3P2b **Symmetries: Examples Sheet 2** Michaelmas 2020

Corrections and suggestions should be emailed to B.C.Allanach@damtp.cam.ac.uk*. Starred questions may be handed in prior to the class to your examples class supervisor, for feedback.*

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1*. This question regards Pauli matrices, $SO(3)$ and $SU(2)$

(a) Verify the following properties of the Pauli matrices $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$:

(i) $\sigma_i\sigma_j = I\delta_{ij} + i\epsilon_{ijk}\sigma_k$,

(ii) $\sigma_2\boldsymbol{\sigma}\sigma_2 = -\boldsymbol{\sigma}^*$,

(iii) $\exp(-i\theta\mathbf{n}\cdot\boldsymbol{\sigma}/2) = I\cos(\theta/2) - i\mathbf{n}\cdot\boldsymbol{\sigma}\sin(\theta/2)$.

(b) Three 3×3 matrices $\mathbf{T} := (T_1, T_2, T_3)$ are defined by $(T_i)_{jk} = -i\epsilon_{ijk}$.

(i) Prove $[T_i\,, T_j] = i\epsilon_{ijk}T_k$.

(ii) Prove $(\mathbf{n}\cdot\mathbf{T})^3 = |\mathbf{n}|^2\mathbf{n}\cdot\mathbf{T}$.

(iii) What are the possible eigenvalues of $\mathbf{n}\cdot\mathbf{T}$ if $\mathbf{n}$ is a unit vector?

(iv) We may represent a rotation by an angle θ about an axis that points along the unit vector n by the member of $SO(3)$ $R_{ij}(\mathbf{n},\theta) := \exp(-i\theta\mathbf{n}\cdot\mathbf{T})_{ij}$. By convention, $\mathbf{n}$ points in any direction and $0 \le \theta \le \pi$. Evaluate R_{ij} explicitly by summing the Taylor series of the exponential, and show that

$$R_{ij}(\mathbf{n},\theta) = n_in_j + (\delta_{ij} - n_in_j)\cos\theta - \epsilon_{ijk}n_k\sin\theta\,.$$

(c) Verify the formula: $e^{-i\theta\mathbf{n}\cdot\boldsymbol{\sigma}/2}\,\sigma_j\,e^{i\theta\mathbf{n}\cdot\boldsymbol{\sigma}/2} = R_{ij}(n,\theta)\,\sigma_i\,.$

(d) The set of matrices $\exp(-i\theta\mathbf{n}\cdot\boldsymbol{\sigma}/2)$ constitutes the defining representation of SU(2). Prove that this representation is *pseudoreal* (defined to be that the complex conjugate returns minus the initial matrix: see later in the course).

2. Let $T^{\alpha_1\ldots\alpha_{2j}}$ be a symmetric $SU(2)$ tensor for $j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

(a) Show that the action of the spin operator is given by

$$\mathbf{S}^{\alpha_1\ldots\alpha_{2j}}{}_{\beta_1\ldots\beta_{2j}}T^{\beta_1\ldots\beta_{2j}} = \sum_{i=1}^{2j}\tfrac{1}{2}(\boldsymbol{\sigma})^{\alpha_i}{}_{\beta}T^{\alpha_1\ldots\alpha_{i-1}\beta\alpha_{i+1}\ldots\alpha_{2j}}\ ,$$

where $\boldsymbol{\sigma}$ are the Pauli matrices.

(b) Defining (for $m = -j, -j+1, \ldots, j$)

$$T^{(jm)} = \big[(j+m)!(j-m)!\big]^{-\frac{1}{2}}T^{\overbrace{1\ldots1}^{j+m}\overbrace{2\ldots2}^{j-m}},$$

calculate $S_\pm T^{(jm)}$ and $S_3T^{(jm)}$.

(c) Show that $\bar{T}_{\alpha_1\ldots\alpha_{2j}}T^{\alpha_1\ldots\alpha_{2j}} = (2j)!\sum_m T^{(jm)*}T^{(jm)}$ for $\bar{T}_{\alpha_1\ldots\alpha_{2j}}$ being the conjugate tensor.

3*. A field $\phi(x)$ transforms under the action of a Poincaré transformation (Λ, a) such that $U[\Lambda,a]\phi(x)U[\Lambda,a]^{-1} = \phi(\Lambda x + a)$. For an infinitesimal transformation, $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ and correspondingly $U[\Lambda,a] = 1 - i\frac{1}{2}\omega^{\mu\nu}M_{\mu\nu} - ia^\mu P_\mu$.

