Mathematical Methods III
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Natural Sciences Tripos, Part IB
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Examples Sheet 1

1. Carefully verify that, in the coupled pendula example (Fig. 1), where each is of length ℓ, the extension of the spring connecting them is approximately $\ell(\theta_2 - \theta_1)$, at leading order in the small angles θ_1 and θ_2

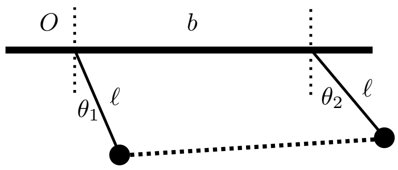

Fig. 1

2. Consider a model of a linear molecule AAB where the atom B is at one end. The masses are $M_A = m$ and $M_B = 2m$. Let the spring constants for the forces between neighbouring atoms be given by $k_{AA} = k$ and $k_{AB} = 2k$. Find the equations of motion and the normal frequencies for linear oscillations along the axis of the molecule. Verify the orthogonality relation for the normal mode vectors $\mathbf{Q}^{(m)}$, and find the normalized form of these generalized eigenvectors.

   At $t = 0$ all atoms are initially at their equilibrium positions, the atoms A are at rest, and atom B is given an initial velocity u_0. Derive expressions for the positions of the 3 molecules as functions of t, and in terms of the normal modes.

3. Three equal masses are connected by equal springs as shown in Fig. 2, the walls being fixed. Assume the motion is constrained to be 1-dimensional along the line of the springs. Find the normal modes of oscillation and the ratios of the normal frequencies. Are there any periodic solutions other than the pure normal modes?

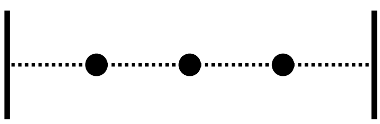
Fig. 2

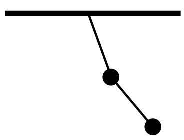
Fig. 3

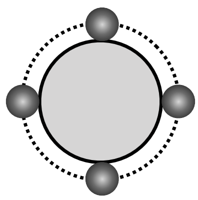
Fig. 4

4. Consider the double pendulum shown in Fig. 3, with two balls of equal mass m at the ends of identical rods, length ℓ, whose masses can be neglected. The motion is constrained to be in a fixed vertical plane. Find the two normal modes of small oscillations and their frequencies. *[Hint: Instead of using Newton's second law, it is easier to determine the kinetic and potential energies and then use Lagrange's equations.]*

5. Four point-size beads of mass m are constrained to move on a frictionless circle of radius R (Fig. 4). Adjacent pairs of beads are connected by springs of spring constant and equilibrium length $\frac{\pi R}{2}$. Write down the Lagrangian for the system. Find the normal modes and normal frequencies. Describe the motions corresponding to each normal mode.

6. Let $T = \frac{1}{2}T_{ij}\dot{q}_i\dot{q}_j$ and $V = \frac{1}{2}V_{ij}q_iq_j$. Verify that the equations of motion $T_{ij}\ddot{q}_j + V_{ij}q_j = 0$ imply that the energy $T + V$ is conserved. Can the constancy of $T + V$ be used to deduce the equations of motion?

7. The six functions f_1, f_2, ..., f_6 are defined by

$$f_1(z) = z\,, \qquad f_2(z) = \frac{1}{1-z}\,, \qquad f_3(z) = 1 - \frac{1}{z}\,,$$
$$f_4(z) = \frac{1}{z}\,, \qquad f_5(z) = 1 - z\,, \qquad f_6(z) = \frac{z}{z-1}\,.$$

Show that these form a group under function composition, i.e. function of a function: the group 'product' operation f_1f_2 is defined by $(f_1f_2)(z) \equiv f_1(f_2(z))$. Construct the group table, find all subgroups and determine which of them are normal. You should find that there are three order-2 subgroups, none of which are normal, and one order-3 normal subgroup. Is this group isomorphic to any other group mentioned in the lectures?

What happens when $z = 0$, 1, or ∞ for each of the functions f_1, f_2, ..., f_6?

8. Find the group table for the cyclic group C_4, consisting of rotations in a plane by angles $\frac{n\pi}{2}$, where $n = 0$, 1, 2, or 3. Find the group table for the so-called Vierergruppe V (also referred to as Klein's four-group and denoted K_4), consisting of the identity and the rotations by π about the x, y, and z axes in 3-dimensional space. Show that both C_4 and V are Abelian, but that they are not isomorphic to each other.

9. The symmetry group of an N-gon is generated by a single rotation R by an angle $2\pi/N$ and a reflection m (being any single one of the possible reflection symmetries). Show by means of sketches that the relations $R^N = I$, $m^2 = I$ and $Rm = mR^{-1}$ are always obeyed. Deduce that the elements of the group are R^n and R^nm, where $n \in \{1, 2, \ldots, N\}$. Use these relations to express the product of two arbitrary elements of the group, either in the form R^n or R^nm.

10. Show that the order of a permutation P is the lowest common multiple of the orders of its component cycles. Resolve

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 9 & 7 & 2 & 5 & 8 & 1 & 3 \end{pmatrix}$$

into cycles and find its order.

11. A permutation $P \in S_N$ may be regarded as permuting the components of an N-vector and is represented by a permutation matrix. For example, the permutation

(12)(34) is represented by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

Show that the determinant of a permutation matrix is equal to ± 1, and show that the permutations whose matrices have determinant $+1$ form a subgroup of S_N with $\frac{1}{2}N!$ elements. Verify that this is the subgroup of even permutations.

12. $GL(n, \mathbb{R})$ is the group of all invertible $n \times n$ real matrices (under matrix multiplication), and $\mathbb{R}^*$ represents the *nonzero*, real numbers, which form a group under multiplication. Show that the map $\Phi : GL(n, \mathbb{R}) \mapsto \mathbb{R}^*$ defined by $\Phi(M) = \det(M)$ is a homomorphism. What is the kernel of Φ? Show that the kernel is a normal subgroup of $GL(n, \mathbb{R})$ and describe its cosets. Show that the product of two cosets is well-defined and produces another coset. Deduce that the set of all cosets forms a group. [This is an example of a quotient group.]

Please e-mail me at M.Wingate@damtp.cam.ac.uk with any comments, especially any errors.