Two parabolic contact geometries in five dimensions

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Contact geometry

Notation

- *M a five-dimensional manifold*
- *$H \subseteq TM$ rank-four distribution*
- *Let $\wedge^k = \wedge^k T^*M$ and $\wedge^k_H = \wedge^k H^*$*
- *$L := Ann(H) \subseteq \wedge^1$ the line bundle of contact forms $\alpha \in L$.*

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Contact condition: $d\alpha|_H$ is a non-degenerate skew-form.

Partial connections and differential invariants

Definition (Partial connection)

A partial connection (with respect to H) on a vector bundle E is a linear differential operator $\nabla : E \to \wedge^1_H \otimes E$ satisfying

$$\nabla(fs) = df|_H \otimes s + f\nabla s.$$

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and for general E the partial curvature:

$$R_{ab}{}^\mu{}_\nu := (\nabla_{[a}\nabla_{b]} s^\mu)_\perp - \tau_{ab}{}^c \nabla_c s^\mu.$$

Promoting partial connections

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Lemma (Eastwood, Gover, 2011)

Let $\nabla : E \to \Lambda^1_H \otimes E$ be a partial connection on a contact manifold.
There exists a unique connection $\tilde{\nabla} : E \to \Lambda^1 \otimes E$ extending ∇ such that the 2-form part of $\tilde{\kappa}|_H$ is trace-free, where $\tilde{\kappa}$ is the curvature of $\tilde{\nabla}$.
Furthermore:

- $\tilde{\nabla}\sigma = 0 \iff \nabla\sigma = 0.$
- *$\tilde{\nabla}$ is flat if and only if the partial curvature of ∇ vanishes.*

Legendrean contact geometry

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A Legendrean contact geometry is a contact manifold of dimension $2n+1$ *with a decomposition* $H = P \oplus V$ *where* P, V *are of rank-*n *and isotropic for* $d\alpha|_H$.

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- Standard contact structure with coordinates (t, q^i, p_i) where $P = \text{span}\{p_i\partial_t + \partial_{q^i}\}$, $V = \text{span}\{\partial_{p_i}\}$

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  P consisting of velocities fixing the hyperplane,
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  P consisting of velocities fixing the hyperplane,
  V consisting of velocities fixing the line.
- Second-order overdetermined PDE in 1 unknown

$$\frac{\partial^2 t}{\partial q^i \partial q^j} = f_{ij}(t, q^i, \partial_{q^i} t)$$

See [Doubrov, Medvedev, The, 2020].

Contact Legendrean as a parabolic geometry

In dimension five Legendrean contact geometries are the Cartan (parabolic) geometries of type $(SL(4,\mathbb{R}), P)$ where

$$P = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\}$$

In particular since P is parabolic there will be no canonical connection on the tangent bundle.

A non-canonical connection

Theorem (Basically (5.2.11) (Čap, Slovák, 2008))

Suppose that P, V *are integrable. Given a choice of contact form* $\alpha \in L$ *there is a unique partial connection* $V^* \to \Lambda^1_H \otimes V^*$ *such that the induced affine partial connection* $\Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$ *has vanishing partial torsion.*

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The freedom in choosing a partial connection $V^* \to \Lambda^1_H \otimes V^*$ is the bundle:

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A priori the partial torsion lies in the bundle

$$Hom(\Lambda^1_H \otimes \Lambda^2_{H\perp}) \qquad rank: \ 4 \times 5 = 20$$

but integrability of P, V further reduces the rank of the bundle in which the torsion lies to 16.

Formulae for change of contact scale

Rescale the contact form by $\hat{\alpha} = \Omega\alpha$ and let $\Upsilon_\alpha = \nabla_\alpha \log \Omega$ and $\bar{\Upsilon}_{\bar{\alpha}} = \bar{\nabla}_{\bar{\alpha}} \log \Omega$.

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Rescale the contact form by $\hat{\alpha} = \Omega\alpha$ and let $\Upsilon_\alpha = \nabla_\alpha \log \Omega$ and $\bar{\Upsilon}_{\bar{\alpha}} = \bar{\nabla}_{\bar{\alpha}} \log \Omega$. We have *change of connection* formulae for sections of V^*:

$$(\hat{\bar{\nabla}}_{\bar{\alpha}}\omega_\beta, \hat{\nabla}_\alpha\omega_\beta) = (\bar{\nabla}_{\bar{\alpha}}\omega_\beta + J_{\bar{\alpha}\beta}\bar{\Upsilon}_{\bar{\gamma}}\omega^{\bar{\gamma}}, \nabla_\alpha\omega_\beta - 2\Upsilon_{(\alpha}\omega_{\beta)}),$$

for sections of P^*

$$(\hat{\bar{\nabla}}_{\bar{\alpha}}\omega_{\bar{\beta}}, \hat{\nabla}_\alpha\omega_{\bar{\beta}}) = (\bar{\nabla}_{\bar{\alpha}}\omega_{\bar{\beta}} - 2\bar{\Upsilon}_{(\bar{\alpha}}\omega_{\bar{\beta})}, \nabla_\alpha\omega_{\bar{\beta}} + J_{\bar{\beta}\alpha}\Upsilon_\gamma\omega^\gamma)$$

for sections of L we have

$$(\hat{\bar{\nabla}}_{\bar{\alpha}}f, \hat{\nabla}_\alpha f) = (\bar{\nabla}_{\bar{\alpha}}f - \bar{\Upsilon}_{\bar{\alpha}}f, \nabla_\alpha f - \Upsilon_\alpha f)$$

These are basically the same formulae for how the Tanaka-Webster connection changes in the CR case (see [Gover, Graham, 2003])

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On a particular tensor product of roots of L and $\wedge^2 P^*$ (densities):

$$\bar{\nabla}_{\bar{\alpha}} f = 0, \ \nabla_\alpha \nabla_\beta f - 2 Z_{\alpha\beta} f = 0$$

where $Z_{\alpha\beta}$ is a particular component of the partial curvature.

