible arguement that the phase diagram of the 3D spin model with Hamiltonian

$$H = -J\sum_{<ij>}\mathbf{s}_i\cdot\mathbf{s}_j + \frac{1}{2}g\sum_i\left((\mathbf{s}_i^z)^2 - \frac{1}{2}\left((\mathbf{s}_i^x)^2+(\mathbf{s}_i^y)^2\right)\right)$$

has the form



where $-\infty < g < \infty$, $< i,j >$ means nearest neighbour pairs, and $\mathbf{s}_i$ is a vector at the i-th site with $|\mathbf{s}_i| = 1$.

You should consider the nature of the dominant configurations for low and high temperature for $|g|$ very large, and the type of transition that is likely to separate them. Then ask what happens for low T as g changes sign. Note that in $D = 3$ the $O(2)$, plane rotator, model (the spin at each site, $\mathbf{s}_i$, is a unit vector lying in the xy-plane) exhibits a continuous phase transition.

[*If you look in the extra material at the end of the notes on the web then you will see the answer.*]