Mathematical Methods I
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Natural Sciences Tripos, Part IB
Michaelmas Term 2019

Example Sheet 1

1. In the following, the indices i, j, k, ℓ, m take the values 1, 2, 3, and the summation convention applies. In particular let $\boldsymbol{n}$ be a unit vector, i.e. $n_i n_i = 1$.

   (a) Simplify the following expressions

   $$\delta_{ij}a_j\,,\quad \delta_{ij}\delta_{jk}\,,\quad \delta_{ij}\delta_{ji}\,,\quad \delta_{ij}n_in_j\,,\quad \epsilon_{ijk}\delta_{jk}\,,\quad \epsilon_{ijk}\epsilon_{ij\ell}\,,\quad \epsilon_{ijk}\epsilon_{ikj}\,,\quad \epsilon_{ijk}(\boldsymbol{a}\times\boldsymbol{b})_k\,.$$

   (b) For each of the following equations, either give the equivalent in vector or matrix notation or explain why the equation is invalid.

   $$x_i = a_ib_kc_k + d_i\,,\quad x_i = a_jb_i + c_kd_ie_kf_j\,,\quad u = \epsilon_{jk\ell}v_kw_\ell x_j\,,$$

   $$\epsilon_{ijk}x_jy_k\epsilon_{i\ell m}x_\ell y_m = \mu\,,\qquad A_{ik}B_{k\ell} = T_{ik}\delta_{k\ell}\,,\qquad x_k = A_{ki}B_{ji}y_j\,.$$

   (c) Write the following equations in suffix notation using the summation convention.

   $$(\boldsymbol{x}+\mu\boldsymbol{y})\cdot(\boldsymbol{x}-\mu\boldsymbol{y}) = \kappa\,,\qquad \boldsymbol{x} = |\boldsymbol{a}|^2\boldsymbol{b} - |\boldsymbol{b}|^2\boldsymbol{a}\,,\qquad (2\boldsymbol{x}\times\boldsymbol{y})\cdot(\boldsymbol{a}+\boldsymbol{b}) = \lambda\,.$$

   (d) Given that $A_{ij} = \epsilon_{ijk}a_k$ (for all i, j), show that $2a_k = \epsilon_{kij}A_{ij}$ (for all k).

   (e) Show that $\epsilon_{ijk}s_{ij} = 0$ (for all k) if and only if $s_{ij} = s_{ji}$ (for all i, j).

   (f) Given that $N_{ij} = \delta_{ij} - \epsilon_{ijk}n_k + n_in_j$ and $M_{ij} = \delta_{ij} + \epsilon_{ijk}n_k$, show that $N_{ij}M_{jk} = 2\delta_{ik}$.

2. A fluid flow has the constant velocity vector (in Cartesian coordinates)

   $$\boldsymbol{v}(\boldsymbol{r}) = (0, 0, W)\,.$$

   Explicitly calculate the volume flux of fluid,

   $$Q = \int \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{S}\,,$$

   flowing across (a) the open hemispherical surface $r = a$, $z \geqslant 0$, and (b) the disc $r \leqslant a$, $z = 0$. Verify that the divergence theorem holds.

3. For a surface S enclosing a volume V, apply the divergence theorem to a vector field $\boldsymbol{F} = \boldsymbol{a}p$, where $\boldsymbol{a}$ is an arbitrary constant vector and $p(\boldsymbol{r})$ is a scalar field. Deduce that

   $$\int_V \boldsymbol{\nabla} p\,\mathrm{d}V = \oint_S p\,\mathrm{d}\boldsymbol{S}\,.$$

4. A time-independent magnetic field $\boldsymbol{B}(\boldsymbol{r})$ is given by

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi} \frac{\boldsymbol{e}_z \times \boldsymbol{r}}{x^2 + y^2},$$

where μ_0 is the magnetic permeability and I is a constant. Using Cartesian coordinates, calculate the electric current density $\boldsymbol{J}$ given by the steady Maxwell equation $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$. Also evaluate $\oint_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r}$, where C is a circle of radius a in the plane $z = 0$ and centred on $x = y = 0$. Discuss whether Stokes's theorem applies in this situation.

5. Show that, in Cartesian coordinates,

$$\nabla^2 \boldsymbol{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{F}).$$

This vector identity remains true for all coordinate systems; however, for non-Cartesian coordinates,

$$(\nabla^2 \boldsymbol{F})_i \neq \nabla^2 F_i.$$

Why is this the case? Illustrate this point by evaluating $\nabla^2 \boldsymbol{F}$ for $\boldsymbol{F} = f(\rho)\, \boldsymbol{e}_\phi$ in cylindrical polar coordinates (ρ, ϕ, z) and comparing the result with $\nabla^2 f$.

6. Find the general circularly symmetric solution to the fourth-order equation

$$\nabla^4 \Psi \equiv \nabla^2(\nabla^2 \Psi) = 0.$$

Hint: use plane polar coordinates (ρ, ϕ), and do not be too eager to expand everything out.

Find those circularly symmetric solutions in the unit disc that are equal to unity at the centre $\rho = 0$ and vanish on the boundary $\rho = 1$. Give a further condition to render the solution unique.

