Example Sheet 4: Complex Analysis and Fourier Transforms

1. Show that the function on the plane defined by

$$\Phi(x,y) = \mathrm{Im}\left\{\frac{2}{\pi}\ln\tanh(x+iy)\right\}$$

satisfies Laplace's equation for $x > 0$. Show also that $\Phi = 0$ on both $y = 0$ and on $y = \pi/2$, and that $\Phi = 1$ on $x = 0$ for $0 < y < \pi/2$. Deduce the steady-state temperature distribution in a semi-infinite two-dimensional bar of width L, with the (infinitely) long sides held at zero temperature and short side held at temperature T_0.

2. (i) State and prove Cauchy's Theorem.

(ii) Let C be a closed contour that encloses, in a positive sense, the point z_0 in the complex z-plane, and let n be an integer. Show that

$$\oint_C (z-z_0)^n\,dz = \begin{cases} 2\pi i & n = -1 \\ 0 & n \neq -1\ . \end{cases}$$

(iii) Given that $f(z)$ is an analytic function, show that $\oint_C \frac{f'(z)dz}{z-z_0} = \oint_C \frac{f(z)dz}{(z-z_0)^2}$.

3. Establish the following general methods for calculating residues. [*These are all very useful in practice and you are advised to memorize them.*]

(i) If $f(z)$ has a simple pole at $z = z_0$ then the residue of f at $z = z_0$ is gvein by $\lim_{z\to z_0}\{(z-z_0)\,f(z)\}$.

(ii) If $f(z)$ is analytic then the residue of $f(z)/(z-z_0)$ at $z = z_0$ is $f(z_0)$.

(iii) If $f(z)$ has a simple zero at $z = z_0$ then the residue of $1/f(z)$ at $z = z_0$ is $1/f'(z_0)$.

(iv) If $h(z)$ has a simple zero at $z = z_0$ and $g(z)$ is analytic then the residue of $g(z)/h(z)$ at $z = z_0$ is $g(z_0)/h'(z_0)$.

(v) If $f(z)$ has a pole of order N at $z = z_0$ then the residue of $f(z)$ at $z = z_0$ is

$$\lim_{z\to z_0}\left\{\frac{1}{(N-1)!}\frac{d^{N-1}}{dz^{N-1}}\left[(z-z_0)^N f(z)\right]\right\}$$

4. Find the poles of the following functions and the residues at each pole:

$$(i)\ \ \frac{z+1}{z^2}\,,\qquad (ii)\ \ \frac{e^{-z}}{z^3}\,,\qquad (iii)\ \ \frac{\sin^2 z}{z^5}\,,\qquad (iv)\ \ \cot z\,,\qquad (v)\ \ \frac{z^2}{(1+z^2)^2}\,.$$

5. Let $f(z)$ be a function that is analytic within and on the circle $|z - z_0| = r$ in the z-plane. For non-negative integer n, show that

$$\left|f^{(n)}(z_0)\right| \leqslant \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)| .$$

Hence prove *Liouville's Theorem*: a function $f(z)$ that is analytic and bounded for all z is constant. Deduce that any polynomial $p(z)$ of degree at least one has at least one zero. [*Hint: consider* $1/p(z)$.]

6. Explain how the calculus of residues may be used to evaluate the integral of a function $f(z)$ around a closed contour C in the complex z-plane. Let

$$f(z) = \frac{z^n}{(z-a)\left(z-a^{-1}\right)} ,$$

for real constant $a > 1$ and non-negative integer n. By choosing C to be a circle of unit radius, and using the calculus of residues, evaluate the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(n\theta)\, d\theta}{1 - 2a\cos\theta + a^2} .$$

7. By integrating around a rectangular contour with vertices at $\pm R$ and $i\pi \pm R$, where R is a large real constant, show that $\int_0^\infty \operatorname{sech} x\, dx = \pi/2$.

8. Let a be a real non-zero constant. Show that

$$(i) \qquad \int_0^\pi \frac{d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{a\sqrt{1+a^2}} \qquad (a > 1) .$$

Show also that

$$(ii) \qquad \int_0^\infty \frac{x^4\, dx}{1+x^8} = \frac{\pi}{8}\sqrt{\frac{2\sqrt{2}}{1+\sqrt{2}}} = \frac{\pi}{4}\sqrt{1 - \frac{1}{\sqrt{2}}} ,$$

$$(iii) \qquad \int_0^\infty \cos\left(\frac{1}{2}ax^2\right) dx = \sqrt{\frac{\pi}{4|a|}} .$$

[*Hint: use a sector of a circle with angle* $\pi/4$.]

Now show that

$$(iv) \qquad \int_0^\infty \frac{x^{-a}\, dx}{x+1} = \frac{\pi}{\sin(\pi a)} \qquad (0 < a < 1) , \tag{1}$$

$$(v) \qquad \int_0^\infty \frac{(\ln x)^2\, dx}{1+x^2} = \frac{\pi^3}{8} . \tag{2}$$

[*Hint: (iv) and (v) require you to consider a branch cut; for (v) use a semi-circular contour with an appropriately chosen branch cut.*]

9. Sketch possible branch cuts for the following complex functions, giving the values on either side of each cut:

$$(i) \quad \left(z^2+1\right)^{\frac{1}{2}}, \qquad (ii) \quad \left(z^2+1\right)^{\frac{1}{3}}, \qquad (iii) \quad \ln\left[\left(\frac{z-i}{z+i}\right)^2\right].$$

10. By considering the integral $\oint \left(z^2+1\right)^{-1} e^{ikz}\, dz$ taken around a large semicircle, show that for real positive k

$$\int_{-\infty}^{\infty} \frac{\cos(kx)\, dx}{x^2+1} = \pi e^{-k}$$

What is the value of the integral for $k \leqslant 0$?

11. Find the function $f(x)$ that has Fourier transform $\tilde{f}(k) = \left(1+k^4\right)^{-1}$.

12. The motion of an overdamped harmonic oscillator subject to an impulsive force is described by the equation $\ddot{x} + 2\gamma\dot{x} + p^2 x = \delta(t)$ for a function $x(t)$ and constants $\gamma > p > 0$. Given that $x = 0$ for $t < 0$, show by Fourier transform methods that for $t > 0$

$$x(t) = \frac{e^{-\gamma t}}{\sqrt{\gamma^2 - p^2}} \sinh\left(\sqrt{\gamma^2-p^2}\, t\right).$$

13. A function $u(x,t)$, defined for all $t \geqslant 0$, satisfies the diffusion equation $\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2}$ subject to the initial conditions

$$u(x,0) = \begin{cases} -e^{x} & x < 0 \\ 0 & x = 0 \\ e^{-x} & x > 0\,. \end{cases}$$

Using Fourier transform methods, show that for $t > 0$

$$u(x,t) = \frac{2}{\pi} \int_0^{\infty} \frac{k\, e^{-\lambda k^2 t} \sin(kx)}{1+k^2} dk\,.$$

Suppose now that $u(x,t)$ is defined only for $x \geqslant 0$, that $u(0,t) = 0$ for all $t \geqslant 0$, and that the initial condition is $u(x,0) = e^{-x}$ for $x > 0$. Write down the solution of this modified problem.

This example sheet is available on `Moodle`*. Hints and answers will also be posted there. Comments/corrections to* `S.J.Cowley@maths.cam.ac.uk`*.*