Mathematical Methods I — Natural Sciences Tripos, Part IB

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Example Sheet 3

1. Use the Cauchy–Schwarz inequality and the properties of the inner product to prove the triangle inequality

$$|\boldsymbol{x}+\boldsymbol{y}| \leqslant |\boldsymbol{x}| + |\boldsymbol{y}|$$

for a complex vector space, where $|\boldsymbol{x}|$ is the norm of the vector $\boldsymbol{x}$. Under what conditions does equality hold?

2. Given a set of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ $(m \geqslant n)$ that span an n-dimensional vector space, show that an orthogonal basis may be constructed by the Gram–Schmidt procedure

$$\boldsymbol{e}_1 = \boldsymbol{u}_1\,,$$

$$\boldsymbol{e}_r = \boldsymbol{u}_r - \sum_{s=1}^{r-1} \frac{\boldsymbol{e}_s \cdot \boldsymbol{u}_r}{\boldsymbol{e}_s \cdot \boldsymbol{e}_s}\,\boldsymbol{e}_s \quad \text{for } r > 1\,.$$

What is the interpretation if any of the vectors $\boldsymbol{e}_r$ vanishes?

Find an orthonormal basis for the subspace of a four-dimensional Euclidean space spanned by the three vectors with components $(1,1,0,0)$, $(0,1,2,0)$ and $(0,0,3,4)$.

3. What does it mean to say that the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ are linearly independent?

Let $\mathbf{A}$ be a linear operator on an n-dimensional vector space, having n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. Consider the action of the operator $\mathbf{A} - \lambda_i \mathbf{1}$ (where $\mathbf{1}$ is the identity operator) on the vector $\boldsymbol{e}_j$ in the cases $i = j$ and $i \neq j$. Hence, or otherwise, show that the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ are linearly independent.

4. An $n \times n$ complex matrix A is such that each row and each column has exactly one non-zero element. The Hermitian conjugate of A is $\mathsf{A}^\dagger = (\mathsf{A}^{\mathrm{T}})^*$ (where A^{T} is the transpose of A, and A^* is its complex conjugate). Show that $\mathsf{A}^\dagger \mathsf{A}$ is a real diagonal matrix.

5. An Hermitian matrix A is one for which $\mathsf{A}^\dagger = \mathsf{A}$. Suppose that A and B are both Hermitian matrices. Show that $\mathsf{AB} + \mathsf{BA}$ is Hermitian. Also show that AB is Hermitian if and only if A and B commute.

6. Find the eigenvalues and eigenvectors of the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where neither of the complex constants α and β vanishes. Find the conditions for which (a) the eigenvalues are real, and (b) the eigenvectors are orthogonal. Hence show that both conditions are jointly satisfied if and only if A is Hermitian.

7. For an Hermitian matrix H, explain how to construct a unitary matrix U such that $\mathsf{U}^\dagger\mathsf{H}\mathsf{U} = \mathsf{D}$, where D is a real diagonal matrix. Illustrate the procedure with the matrix

$$\mathsf{H} = \begin{bmatrix} 4 & 3\mathrm{i} \\ -3\mathrm{i} & -4 \end{bmatrix} .$$

8. An anti-Hermitian matrix A is one for which $\mathsf{A}^\dagger = -\mathsf{A}$. What can be said about the eigenvalues of A?

   If S is real symmetric and T is real antisymmetric, show that $\mathsf{T} \pm \mathrm{i}\mathsf{S}$ are anti-Hermitian. Deduce that

$$\det(\mathsf{T} + \mathrm{i}\mathsf{S} - 1) \neq 0 .$$

   Show that the matrix

$$\mathsf{U} = (1 + \mathsf{T} + \mathrm{i}\mathsf{S})(1 - \mathsf{T} - \mathrm{i}\mathsf{S})^{-1}$$

   is unitary.

   For

$$\mathsf{S} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} , \qquad \mathsf{T} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ,$$

   show that the eigenvalues of U are $\pm(1 - \mathrm{i})/\sqrt{2}$.

9. Show that the eigenvalues of a real orthogonal matrix have unit modulus and that if λ is an eigenvalue then so is λ^*. Hence argue that the eigenvalues of a 3×3 real orthogonal matrix R must be a selection from

$$+1 , \quad -1 \quad \text{and} \quad \mathrm{e}^{\pm \mathrm{i}\alpha} .$$

   Verify that $\det \mathsf{R} = \pm 1$. What is the effect of R on vectors orthogonal to an eigenvector with eigenvalue ± 1?

10. Let H be an $n \times n$ Hermitian matrix with n distinct eigenvalues $\{\lambda_i\}$ and orthonormal eigenvectors $\{\mathsf{e}_i\}$. Now consider the slightly perturbed matrix $\mathsf{H} + \delta\mathsf{H}$, where $\delta\mathsf{H}$ is small and Hermitian. Let the eigenvalues and orthonormal eigenvectors of $\mathsf{H} + \delta\mathsf{H}$ be $\{\lambda_i + \delta\lambda_i\}$ and $\{\mathsf{e}_i + \delta\mathsf{e}_i\}$, respectively. By working to first order in the small quantities, show that

$$\delta\lambda_i = \mathsf{e}_i^\dagger(\delta\mathsf{H})\mathsf{e}_i ,$$

$$\delta\mathsf{e}_i = \sum_{j \neq i} \frac{\mathsf{e}_j^\dagger(\delta\mathsf{H})\mathsf{e}_i}{\lambda_i - \lambda_j} \, \mathsf{e}_j .$$

   Why is it permissible to omit any contribution to $\delta\mathsf{e}_i$ parallel to e_i? [Write out the eigenvector equation and the orthonormality condition for the eigenvectors of the perturbed matrix. Expand, neglect products of small quantities, and simplify. Use the fact that $\{\mathsf{e}_i\}$ is a basis.]

11. Find the eigenvalues and normalized eigenvectors of the symmetric matrices

$$\mathsf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathsf{B} = \begin{bmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{bmatrix}, \qquad \mathsf{C} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{bmatrix}.$$

Describe the related quadratic surfaces.

This example sheet is available on the Cambridge University Moodle site.
Hints and answers will also be posted on the Moodle site.

Supervisors may request these early. Comments/corrections to `n.peake@damtp.cam.ac.uk`