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Statistical Field Theory

Examples 2

Statistical Models and Landau Theory

(1) By considering $<\sigma_0>$ in equilibrium show that in the mean field approximation to the Ising model in D-dimensions the equilibrium magnetisation is given by the solution to

$$M = \tanh \beta(qJM + h)$$

where $\beta = 1/kT$, $q = 2D$, $J(>0)$ is the nearest neighbour coupling constant and h is the applied magnetic field.

The free energy in this approach is

$$A = -kT \log\left[2\cosh\beta(qJM + h)\right] \ + \ \frac{1}{2}qJM^2.$$

Show that the expression for the equilibrium magnetization above can also be obtained by minimizing A with respect to M.

The critical exponent α governs the divergence in the specific heat:

$$C \ = \ T\left(\frac{\partial^2 A}{\partial T^2}\right)_{h=0}, \qquad C \sim |T - T_c|^{-\alpha} \quad T \to T_c. \tag{1}$$

Using the expression for M above show that the free energy in equilibrium for $h = 0$ is

$$A \ = \ -kT\log 2 \ + \ \frac{1}{2}kT\log\left(1 - M^2\right) \ + \ \frac{1}{2}qJM^2\,, \tag{2}$$

By assuming the expansion $M^2 = Ct + Dt^2 + \ldots$ where $t \equiv (T - T_c)/T_c$, derive that

$$A \ = \ -kT\log 2 \ + \ \frac{1}{2}kT\log\left(1 - Ct - Dt^2 - \ldots\right) \ + \ \frac{1}{2}qJ(Ct + Dt^2 + \ldots)\,.$$

By expanding A in t show that

$$A \ = \ a_0 + a_1 t + \frac{1}{2}a_2 t^2 + \ldots\,,$$

for constants a_i, and consequently that

$$C \ = \ a_2 \ + \ O(t),$$

and hence that the exponent α is given by $\alpha = 0$.

We note also that, except for α we can get all exponents from the equation of state by expanding tanh near $t = 0$:

$$M \ = \ (\beta qJM + \beta h) \ - \ \frac{1}{3}\ (\beta qJM + \beta h)^3 \ + \ \ldots\,,$$

or, keeping only linear terms in h (ignore e.g. hM^2 since this gives non-leading singular behaviour – check it),

$$\frac{1}{3}(\beta qJ)^3 M^3 \ + \ (1 - \beta qJ)M \ - \ \beta h \ = \ 0\,,$$

as we find from minimizing A.

(2) In the Blume-Capel model in D-dimensions the spins σ_n take values $\sigma_n = 1, 0, -1$. The Hamiltonian is an extension of the Ising-like one discussed in the previous question:

$$H \; = \; -J\sum_n \sigma_n\sigma_{n+\mu} \; + \; g\sum_n \sigma_n^2 \; - \; h\sum_n \sigma_n \; .$$

Use the mean field approach to show that the free energy of this system is approximated by

$$A \;\equiv\; \frac{F}{N} \;=\; \frac{1}{2}Jq\, M^2 - kT\; \log\left(1 + 2\kappa\, \cosh\beta(JqM + h)\right) \; .$$

where $\kappa = \exp(-\beta g)$. (*Hint: do not approximate the* $g\sum_\sigma\, \sigma_n^2$ *term.*)

For $h = 0$ expand A as a power series in M. For what values of (T, κ) does mean field theory predict (i) ordinary critical behaviour, (ii) tricritical behaviour, (iii) a first order transition? In each case find the value of the critical temperature $T_c(\kappa)$.

Calculate the critical exponent α for both critical and tricritical behaviours.

(3) The free energy, A, of an Ising system with order parameter, M, is given by

$$A = -hM + A_2M^2 + A_4M^4 + A_6M^6,$$

where it is assumed that $A_6 > 0$ and that A_2 and A_4 are functions of the external fields T and g, with $A_2 \sim (T - T_c(g))$ and where h is the applied magnetic field. (Note, this is similar to Q2 where $\kappa = \exp(-\beta g)$.)

On dimensional grounds argue that at equilibrium A may be expressed as

$$A = \frac{|A_2|^{\frac{3}{2}}}{A_6^{\frac{1}{2}}} F\left(\frac{A_4}{2|A_2|^{\frac{1}{2}}A_6^{\frac{1}{2}}}, \frac{hA_6^{\frac{1}{4}}}{|A_2|^{\frac{5}{4}}}\right),$$

where $F(0, 0)$ is finite and non-zero.

