Mathematical Tripos Part III — Michaelmas Term 2013

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Statistical Field Theory

Examples 3

Field Theory

(1) In the Gaussian model with $\alpha = 1$

$$\Delta(p) \;=\; \frac{1}{p^2+m^2}\;.$$

Let $x = (s, \mathbf{x})$ where $\mathbf{x}$ is a $D-1$ dimensional vector. Show that

$$\int d^{D-1}\mathbf{x}\;\Delta(s,\mathbf{x}) \;\propto\; \xi\, e^{-|s|/\xi}\;,$$

where $\xi = 1/m$ is the correlation length.

An optional and much harder result to obtain is

$$G(r) \;=\; \int \frac{d^Dp}{(2\pi)^D}\, e^{-i\mathbf{p}\cdot\mathbf{x}} \frac{\alpha}{p^2+\alpha m^2} \;\sim\; \begin{cases} \dfrac{\xi e^{-r/\xi}}{(r\xi)^{(D-1)/2}} & a \ll \xi \ll r\;, \\[2ex] \dfrac{1}{r^{(D-2)}} & a \ll r \ll \xi\;, \end{cases}$$

with $\xi^{-2} = \alpha m^2$ and α constant. A hint is to heavily approximate the integral in the two regions by recognizing that for $a \ll \xi \ll r$ it is dominated by $p \ll m$ and for $a \ll r \ll \xi$ it is dominated by $p \gg m$.

In an interacting theory α depends on p and we write

$$\frac{\alpha(p,\xi)}{p^2+\alpha(p,\xi)m^2} \;=\; \frac{Z(p,\xi)}{p^2+\xi^{-2}}\;,$$

which **defines** $Z(p,\xi)$. For large enough ξ we find that

$$Z(p,\xi) \;\sim\; \begin{cases} (ap)^{-\sigma} & p \gg \xi^{-1}\;, \\ (a/\xi)^{-\sigma} & p \ll \xi^{-1}\;, \end{cases}$$

In this case, can you infer the asymptotic forms for $G(r)$ similar to those stated above?

(2) The partition function of a scalar $D = 0$ field theory is given by

$$\mathcal{Z} \;=\; \int dx\; e^{-\left(\frac{1}{2}\lambda x^2 + \frac{g}{4!}x^4\right)}\;.$$

Derive the Feynman rules for the perturbation expansion in g for the connected r-point function $\langle x^r\rangle_c$. To $O(g^2)$ evaluate $\langle x^4\rangle_c$ and verify your answer by explicit calculation. Try some other low order calculations of your choice.

(3) The partition function for a Euclidean scalar field theory is defined in terms of the action $S(\phi)$ by

$$Z = \int \{d\phi\} e^{-S(\phi)/\lambda}.$$

$S(\phi)$ can be written explicitly in the form

$$S(\phi) = \int d^Dx \frac{1}{2}(\nabla\phi)^2 \ + \ \frac{1}{2}m^2\phi^2 \ + \ \sum g_n \frac{\phi^{2n}}{2n!} \ + \ \ldots$$

More interactions such as derivative interactions can be included in principle.

By inspecting the rules for generating the diagrammatic expansion of the theory show that

(i) every vertex carries a factor λ^{-1},

(ii) every propagator line carries a factor λ.

Now consider the graphical expansion for the one-particle irreducible (1PI), truncated n-point function. Prove by induction or otherwise that the contribution of a given diagram carries a factor λ^{L-1}, where L is the number of loops in that diagram. Hence show that the result is also true for all connected diagrams. [See Itzykson and Zuber Section 6.2 for one way to do this.]

Also show that the contribution to $W(J) = -\lambda \ \log Z(J)$ from diagrams with L loops is $O(\lambda^L)$.

Note: in quantum field theory λ is identified with $\hbar$ and so the expansion in the number of loops is the **same** as an expansion in powers of $\hbar$ which measures the size of quantum corrections.

