3P2c **Symmetries: Examples Sheet 3** Michaelmas 2020

Corrections and suggestions should be emailed to `B.C.Allanach@damtp.cam.ac.uk`. *Starred questions may be handed in prior to the class to your examples class supervisor, for feedback.*

1. A Lie group has group elements $g(a)$ depending on group parameters a^r, with $g(0) = e$, the identity, and under group multiplication $g(a)g(b) = g(\varphi(a,b))$ for some $\varphi^r(a,b)$. Let $g(a)^{-1} = g(\bar{a})$ where $\varphi(\bar{a}, a) = 0$.

   (a) Why must $\varphi^r(a,0) = a^r, \varphi^r(0,b) = b^r$?

   (b) Assume $\varphi^r(a,b)$ is expanded near the origin according to

   $$\varphi^a(a,b) = a^a + b^a + c^a{}_{bc} a^b b^c + \mathrm{O}(a^2 b, ab^2).$$

   Use this to find $\bar{a}(a)$ for a small.

   (c) Let $g(d) = g(a)^{-1}g(b)^{-1}g(a)g(b)$ and show that for a, b small $d^a = f^a_{bc} a^b b^c$ where $f^a_{bc} = c^a_{bc} - c^a_{cb}$.

   (d) Using an expansion to one higher order show that the associativity condition $\varphi(\varphi(a,b),c) = \varphi(a, \varphi(b,c))$ leads to the Jacobi identity.

   (e) Assume the Lie group has generators T_a satisfying $[T_a, T_b] = f^a_{bc} T_c$. For an element of the Lie algebra $a^a T_a$ there is an associated group element given by $g(a) = \exp(a^a T_a)$. Use the Baker-Campbell-Hausdorff formula $\exp A \exp B = \exp(A + B + \frac{1}{2}[A,B] + \frac{1}{12}([A,[A,B]] + [B,[B,A]]) + \dots)$ to obtain $\varphi(a,b)$ for small a, b and verify that this is compatible with the general expansion of φ.

2. Using the same notation as the previous question, where $c^r = \varphi^r(a,b)$, obtain

   $$\frac{\partial c^r}{\partial b^s} = \lambda_s{}^a(b)\mu_a{}^r(c), \quad \mu_a{}^r(c) = \frac{\partial}{\partial b^a}\varphi^r(c,b)\bigg|_{b=0}, \quad \mu_a{}^r(c)\lambda_r{}^b(c) = \delta_a{}^b.$$

   Show that the equation for the structure constants f^a_{bc} may also be expressed as

   $$\frac{\partial}{\partial b^r}\lambda_s{}^a(b) - \frac{\partial}{\partial b^s}\lambda_r{}^a(b) = -f^a_{bc}\lambda_r{}^b(b)\lambda_s{}^c(b).$$

3*. An $SU(2)$ matrix may be represented by

   $$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

   Let $\alpha = |\alpha|e^{i\lambda}$ where $|\alpha|^2 = 1 - |\beta|^2$ so that A is regarded as depending on the real parameter λ and the complex parameter β.

   (a) Determine $\mathrm{d}\lambda, \mathrm{d}\beta, \mathrm{d}\bar{\beta}$ in terms of infinitesimal $\phi, \theta, \bar{\theta}$ where

   $$\begin{pmatrix} \alpha + \mathrm{d}\alpha & \beta + \mathrm{d}\beta \\ -\bar{\beta} - \mathrm{d}\bar{\beta} & \bar{\alpha} + \mathrm{d}\bar{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 1 + i\phi & \theta \\ -\bar{\theta} & 1 - i\phi \end{pmatrix}.$$

   (b) Hence show that the invariant measure for $SU(2)$ with these parameters is $\mathrm{d}\lambda \mathrm{d}^2\beta$. What are the ranges of λ and β?

(c) Calculate $\int \mathrm{d}\lambda \mathrm{d}^2\beta$. (For $\mathrm{d}\beta = \mathrm{d}\beta_1 + i\mathrm{d}\beta_2$, $\mathrm{d}^2\beta = \mathrm{d}\beta_1\mathrm{d}\beta_2$.)

4. Consider two $SU(2)$ algebras, $I_\pm, I_3$ and $U_\pm, U_3$, where $[I_3, I_\pm] = \pm I_\pm,\ [I_+, I_-] = 2I_3$ with $I_3{}^\dagger = I_3,\ I_+{}^\dagger = I_-$, and also similarly for $I \to U$. Assume

$$[I_-, U_+] = 0, \quad [I_3, U_+] = -\frac{1}{2}U_+, \qquad [U_-, I_+] = 0 \quad [U_3, I_+] = -\frac{1}{2}I_+.$$

Define $V_+ = [I_+, U_+]$ and $V_- = V_+{}^\dagger$.

(a) Explain why $[I_+, V_+] = [U_+, V_+] = 0$.

(b) Evaluate $[V_+, V_-]$ and show that $V_\pm, V_3 = I_3 + U_3$ form a $SU(2)$ algebra.

(c) Let $U_3 = -\frac{1}{2}I_3 + \frac{3}{2}Y$. Work out the various commutators involving Y.

(d) Explain how the **I**-spin and **U**-spin operators with the above relations generate a $SU(3)$ algebra.