Compare this expression with the **generic** form for the free energy, A, near the tri-critical point, namely

$$A = |T - T_c(\tilde{g})|^{2-\alpha} F\left(\frac{\tilde{g}}{|T - T_c(\tilde{g})|^{\phi}}, \frac{h}{|T - T_c(\tilde{g})|^{\Delta}}\right),$$

where $\tilde{g} \propto A_4$ and $\tilde{g}$ has been substituted for g as one of the independent external fields. Deduce that

$$\alpha = \frac{1}{2}, \quad \phi = \frac{1}{2}, \quad \Delta = \frac{5}{4}.$$

Define the critical temperature at the tricritical point to be $T_{TCP} \equiv T_c(\tilde{g} = 0)$.

(i) For $h = 0$ consider the trajectory in $(T, \tilde{g})$ space defined by the limit

$$\tilde{g} \to 0, \qquad T \to T_{TCP},$$

with

$$\frac{\tilde{g}}{|T - T_c(\tilde{g})|^{\phi}} = x \qquad \text{fixed.}$$

Observe that $A \sim |T - T_{TCP}|^{\frac{3}{2}} \;\; \Rightarrow \alpha = \frac{1}{2}$, The trajectory lies in the **tricritical region**, i.e., we see tricritical exponents as we approach the transition. ϕ is known as the crossover exponent since it controls the shape of the trajectory nd hence defines the boundary of the tricritical region.

(ii) For $h = 0$, $\tilde{g}$ fixed and $T \to T_c$ show that $\alpha = 0$ (i.e. normal critical behaviour) as long as it can be assumed that the function G defined by

$$yG(y, 0) = F(\frac{1}{y}, 0)$$

is finite and non-zero at $y = 0$.

The crucial point is that to use dimensional analysis the existence of scaling functions such as F and G must be assumed and that these functions are finite and non-zero when their arguments are set to zero.

The Renormalization Group

(4) Derive the RG transformation equations for the 1D Ising model as given in the notes:

$$\begin{aligned} x' &= \frac{x(1+y)^2}{(x+y)(1+xy)} \\ y' &= \frac{y(x+y)}{(1+xy)} \\ w' &= \frac{w^2xy^2}{(1+y)^2(x+y)(1+xy)} \, . \end{aligned}$$

where

$$x = e^{-4\beta J} \, , \qquad y = e^{-2\beta h} \, , \qquad w = e^{4\beta C} \, .$$

Show that there is a fixed point at $x = 0, y = 1$. Linearize the transformation about this fixed point, derive the fixed point eigenvalues and the two associated critical indices. Hence, deduce that the singular part of the free energy per spin satisfies

$$f(x, \rho) \;=\; b^{-1}f(b^2x, b\rho),$$

for a scale change $b = 2^p$ and where $y = 1 - \rho$. Use this result to show that

$$f(x, \rho) \;=\; \sqrt{x}\tilde{f}(\rho/\sqrt{x}) \, ,$$

where $\tilde{f}(z) = f(1, z)$. Verify that this is consistent with the exact result for the free energy if we choose

$$\tilde{f}(z) \;=\; -\, kT\sqrt{1 + z^2/4} \, .$$

for the **singular** part.

(5) Compare the complete expression for the free energy F, derived in the notes, with its scaling form. What plays the rôle of the inhomogeneous part?

For fixed J and $h = 0$ find an expression as $T \to 0$ for the leading singularity in the analogue of the specific heat $C = \partial^2F/\partial t^2$. Comment on what this implies for the value of α and the validity of the scaling relation $\alpha + 2\beta + \gamma = 2$.

Suppose now $T > 0$ and fixed with $h = 0$ and let $J \to \infty$. What is the value of α in this case?

(6) (*This is Q5.2 from Binney et al. "The Theory of Critical Phenomena"*) Consider the two-dimensional Ising model on a square lattice with nearest- and next-nearest-neighbour interactions only, and denote the couplings by K and L, respectively. Perform a thinning of degrees of freedom by summing over the spins on every second

site in a ‘checker-board’ fashion. Rescale the lattice by factor $b = \sqrt{2}$ to recover the original lattice spacing. Calculate the interactions on the blocked lattice keeping only terms up to $O(K^2)$ and $O(L)$. Show that there are only two such interactions to this order which are the same operators as the original ones but with couplings

$$K' \;=\; 2K^2 + L, \qquad L' \;=\; K^2\,.$$

Find the critical points for these RG equations and identify the non-trivial one. Linearizing about this point, find a value for the exponent ν.

Sketch the RG flows in the K, L plane with $K > 0, L > 0$.