Isomonodromic deformations and twistor geometry

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and
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Introduction

Slides available at `damtp.cam.ac.uk/tjahm2`

`arXiv:2509.05275` in one sentence:
Given a tuple of positive integers μ we construct a (meromorphic) complex hyper-Kähler structure called a Joyce structure on a "natural" torus bundle $\mathbb{T}$ fibreing over Quad(μ), the moduli of quadratic differentials on $\mathbb{CP}^1$ with simple zeroes and poles of order μ.

Caveats: μ should consist of odd integers, one of which ≥ 5.

Motivation I

The why (and a link to mathematical physics):

- Bridgeland conjectures: Complex hyper-Kähler metrics on the "natural" torus bundle $\mathbb{T}$ over *spaces of stability conditions M* of CY3 categories. Geometry called: "Joyce structure"
- The metrics arise due to dependence of Donaldson-Thomas invariants on a choice of stability condition (WCF).
- Bridgeland-Smith correspondence:
  Special spaces of stability conditions are moduli spaces of quadratic differentials on Riemann surfaces.

Motivation II

A geometic axiomatisation (without reference to stability conditons) of Joyce structures is given in Bridgeland-Strachan (2020). Complex hyper-Kähler metric on torus bundle over symplectic manifold M + many symmetries

- They are complex hyperkähler metrics with *CKV* ($\mathcal{L}_W g = g$), a type of metric long studied by Tod, Jones, Calderbank, Pedersen, Dunajski et. al.
- Our examples arise from *isomonodromy problems*: Some correspond to known integrable systems/Painlevé equations. See Bridgeland-del Monte (2025)
- Link to Frobenius structures? Starting from defn. there is a procedure to construct a commutative algebra and a compatible flat bilinear form on M. Open problem: When is this a Frobenius structure?

Twistor distributions

In what follows let X be a complex manifold of dimension $4n$ and TX the holomorphic tangent bundle. For our purposes:

Definition (Twistor distribution)

A (hyper-Hermitian) twistor distribution is a one-parameter family of subbundles of TX

$$L(\hbar) = \text{span}\left\{\hbar U_a + V_a\right\}_{a=1}^{2n} \tag{1}$$

depending on a parameter $\hbar \in \mathbb{C}$*, where* U_a, V_a *are some vector fields on* X *such that* $TX = span\{U_a, V_a\}_{a=1}^{2n}$.

Quaternionic structure

$L(\hbar) = \text{span}\left\{\hbar U_a + V_a\right\}_{a=1}^{2n}$ determines a quaternionic structure given by

$$J(U_a) = V_a, \quad K(U_a) = iV_a \tag{2}$$

and the relations $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}_{TX}$. Thinking of $\hbar = u^1/u^0$ the usual affine coordinate on $\mathbb{CP}^1$ think of the $L(\hbar)$ as a distribution L on $X \times \mathbb{CP}^1$.

L Frobenius integrable $\iff$ complex structures I, J, K integrable $\iff$ $\exists 2n+1$-dimensional twistor space

$$\mathcal{Z} = (X \times \mathbb{CP}^1)/L. \tag{3}$$

Associated family of hyper-Hermitian metrics

$L(\hbar) = \text{span}\left\{U_a + \frac{V_a}{\hbar}\right\}_{a=1}^{2n}$ also determines a family of holomorphic metrics satisfying the *hyper-Hermitian condition* $I^*g = J^*g = K^*g = g$:

$$g = \sum_{i,j=1}^{2n} e_{ab} U^a \odot V^a, \quad \{U^a, V^a\}_{a=1}^{2n} \text{ dual basis for } T^*X \quad (4)$$

parametrised by $2n \times 2n$ non-degenerate skew matrices e_{ab} of holomorphic functions ($\odot$ means symmetrised tensor product!)

- $n = 1$: Classical twistor theory e.g. Penrose (1976). A conformal class of metrics on 4-dimensional X. L is Frobenius integrable $\iff$ g has ASD Weyl tensor
- $n > 1$: almost Grassmannian (paraconformal) geometries. See (Bailey-Eastwood (1991) or Čap-Slovák (2009))

Complex hyper-Kähler metrics

Definition (Complex hyper-Kähler)

A complex hyper-Kähler structure is a holomorphic metric g and triple of holomorphic endomorphisms I, J, K of TX satisfying the quaternion relations such that g is Hermitian for each and $\nabla I = \nabla J = \nabla K = 0$.

When is there a hyper-Kähler metric in the class?

$$g = \sum_{i,j=1}^{2n} e_{ab} U^a \odot V^a$$

has associated

$$\Omega = \sum_{a,b=1}^{2n} \frac{e_{ab} U^a \wedge U^b}{\hbar^2} - \frac{2 e_{ab} U^a \wedge V^b}{\hbar} + e_{ab} V^a \wedge V^b. \tag{5}$$

$\mathcal{O}(2)$-valued relative 2-form

g is hyper-Kähler if and only if

$$\Omega = \sum_{a,b=1}^{2n} \frac{e_{ab} U^a \wedge U^b}{\hbar^2} - \frac{2 e_{ab} U^a \wedge V^b}{\hbar} + e_{ab} V^a \wedge V^b \tag{6}$$

is a d_X-closed 2-form. (Not surprising)
(More interestingly) Every d_X-closed 2-form on $X \times \mathbb{CP}^1$ of the form

$$\Omega = \frac{\Omega_-}{\hbar^2} + \frac{2i\Omega_I}{\hbar} + \Omega_+ \tag{7}$$

which has rank-$2n+1$ kernel containing L yields a complex hyper-Kähler metric.
The $\mathcal{O}(2)$-valued 2-form characterisation appears in Hitchin-Karlhede-Lindström-Roček (1987), Bailey-Eastwood (1991).