(4*) For the ϕ^4 field theory for $D = 4 - \epsilon$ the RG evolution equations for the coupling constant g and mass m are

$$\begin{aligned}
\frac{du^2}{db} &= 2u^2 + \frac{\Omega_D}{2(2\pi)^D}\frac{\lambda}{1+u^2} \\
\frac{d\lambda}{db} &= \epsilon\lambda - \frac{3\Omega_D}{2(2\pi)^D}\frac{\lambda^2}{(1+u^2)^2}
\end{aligned}$$

where

$$u^2(b,T) = \Lambda^{-2}m^2(\Lambda,T) , \quad \lambda(b,T) = \Lambda^{-\epsilon}g(\Lambda,T) ,$$

and Λ is the Ultra-Violet cutoff. Note that both u and λ are dimensionless.'

Verify that the non-trivial fixed point is at

$$u^{*2} = -\epsilon/6 , \quad \lambda^* = 16\pi^2\epsilon/3 .$$

Draw a typical trajectory flow for a theory near to $T = T_c$ for $t > 0, \ t < 0$, calculate the relevant eigenvalue λ_t and so derive the related critical exponents.

1,

$$\int d^{D-1}x\, \Delta(\xi, x)$$

$$= \int d^{D-1}x \; \frac{e^{ip\cdot x + ip_0\xi}}{p^2+p_0^2+m^2} \frac{d^D p}{(2\pi)^D}$$

$$= \int \frac{dp_0}{2\pi} \frac{e^{ip_0\xi}}{p_0^2+m^2}$$

Choose correct contour

$$\sim \frac{1}{m} \underline{\underline{e^{-m\xi}}}$$

$$\int \frac{d^D p}{(2\pi)^D} e^{-ipx} \frac{1}{p^2+m^2}$$

$r \gg \xi$ then only need $p \ll m$. Then*

$$\sim \int |\underline{p}|^{D-2} d\underline{p} \quad e^{-\sqrt{\underline{p}^2+m^2}\, x} \cdot \frac{1}{m}$$

look at exponent

$$-\sqrt{\underline{p}^2+m^2}\, x + (D-2)\log|\underline{p}|$$

* $x \parallel p_0$

(2)

Minimum ~~le~~

$$-|p|\frac{x}{m} + \frac{D-2}{|p|} = 0$$

$$\Rightarrow |p| \sim \left(\frac{m}{x}\right)^{1/2}$$

2nd derivative $\sim \left(\frac{x}{m}\right)$. ~~Let~~ $u =$

Let $|p| = \left(\frac{m}{x}\right)^{1/2} + u$.

$$\int \sim \frac{1}{m}\int \left(\frac{m}{x}\right)^{\frac{D-2}{2}} du\, e^{-mx} e^{-\frac{1}{2}\left(\frac{x}{m}\right)u^2}$$

$$\sim \frac{1}{m}\left(\frac{m}{x}\right)^{\frac{D-2}{2}}\left(\frac{m}{x}\right)^{1/2} e^{-mx}$$

$$= \underline{\underline{\frac{\xi}{(x\xi)^{\frac{D-1}{2}}} e^{-x/\xi}}}$$

$r \ll \xi$ can ignore m^2

$$\int \sim \int p^{D-3}\, dp\, e^{ipx}$$

$$\sim \frac{1}{r^{(D-2)}}$$ on dimensional grounds

$$Z \sim (a\,p)^{-\sigma} \;,\; (a/\xi)^{-\sigma}$$

In ~~first~~ second case just get

$$\frac{\xi^{1+\sigma}}{(r\xi)^{\frac{D-1}{2}}}\, e^{-r/\xi}$$

In ~~second~~ first case get

$$\frac{1}{r^{D-1+\sigma}}$$ on dimensional grounds

Note: a is "hidden" as an overall constant

2/ $G_0 = \frac{1}{\lambda}$.

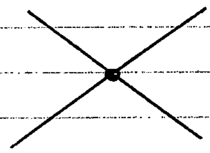
Rules are

$$\frac{1}{\lambda}$$

$\frac{1}{3}$ symmetry factor

$-g$ on each vertex.

$$\langle x^4 \rangle_c = \langle x^4 \rangle - 3 \langle x^2 \rangle^2$$

$O(g)$  $\dfrac{-g}{\lambda^4}$ $\alpha = \dfrac{g}{4!}$

$$\langle x^4 \rangle = \frac{\langle x^4 \rangle_0 - \alpha \langle x^8 \rangle_0}{1 - \alpha \langle x^4 \rangle_0}$$

$$\langle x^2 \rangle = \frac{\langle x^2 \rangle_0 - \alpha \langle x^6 \rangle_0}{1 - \alpha \langle x^4 \rangle_0}$$

$$\langle x^2 \rangle_0 = \frac{1}{\lambda} \qquad \langle x^4 \rangle_0 = \frac{3}{\lambda^2}$$

$$\langle x^6 \rangle_0 = \frac{15}{\lambda^3} \qquad \langle x^8 \rangle_0 = \frac{105}{\lambda^4}$$