(a) Show that

$$[M_{\mu\nu}, \phi(x)] = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi(x)\,, \qquad [P_\mu, \phi(x)] = i\partial_\mu \phi(x)\,.$$

(b) Verify that $M_{\mu\nu} \to i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ and $P_\mu \to -i\partial_\mu$ satisfy the algebra for $[M_{\mu\nu}, M_{\sigma\rho}]$ and $[M_{\mu\nu}, P_\sigma]$ expected for the Poincaré group.

4. (a) Show how $B(\theta, \mathbf{n}) \in SL(2, \mathbb{C})$ where

$$B(\theta, \mathbf{n}) = I \cosh \frac{\theta}{2} + \sigma \cdot \mathbf{n}\ \sinh \frac{\theta}{2}, \qquad \mathbf{n}^2 = 1,$$

corresponds to a Lorentz boost with velocity $\mathbf{v} = \tanh\theta\, \mathbf{n}$.

(b) Show that

$$\big(1 + \frac{1}{2}\sigma \cdot \delta\mathbf{v}\big) B(\theta, \mathbf{n}) = B(\theta', \mathbf{n}')R,$$

where, to first order in $\delta\mathbf{v}$,

$$\theta' = \theta + \delta\mathbf{v} \cdot \mathbf{n}, \quad \mathbf{n}' = \mathbf{n} + \coth\theta(\delta\mathbf{v} - \mathbf{n}\, \mathbf{n} \cdot \delta\mathbf{v}),$$

and R is an infinitesimal rotation given by

$$R = 1 + \frac{i}{2} \tanh \frac{\theta}{2}(\delta\mathbf{v} \times \mathbf{n}) \cdot \sigma = 1 + \frac{i}{2}\frac{\gamma}{\gamma + 1}(\delta\mathbf{v} \times \mathbf{v}) \cdot \sigma, \qquad \gamma = (1 - \mathbf{v}^2)^{-\frac{1}{2}}.$$

(c) Show that we must have $\mathbf{v}' = \mathbf{v} + \delta\mathbf{v} - \mathbf{v}\ \mathbf{v}\cdot\delta\mathbf{v}$. [NB $\sigma\cdot\mathbf{a}\ \sigma\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{b}\ 1 + i\sigma\cdot(\mathbf{a}\times\mathbf{b})$.]

5. The group of four dimensional space-time symmetries may be expanded to conformal transformations $x \to x'$ defined by the requirement

$$\mathrm{d}x'^2 = \Omega(x)^2 \mathrm{d}x^2\,,$$

where $\mathrm{d}x^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. For an infinitesimal transformation $x'^\mu = x^\mu + f^\mu(x)$, $\Omega(x)^2 = 1 + 2\sigma(x)$.

(a) Obtain in this case

$$\partial_\mu f_\nu + \partial_\nu f_\mu = 2\sigma\, g_{\mu\nu} \quad \Rightarrow \quad 4\sigma = \partial \cdot f\,.$$

(b) Hence obtain

$$4\, \partial_\sigma \partial_\mu f_\nu = g_{\mu\nu}\, \partial_\sigma \partial \cdot f + g_{\sigma\nu}\, \partial_\mu \partial \cdot f - g_{\sigma\mu}\, \partial_\nu \partial \cdot f\,.$$

From this obtain $2\partial_\sigma \partial_\mu \partial \cdot f = -g_{\sigma\mu}\, \partial^2 \partial \cdot f$ and hence show that we must have $\partial_\sigma \partial_\mu \partial \cdot f = 0$.

(c) Why does it then follow that $f_\mu(x)$ can only be quadratic in x? Show that $f^\mu(x)$ must then have the general form

$$f^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2b \cdot x\, x^\mu\,, \quad \omega_{\mu\nu} = -\omega_{\nu\mu}\,.$$

(d) Show also that an inversion $x'^\mu = x^\mu / x^2$ is a conformal transformation. Calculate the finite conformal transformation obtained by an inversion followed by a translation by b^μ followed by another inversion and show that it is compatible with the result for $f^\mu(x)$.

6. Verify the Baker-Campbell-Hausdorff formula

$$\exp A\ \exp B = \exp\left(A + B + \frac{1}{2}[A, B] + \frac{1}{12}\left\{[A, [A, B]] + [B, [B, A]]\right\} + \ldots\right)$$

to the order shown.

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