Spaces of meromorphic quadratic differentials

Consider the moduli space $M = \text{Quad}(\mu)$. We have $\dim M = 2n$.

- $\phi \in M$ are of the form $\phi = Q_0(x)dx^2$.
- We identify them up to Möbius transformation.
- Associated genus n spectral curve (*SW curve*) $\Sigma_0(\phi)$ given by $y_0^2 = Q_0(x)$.
- $T_\phi M \cong H^1(\Sigma_0(\phi), \mathbb{C})$ via differentiation of taut. 1-form

$$T_\phi M \ni V \mapsto [V(y_0)dx].$$

- $\mathbb{T} = TM/\Gamma$ where Γ is the complex lattice of integral cycles

Standard example: $\text{Quad}(\{7\})$ every differential represented by

$$\phi = (x^3 + ax + b)dx^2, \quad 4a^3 + 27b^2 \neq 0 \tag{8}$$

and $\Sigma_0(\phi)$ is an elliptic curve. (a, b) give local coordinates!

The moduli space $X \hookrightarrow \mathbb{T}$

We consider a space X of pairs $\xi = (\overbrace{\phi = Q_0(x)dx^2}^{\in M}, Q_1(x))$ where $Q_1(x)$ is the general meromorphic function of the form:

$$Q_1(x) = R(x) + \sum_{I=1}^{n} \frac{p_I}{x - q_I} \tag{9}$$

where $R(x)$ may have the poles of Q_0 at worst "half as bad rounded-up". Here q_I are distinct points not coinciding with the poles or zeroes of ϕ and $p_I^2 = Q_0(q_I)$. **Why?**
Taking periods of the 1-form on $\Sigma_0(\phi) = \{y_0^2 = Q_0(x)\}$

$$\sigma = \frac{Q_1(x)}{2y_0}dx \quad \text{(residues } \pm 1/2) \tag{10}$$

gives a biholomorphism from X to an open dense subset of $\mathbb{T}$.

ODE determined by a point in $X \hookrightarrow \mathbb{T}$

A point $\xi = (Q_0(x)dx^2, Q_1(x)) \in X$ together with the parameter $\hbar$ determines an ODE (via Hitchin spectral corresondence)

$$Y'' = Q(x)Y \tag{11}$$

where

$$Q(x) = \frac{Q_0(x)}{\hbar^2} + \frac{Q_1(x)}{\hbar} + Q_2(x). \tag{12}$$

$$Q_1(x) = R(x) + \sum_{I=1}^{n} \frac{p_I}{x - q_I}$$

Here $Q_2(x)$ is chosen so that Y is multiplied by -1 after analytic continuation around a $x = q_I$ (in order the equation has isomonodromic deformations!).

Defining PDE for isomonodromic deformations

For the purposes of our problem the output of the general theory is the following theorem:

Theorem (Schlesinger / Jimbo-Miwa-Ueno(1981))

A one-parameter family $Q(t,x)$ of deformations of the potential for the ODE

$$\frac{d^2Y}{dx^2} = Q(x)Y$$

varying holomorphically with the parameter t has constant (generalised) monodromy if and only if

$$\frac{\partial Q(x,t)}{\partial t} = 2Q(x,t)\frac{\partial A(t,x)}{\partial x} + \frac{\partial Q(t,x)}{\partial x}A(t,x) - \frac{1}{2}\frac{\partial^3 A(t,x)}{\partial x^3}$$

for some $A(t,x)$.

Defining PDE geometrically

If we write $y^2 = Q(x)$ then there is a tautological 1-form ydx with constant residues on the genus-$2n$ curve $\Sigma(\xi, \hbar)$. We can define a map $\mu_{\xi,\hbar} : T_\xi X \to H^1(\Sigma(\xi, \hbar), \mathbb{C})$ by

$$V \mapsto [V(y)dx] \in H^1(\Sigma(\xi, \hbar), \mathbb{C}) \quad \text{(not an isomorphism!)}$$

There is a 2-form Ω on X given by pulling the standard cohomology intersection forms back by this map.

$$\Omega(U, V) = 2\pi i \sum_{x \in \Sigma(\xi,\hbar)} \mathrm{Res}_x(d^{-1}(U(y)dx)V(y)dx) \qquad (13)$$

Then the deformation condition $\implies$

$$\left[\frac{dy}{dt}dx\right] = -\left[\frac{1}{4y}\frac{\partial^3 A}{\partial x^3}dx\right] \in H^1(\Sigma(\xi, \hbar), \mathbb{C}). \qquad (14)$$

the RHS turns out to be $\perp$ w.r.t Ω of $\mathrm{im}\mu_{\xi,\hbar}$.

$\mathcal{O}(2)$-valued 2-form

Theorem (M. (2025))

- *The generators of the isomonodromic deformations of the equation* $Y'' = Q(x)Y$ *with potential (12) define a twistor distribution* L.
- Ω *takes the form*

$$\Omega = \frac{\Omega_-}{\hbar^2} + \frac{2i\Omega_I}{\hbar} + \Omega_+ \tag{15}$$

and has rank $2n+1$ *kernel containing* L

Hence we get a hyper-Kähler metric on X.

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