Mathematical Tripos Part III — Michaelmas Term 2014

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Statistical Field Theory

Examples 3

Field Theory

(1) In the Gaussian model with $\alpha = 1$

$$\Delta(p) \;=\; \frac{1}{p^2+m^2}\;.$$

Let $x = (s, \mathbf{x})$ where $\mathbf{x}$ is a $D-1$ dimensional vector. Show that

$$\int d^{D-1}\mathbf{x}\;\Delta(s,\mathbf{x}) \;\propto\; \xi\, e^{-|s|/\xi}\;,$$

where $\xi = 1/m$ is the correlation length.

An optional and much harder result to obtain is

$$G(r) \;=\; \int \frac{d^Dp}{(2\pi)^D}\, e^{-i\mathbf{p}\cdot\mathbf{x}} \frac{\alpha}{p^2+\alpha m^2} \;\sim\; \begin{cases} \dfrac{\xi e^{-r/\xi}}{(r\xi)^{(D-1)/2}} & a \ll \xi \ll r\;, \\[2ex] \dfrac{1}{r^{(D-2)}} & a \ll r \ll \xi\;, \end{cases}$$

with $\xi^{-2} = \alpha m^2$ and α constant. A hint is to heavily approximate the integral in the two regions by recognizing that for $a \ll \xi \ll r$ it is dominated by $p \ll m$ and for $a \ll r \ll \xi$ it is dominated by $p \gg m$.

In an interacting theory α depends on p and we write

$$\frac{\alpha(p,\xi)}{p^2+\alpha(p,\xi)m^2} \;=\; \frac{Z(p,\xi)}{p^2+\xi^{-2}}\;,$$

which **defines** $Z(p,\xi)$. For large enough ξ we find that

$$Z(p,\xi) \;\sim\; \begin{cases} (ap)^{-\sigma} & p \gg \xi^{-1}\;, \\[1ex] (a/\xi)^{-\sigma} & p \ll \xi^{-1}\;, \end{cases}$$

In this case, can you infer the asymptotic forms for $G(r)$ similar to those stated above?

(2) The partition function of a scalar $D = 0$ field theory is given by

$$\mathcal{Z} \;=\; \int dx\; e^{-\left(\frac{1}{2}\lambda x^2 + \frac{g}{4!}x^4\right)}\;.$$

Derive the Feynman rules for the perturbation expansion in g for the connected r-point function $\langle x^r\rangle_c$. To $O(g^2)$ evaluate $\langle x^4\rangle_c$ and verify your answer by explicit calculation. Try some other low order calculations of your choice.

(3) The partition function for a Euclidean scalar field theory is defined in terms of the action $S(\phi)$ by

$$Z = \int \{d\phi\} e^{-S(\phi)/\lambda}.$$

$S(\phi)$ can be written explicitly in the form

$$S(\phi) = \int d^D x \frac{1}{2}(\nabla\phi)^2 \quad + \quad \frac{1}{2}m^2\phi^2 \quad + \quad \sum g_n \frac{\phi^{2n}}{2n!} \quad + \quad \ldots$$

More interactions such as derivative interactions can be included in principle.

By inspecting the rules for generating the diagrammatic expansion of the theory show that

(i) every vertex carries a factor λ^{-1},

(ii) every propagator line carries a factor λ.

Now consider the graphical expansion for the one-particle irreducible (1PI), truncated n-point function. Prove by induction or otherwise that the contribution of a given diagram carries a factor λ^{L-1}, where L is the number of loops in that diagram. Hence show that the result is also true for all connected diagrams. [See Itzykson and Zuber Section 6.2 for one way to do this.]

Also show that the contribution to $W(J) = -\lambda \log Z(J)$ from diagrams with L loops is $O(\lambda^L)$.

Note: in quantum field theory λ is identified with $\hbar$ and so the expansion in the number of loops is the **same** as an expansion in powers of $\hbar$ which measures the size of quantum corrections.

(4*) For the ϕ^4 field theory for $D = 4 - \epsilon$ the RG evolution equations for the coupling constant g and mass m are

$$\frac{du^2}{db} = 2u^2 + \frac{\Omega_D}{2(2\pi)^D} \frac{\lambda}{1+u^2}$$
$$\frac{d\lambda}{db} = \epsilon\lambda - \frac{3\Omega_D}{2(2\pi)^D} \frac{\lambda^2}{(1+u^2)^2}$$

where

$$u^2(b,T) = \Lambda^{-2} m^2(\Lambda, T) , \qquad \lambda(b,T) = \Lambda^{-\epsilon} g(\Lambda, T) ,$$

and Λ is the Ultra-Violet cutoff. Note that both u and λ are dimensionless.‘

Verify that the non-trivial fixed point is at

$$u^{*2} = -\epsilon/6 , \qquad \lambda^* = 16\pi^2\epsilon/3 .$$

Draw a typical trajectory flow for a theory near to $T = T_c$ for $t > 0, \ t < 0$, calculate the relevant eigenvalue λ_t and so derive the related critical exponents.