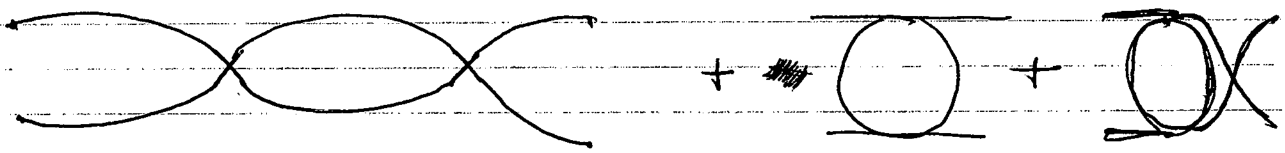
$\langle x^4 \rangle_c =$

$$\left(\frac{3}{\lambda^2} - \frac{105\alpha}{\lambda^4}\right)\left(1 + \frac{3\alpha}{\lambda^2}\right)$$

$$- 3\left(\frac{1}{\lambda} - \frac{15\alpha}{\lambda^3}\right)^2 \left(1 + \frac{3\alpha}{\lambda^2}\right)^2$$

$O(g)$ is $\dfrac{-105 + 9 + 90 - 18}{\lambda^4} \alpha = \dfrac{-24\alpha}{\lambda^4}$

$$= \frac{-g}{\lambda^4}$$



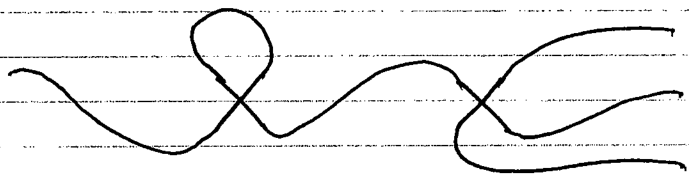
exponential expansion

$$\frac{1}{2}\cdot 3\cdot \frac{1}{4!}\,\frac{1}{4!}\quad 8\cdot 3\cdot 4\cdot 3\cdot 2 \;=\; \frac{3}{2}\,\frac{1}{24}\,\frac{1}{24}\;24\cdot 24$$

pairs

$$= \frac{3}{2}$$

Hence get $\frac{3}{2} g^2 \frac{1}{\lambda^6}$



$$\frac{1}{2}\;\cancel{4}\cdot 2\cancel{4}\cancel{4}\cancel{3}\,\cancel{2}\,\cancel{3}\;\cancel{\frac{1}{4!}}\;\cancel{\frac{1}{4!}} \;=\; 2$$

Hence get $2\, g^2 \frac{1}{\lambda^6}$

Total $= \underline{\underline{\frac{7}{2} g^2 \frac{1}{\lambda^6}}}$

$$\langle x^4\rangle = \frac{\langle x^4\rangle_0 - \alpha\langle x^8\rangle_0 + \frac{1}{2}\alpha^2\langle x^{12}\rangle_0}{1 - \alpha\langle x^4\rangle_0 + \frac{1}{2}\alpha^2\langle x^8\rangle_0}$$

$$= \frac{\frac{3}{\lambda^2} - \frac{105\alpha}{\lambda^4} + \frac{10395}{2\lambda^6}\alpha^2}{1 - \frac{3\alpha}{\lambda^2} + \frac{105}{2\lambda^4}\alpha^2}$$

$$= \left(\frac{3}{\lambda^2} - \frac{105\alpha}{\lambda^4} + \frac{10395\alpha^2}{2\lambda^6} \right)$$

$$\cdot \left(1 + \frac{3\alpha}{\lambda^2} - \frac{105}{2\lambda^4}\alpha^2 + \frac{9\alpha^2}{\lambda^4} \right)$$