7. Parabolic coordinates (u, v, ϕ) are defined in terms of Cartesian coordinates (x, y, z) by

$$x = uv \cos\phi, \qquad y = uv \sin\phi, \qquad z = \tfrac{1}{2}(u^2 - v^2).$$

Show that the surfaces of constant u, and those of constant v, are surfaces obtained by rotating parabolae about the z-axis. What are the surfaces of constant ϕ? Show that the coordinate surfaces intersect at right angles and hence that these coordinates are orthogonal. Find the scale factors (h_u, h_v, h_ϕ) defined by

$$|\mathrm{d}\boldsymbol{r}|^2 = h_u^2\, \mathrm{d}u^2 + h_v^2\, \mathrm{d}v^2 + h_\phi^2\, \mathrm{d}\phi^2.$$

Hence obtain the Laplacian in these coordinates using the formula

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial q_3}\right)\right].$$

8. Consider the two-stage transformation of Cartesian coordinates (x, y, z) given by

$$x = ax' , \qquad y = by' , \qquad z = cz' ,$$

$$x' = r' \sin\theta' \cos\phi' , \qquad y' = r' \sin\theta' \sin\phi' , \qquad z' = r' \cos\theta' ,$$

where a, b and c are positive constants. Calculate the Jacobian matrices of the transformations $(x, y, z) \mapsto (x', y', z')$ and $(x', y', z') \mapsto (r', \theta', \phi')$ and verify explicitly that

$$\frac{\partial(x, y, z)}{\partial(r', \theta', \phi')} = \frac{\partial(x, y, z)}{\partial(x', y', z')} \frac{\partial(x', y', z')}{\partial(r', \theta', \phi')} .$$

Are the coordinates (r', θ', ϕ') orthogonal? What range of these coordinates is required to cover the interior of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,?$$

Express the volume element in coordinates (r', θ', ϕ') and hence calculate the volume of the ellipsoid.

9. In a Cartesian coordinate system (x_1, x_2, x_3), A is the point $(0, 0, -1)$, B is the point $(0, 0, 1)$ and P is an arbitrary point (x_1, x_2, x_3). In a curvilinear coordinate system, the coordinates of P are specified by

$$u_1 = \frac{1}{2}(r_1 + r_2) \qquad u_2 = \frac{1}{2}(r_1 - r_2) \qquad u_3 = \phi,$$

where r_1 and r_2 are the distances AP and BP respectively and ϕ is the angle between the planes ABP and $x_2 = 0$. Show that $x_3 = u_1 u_2$ and that the distance ρ from P to the x_3-axis is equal to $\sqrt{(u_1^2 - 1)(1 - u_2^2)}$. Next evaluate $\partial x_i / \partial u_j$ (with i, j = 1,2,3). Deduce that the curvilinear coordinates are orthogonal and sketch the coordinate surfaces. Show that the metric coefficients are

$$h_1 = \sqrt{\frac{u_1^2 - u_2^2}{u_1^2 - 1}} , \quad h_2 = \sqrt{\frac{u_1^2 - u_2^2}{1 - u_2^2}} , \quad h_3 = \sqrt{(u_1^2 - 1)(1 - u_2^2)} \, .$$

Show that if the function Ψ satisfies Laplace's equation and is independent of u_2 and u_3 then it has the form

$$\Psi = a + b \ln\left(\frac{u_1 - 1}{u_1 + 1}\right)$$

for constant a and b.

10. A uniform stretched string of length L, mass per unit length ρ and tension $T = \rho c^2$ is fixed at both ends. The motion of the string is resisted by the surrounding medium, the resistive force per unit length being $-2\mu\rho\dot{y}$, where $y(x, t)$ is the transverse displacement and $\dot{y} = \partial y / \partial t$. Generalize the argument given in lectures to show that the equation of motion of the string is

$$\frac{\partial^2 y}{\partial t^2} + 2\mu \frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2} .$$

Find $y(x,t)$ if $\mu = \pi c/L$, $y(x,0) = d\sin(\pi x/L)$ and $\dot{y}(x,0) = 0$.

If an extra transverse force $F\sin(\pi x/L)\cos(\pi ct/L)$ per unit length acts on the string, find the resulting forced oscillation.

11. Show that the solution of Laplace's equation, $\nabla^2\Phi = 0$, in the region $0 < x < a$, $0 < y < b$, $0 < z < c$, with $\Phi = 1$ on the surface $z = 0$ and $\Phi = 0$ on the other surfaces, is

$$\Phi = \frac{16}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin[(2m-1)\pi x/a]\sin[(2n-1)\pi y/b]\sinh[k(c-z)]}{(2m-1)(2n-1)\sinh(kc)},$$

where $k^2 = (2m-1)^2\pi^2/a^2 + (2n-1)^2\pi^2/b^2$.

12. The temperature distribution $\theta(x,t)$ along a thin bar of length L satisfies the one-dimensional diffusion equation

$$\frac{\partial\theta}{\partial t} = \nu\frac{\partial^2\theta}{\partial x^2},$$

where the diffusivity ν is a constant, t denotes time and x is the distance from one of the ends. Find $\theta(x,t)$ if the bar is insulated at each end (i.e. if $\partial\theta/\partial x = 0$ at each end), and if the initial temperature distribution is given by

$$\theta(x,0) = 2\theta_0\cos^2\left(\frac{\pi x}{L}\right),$$

where θ_0 is a constant. For large times what is the temperature distribution in the bar? Comment.

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