5. For a simple Lie algebra $\mathcal{L}$, with elements X_i such that $[X_i, X_j] = if_{ijk}X_k$ where f_{ijk} is totally antisymmetric, let T_i be matrices forming a representation R of $\mathcal{L}$ and assume $T_iT_i = C_R I$. Define

$$\langle X_i, X_j \rangle = \mathrm{tr}(T_iT_j)\frac{\mathrm{dim}\mathcal{L}}{C_R \mathrm{dim} R}.$$

(a) Evaluate $\langle J_3, J_3 \rangle$ in the j-th irreducible representation of $SU(2)$ and show that the result is independent of j.

(b) For $SU(3)$ show that the representation given by $T_i = \frac{1}{2}\lambda_i$ gives the same value for $\langle X_i, X_j \rangle$ as does the adjoint representation $(T_i^{\mathrm{ad}})_{jk} = -if_{ijk}$.

6. Let $\{T^i{}_j\}$ be $n \times n$ matrices such that $T^i{}_j$ has a 1 in the i'th row and j'th column and is zero otherwise.

(a) Show that they satisfy the Lie algebra

$$[T^i{}_j, T^k{}_l] = \delta^k{}_j T^i{}_l - \delta^i{}_l T^k{}_j.$$

(b) Define $X = T^i{}_j X^j{}_i$ with arbitrary components $X^j{}_i$. Determine the adjoint matrix $(X^{\mathrm{ad}})^n{}_m,{}^k{}_l$ by

$$[X, T^k{}_l] = T^m{}_n (X^{\mathrm{ad}})^n{}_m,{}^k{}_l,$$

and show that

$$\kappa(X, Y) = \mathrm{tr}\big(X^{\mathrm{ad}}Y^{\mathrm{ad}}\big) = 2\big(n\textstyle\sum_{i,j} X^j{}_i Y^i{}_j - \sum_i X^i{}_i \sum_j Y^j{}_j\big).$$

(c) Show that $1 + \epsilon X \in U(n)$ for infinitesimal ϵ if $(X^j{}_i)^* = -X^i{}_j$.

(d) Hence show that in this case

$$\kappa(X, X) = -2n\textstyle\sum_{i,j}|\hat{X}^j{}_i|^2, \qquad \hat{X}^j{}_i = X^j{}_i - \frac{1}{2}\delta^j{}_i \sum_k X^k{}_k,$$

and therefore $\kappa(X, X) = 0 \ \Leftrightarrow\ X^{\mathrm{ad}} = 0$.

(e) What restrictions must be made for $SU(N)$ and verify that in this case the generators satisfy $\kappa(X, X) < 0$ so the group is semi-simple?

7. For a group with a Lie algebra with a basis $\{T_a\}$ such that $[T_a, T_b] = f^c{}_{ab} T_c$ let $g_{ab} = \langle T_a, T_b \rangle$ where $\langle\ ,\ \rangle$ is an invariant symmetric bilinear form so that $\langle [X, Y], Z \rangle = -\langle Y, [X, Z] \rangle$.

   (a) Show that $f_{abc} = g_{ad} f^d{}_{bc}$ is totally antisymmetric.

   (b) If D_μ is an appropriate covariant derivative involving a gauge field A^a_μ, verify

   $$\partial_\mu \langle X(x), Y(x) \rangle = \langle D_\mu X(x), Y(x) \rangle + \langle X(x), D_\mu Y(x) \rangle.$$

   (c) Let $T^\mu{}_\nu = \langle F^{\mu\sigma}, F_{\nu\sigma} \rangle - \frac{1}{4} \delta^\mu{}_\nu \langle F^{\sigma\rho}, F_{\sigma\rho} \rangle$. Show, using the Bianchi identity, $\partial_\mu T^\mu{}_\nu = \langle D_\mu F^{\mu\sigma}, F_{\nu\sigma} \rangle$.

   (d) For a variation δA^a_μ obtain also $\delta \frac{1}{4} \epsilon^{\mu\nu\sigma\rho} \langle F_{\mu\nu}, F_{\sigma\rho} \rangle = \partial_\mu \epsilon^{\mu\nu\sigma\rho} \langle \delta A_\nu, F_{\sigma\rho} \rangle$.

   (e) By letting $A_\mu \to t A_\mu$ and differentiating with respect to t and then integrating show that

   $$\frac{1}{4} \epsilon^{\mu\nu\sigma\rho} \langle F_{\mu\nu}, F_{\sigma\rho} \rangle = \partial_\mu \epsilon^{\mu\nu\sigma\rho} \big\langle A_\nu, \partial_\sigma A_\rho + \frac{1}{3} [A_\sigma, A_\rho] \big\rangle.$$

8. With notation as in the previous question define a three dimensional Lagrangian

   $$\mathcal{L} = \epsilon^{\mu\nu\rho} \big(g_{ab} A^a_\mu \partial_\nu A^b_\rho + \tfrac{1}{3} f_{abc} A^a_\mu A^b_\nu A^c_\rho \big).$$

   For a a gauge transformation $\delta A^a_\mu = -\partial_\mu \lambda^a - f^a{}_{bc} A^b_\mu \lambda^c$ show that $\delta \mathcal{L} = -\partial_\mu \big(\epsilon^{\mu\nu\rho} g_{ab} \lambda^a \partial_\nu A^b_\rho \big)$ so that $\int \mathrm{d}^3 x \mathcal{L}$ is invariant.