Set $\lambda = 1$ and collect α^2 terms

$$\left(3 - 105\alpha + \frac{10395\alpha^2}{2} \right)$$

$$\left(1 + 3\alpha - \frac{87\alpha^2}{2} \right)$$

$$= 3 \cdot \left(1 - 35\alpha + \frac{3465\alpha^2}{2} \right)\left(1 + 3\alpha - \frac{87\alpha^2}{2} \right)$$

$$= 3 \cdot \left[\frac{3465}{2} - 105 - \frac{87}{2} \right] \alpha^2$$

$$= \underline{\underline{4752\,\alpha^2}}$$

$$\langle x^2 \rangle = \left[\langle x^2 \rangle_0 - \alpha \langle x^6 \rangle_0 + \tfrac{1}{2}\alpha^2 \langle x^{10} \rangle_0 \right]$$

$$\cdot \left(1 + 3\alpha - \frac{87\alpha^2}{2} \right)$$

$$= \left[1 - 15\alpha + \frac{945}{2}\alpha^2 \right]$$

$$\left[1 + 3\alpha - \frac{87\alpha^2}{2} \right]$$

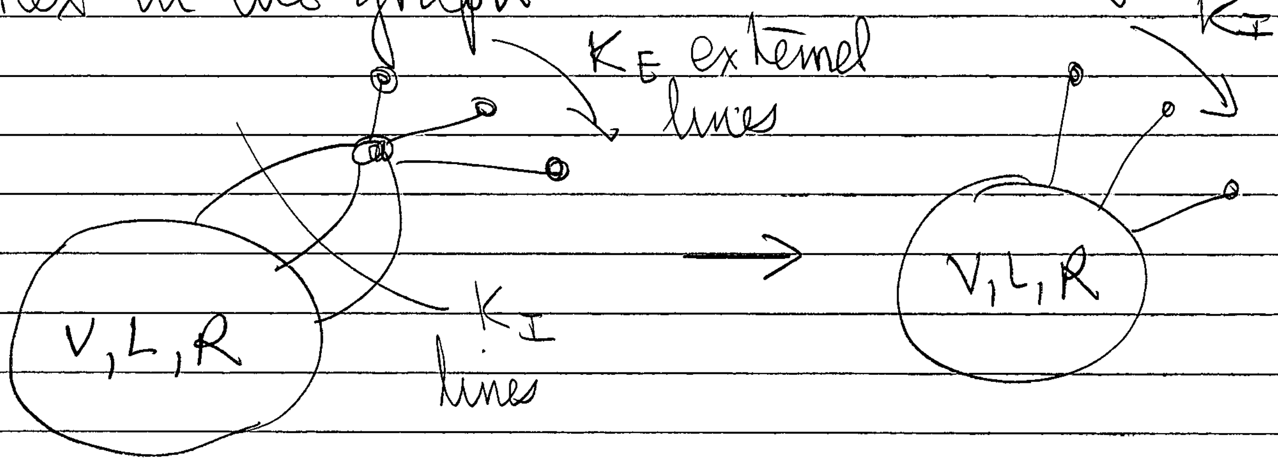
$$= \left[1 - 12\alpha + 384\alpha^2\right]$$

$$\langle x^2\rangle^2 = (144 + 2\cdot 384)\alpha^2 = \underline{\underline{912\alpha^2}}$$

Get $\dfrac{4752 - 3\cdot 912}{24^2}\dfrac{g^2}{\lambda^6} = \underline{\underline{\dfrac{7}{2}\dfrac{g^2}{\lambda^6}}}$

3(

Treat external lines as terminating on a vertex with a single line. Identify a vertex in the graph



L = loops, V = vertices, R = lines.

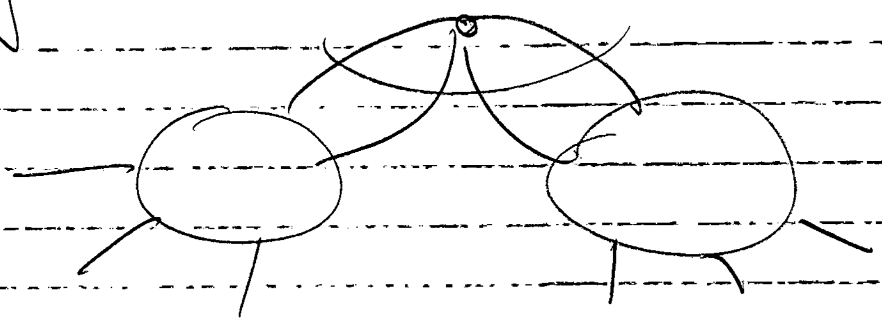
Let N_c = number of <u>disconnected</u> graphs initially $N_c = 1$.

