

3P1b Quantum Field Theory: Example Sheet 2 Michaelmas 2019

Corrections and suggestions should be emailed to B.C.Allanach@damtp.cam.ac.uk. Starred questions may if you wish be handed in to your supervisor for feedback prior to the class.

1. Consider a real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2. \quad (1)$$

Show that, after normal ordering, the conserved four-momentum $P^\mu = \int d^3x T^{0\mu}$ takes the operator form

$$P^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu a_{\vec{p}}^\dagger a_{\vec{p}} \quad (2)$$

where $p^0 = E_{\vec{p}}$ in this expression. From Eq. (2), verify that if $\phi(x)$ is now in the Heisenberg picture, then

$$[P^\mu, \phi(x)] = -i \partial^\mu \phi(x).$$

- 2* Show that in the Heisenberg picture,

$$\dot{\phi}(x) = i[H, \phi(x)] = \pi(x) \quad \text{and} \quad \dot{\pi}(x) = i[H, \pi(x)] = \nabla^2 \phi(x) - m^2 \phi(x).$$

Hence show that the operator $\phi(x)$ satisfies the Klein-Gordon equation.

3. Let $\phi(x)$ be a real scalar field in the Heisenberg picture. Show that the relativistically normalised states $|p\rangle = \sqrt{2E_{\vec{p}}} a_{\vec{p}}^\dagger |0\rangle$ satisfy

$$\langle 0 | \phi(x) | p \rangle = e^{-ip \cdot x}.$$

- 4* In Example Sheet 1, you showed that the classical angular momentum of a field is given by

$$Q_i = \frac{1}{2} \epsilon_{ijk} \int d^3x (x^j T^{0k} - x^k T^{0j}).$$

Write down the explicit form of the angular momentum for a free real scalar field with Lagrangian as in Eq.(1). Show that, after normal ordering, the quantum operator Q_i can be written as

$$Q_i = \frac{i}{2} \epsilon_{ijk} \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}}^\dagger \left(p^j \frac{\partial}{\partial p_k} - p^k \frac{\partial}{\partial p_j} \right) a_{\vec{p}}.$$

Hence confirm that the quanta of the scalar field have spin zero (i.e. a one-particle state $|\vec{p}\rangle$ has zero angular momentum in its rest frame).

5. Show that the time ordered product $T(\phi(x_1)\phi(x_2))$ and the normal ordered product $:\phi(x_1)\phi(x_2):$ are both symmetric under the interchange of x_1 and x_2 . Deduce that the Feynman propagator $\Delta_F(x_1 - x_2)$ has the same symmetry property.

6. Verify Wick's theorem for the case of three scalar fields:

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) = : \phi(x_1)\phi(x_2)\phi(x_3) : + \phi(x_1)\Delta_F(x_2 - x_3) + \phi(x_2)\Delta_F(x_3 - x_1) + \phi(x_3)\Delta_F(x_1 - x_2).$$

7. Examine $\langle 0|S|0\rangle$ to order λ^2 in ϕ^4 theory. Identify the different contributions arising from an application of Wick's theorem. Confirm that to order λ^2 , the combinatoric factors work out so that the vacuum to vacuum amplitude is given by the exponential of the sum of distinct vacuum bubble types,

$$\langle 0|S|0\rangle = \exp\left(\text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots \right)$$

8. Consider the theory of a complex scalar ψ and a real scalar ϕ with Lagrangian

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \mu^2\psi^*\psi - \frac{1}{2}m^2\phi^2 - g\psi^*\psi\phi - h|\psi|^4 - k\phi^3 - l\phi(\partial_\mu\psi)(\partial^\mu\psi^*).$$

Draw and write down momentum-space Feynman space Feynman rules for the propagators and interactions of this theory. What are the mass dimensions of g , h , k , l ? From these, identify a property of the theory.

9. Consider the scalar Yukawa theory given by the Lagrangian

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \mu^2\psi^*\psi - \frac{1}{2}m^2\phi^2 - g\psi^*\psi\phi.$$

In meson decay $\phi \rightarrow \psi\bar{\psi}$, assuming that $m > 2\mu$, show that to lowest order in g^2 that the decay width is

$$\Gamma = \frac{|g|^2}{16\pi m} \sqrt{1 - 4\mu^2/m^2}.$$

Does this make sense, dimensionally?