$\Delta R = -K_E \qquad \Delta V = -K_E - 1 + K_I$

$\Delta L = -K_I + 1 + \Delta N_c$

More on ΔL:

E.g



$\Delta N_c = 1 \qquad \Delta L = -2 \qquad K_I = 4$

Check $\quad -K_I + 1 + \Delta N_c = -4 + 1 + 1 = -2$

Then

$$\Delta L + \Delta V - \Delta R = -K_I + 1 + \Delta N_c - K_E - 1 + K_I + K_E = \underline{\underline{\Delta N_c}}$$

Reduce by this method to sum of N_c disconnected tree diagrams. For any tree diagrams

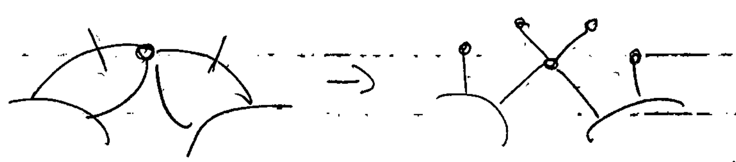
$$L + V - R = 1$$

Thus, for original diagram

$$L + V - R = \underset{\substack{\uparrow \\ L+V-R \text{ for trees}}}{N_c \cdot 1} - \underset{\sum \Delta N_c}{(N_c - 1)}$$

$$\Rightarrow \quad \underline{\underline{L + V - R = 1}}$$

NB In above can cut



$\Delta L = -2, \ \Delta V = 4, \ \Delta R = 2$

$\Rightarrow \quad \underline{\underline{\Delta L + \Delta V - \Delta R = 0}}$

Avoids disconnecting diagrams

I&Z argument.

* L = # independent momenta.

* R = # momenta

* At each vertex impose momentum conservation.

Overall momentum conservation is one constraint already imposed

$$\Rightarrow \# \text{ constraints} = V - 1$$

Thus $\quad L = R - V + 1$

$$\Rightarrow \quad \underline{\underline{L + V - R = 1}}$$

Note can truncate external lines since

$$\Delta (V - R) = 0$$

So result holds for <u>truncated</u> graphs.

In truncated graph

Each vertex has factor λ^{-1}

Each internal line has factor λ

$$\Rightarrow \text{power of } \lambda = \lambda^{-V} \lambda^{R} = \underline{\underline{\lambda^{L-1}}}$$

$W(J) = -\lambda \log Z(\lambda, J)$ (Free energy)

$Z(\lambda, J) = e^{-S(\phi_0)/\lambda} \int \ldots$

$\Rightarrow W(J) = S(\phi_0) + \sum_{L=1}^{\infty} \lambda^L \cdot$ (diags with L loops)

$\lambda \to \hbar$ in QFT

4) In nbhd of fixed point write

$x = u^2 - u^{*2} \qquad y = \lambda - \lambda^*$

to order ε (ie ignoring ε^2 etc)

have $(\Omega_4 = 2\pi^2)$

a) $u^{*2} = \lambda^* = 0$ trivial f p

which is repulsive:

$$\frac{d}{ds}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1/16\pi^2 \\ 0 & \epsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

evals $\lambda = 2, \epsilon \Rightarrow$ repulsive

b) $u^{*2} = -\frac{\epsilon}{6}$ $\lambda^* = \frac{16\pi^2\epsilon}{3}$

$$\frac{d}{db}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2-\frac{1}{3}\epsilon & \frac{1}{3}\epsilon \\ 0 & -\epsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Thus λ, and hence g, is irrelevant since eigenvalue is $\underline{\underline{-\epsilon}}$

u^2 and hence m^2 is relevant and have $\lambda_t = 2 - \frac{1}{3}\epsilon$

$\Rightarrow$ $\nu = \frac{1}{\lambda_t} = \underline{\underline{\frac{1}{2} + \frac{1}{12}\epsilon}}$

$\Rightarrow$ $\alpha = 2 - D\nu$ $\Rightarrow$ $\underline{\underline{\alpha = \frac{5}{6}\epsilon}}$