A canonical connection

This invariant PDE can be prolonged to:

$$\bar{\nabla}_{\bar{\alpha}} \begin{bmatrix} f \\ \phi_\beta \\ g \end{bmatrix} = \begin{bmatrix} \bar{\nabla}_{\bar{\alpha}} f \\ \bar{\nabla}_{\bar{\alpha}} \phi_\beta - J_{\bar{\alpha}\beta} g + \frac{1}{2} Y_{\bar{\alpha}\beta} f \\ \bar{\nabla}_{\bar{\alpha}} g - 2P_{\bar{\alpha}}^{\beta} \phi_\beta + \frac{1}{2} \bar{\nabla}_{\bar{\gamma}} Y_{\bar{\alpha}}^{\bar{\gamma}} f \end{bmatrix} = 0$$

$$\nabla_\alpha \begin{bmatrix} f \\ \phi_\beta \\ g \end{bmatrix} = \begin{bmatrix} \nabla_\alpha f - \phi_\alpha \\ \nabla_\alpha \phi_\beta - K_{\alpha\beta} f \\ \Big\{ \nabla_\alpha g - \frac{1}{3} \bar{\nabla}_{\bar{\beta}} K_\alpha^{\bar{\beta}} f + \frac{4}{3} Y^{\bar{\beta}\gamma}_{\bar{\beta}\alpha} \phi_\gamma \\ + \frac{1}{6} (\nabla^{\bar{\gamma}} Y_{\bar{\gamma}\alpha}) f - \frac{1}{6} Y_{\bar{\gamma}\alpha} \phi^{\bar{\gamma}} \Big\} \end{bmatrix} = 0$$

$K_{\alpha\beta}, P_{\bar{\alpha}\bar{\beta}}, {Y_{\bar{\alpha}\beta}}^{\bar{\gamma}\nu}, Y_{\bar{\alpha}\beta}$ are parts of the partial curvature of $(\nabla_{\bar{\alpha}}, \nabla_\alpha)$ in a given scale.

The prolongation bundle

The change of splitting / prolongation variables given a change of scale is given by:

$$\widehat{\begin{bmatrix} f \\ \phi_\alpha \\ g \end{bmatrix}} = \begin{bmatrix} f \\ \phi_\alpha + \Upsilon_\alpha f \\ g + \bar{\Upsilon}_{\bar{\gamma}} \phi^{\bar{\gamma}} + \frac{1}{2}(\bar{\nabla}_{\bar{\gamma}} \Upsilon^{\bar{\gamma}}) f + \bar{\Upsilon}_{\bar{\gamma}} \Upsilon^{\bar{\gamma}} f \end{bmatrix} \qquad (1)$$

These formulae can be checked (after a change of variables) to be consistent with the standard tractor bundle given in [Čap, Slovák, 2008].

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Flatness of the tractor partial connection implies a (local) isomorphism with the flat model, by general theory [ČS] or constructing the isomorphism directly [M. 2021].

G_2 contact geometry

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These are the parabolic geometries of type (G_2, P_2) where (thinking of $G_2 \subseteq SO(3,4)$ the usual way) P_2 is the the stabiliser of a plane null with respect to the indefinite metric and which inserts trivially into the 3-form ω.

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Theorem (Eastwood, Nurowski, 2019)

Given a choice of contact form there exists a unique partial connection $\nabla_{ABC} : S \to \odot^3 S \otimes S$ such that the induced partial connection on $\odot^3 S = \Lambda^1_H$ has partial torsion $\tau_{ABCDEFG}$ totally symmetric.

Another invariant PDE

For the standard tractor bundle, P_2 preserves a plane in the standard representation (so also its orthogonal completement of rank-five) $\implies$ the projecting part of the tractor bundle is a rank-two subbundle.
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So to obtain it by prolongation we need an invariant differential operator on (weighted) spinors.
Using the formulae for a change of scale:

$$\hat{\nabla}_{ABC}\phi_D = \nabla_{ABC}\phi_D + (1/3 + w)\Upsilon_{ABC}\phi_D - \Upsilon_{D(AB}\phi_{C)}$$

we obtain the invariant PDE (G_2 contact 'twistor equation')

$$\nabla_{(ABC}\phi_{D)} = 0$$

The prolongation connection

Which has prolongation (in the case of vanishing partial torsion):

$$\nabla_{ABC}\begin{bmatrix}\phi_D\\ \mu_{DE}\\ \rho_D\end{bmatrix} = \begin{bmatrix}\nabla_{ABC}\phi_D - \mu_{(AB^{\varepsilon}C)D}\\ \nabla_{ABC}\mu_{DE} + P_{ABCDEF}\phi^F\\ \nabla_{ABC}\rho_D + X_{ABC}{}^{D}\phi_D + Y_{ABC}{}^{DE}\mu_{DE}\end{bmatrix} = 0$$

where, $P_{ABC\ DEF} = P_{DEF\ ABC}$, and P, X, Y have long expressions in terms of the partial curvature and its derivatives.

The prolongation bundle

The change of splitting / prolongation variables given a change of scale is given by:

$$\widehat{\begin{bmatrix} \phi_D \\ \mu_{DE} \\ \rho_D \end{bmatrix}} = \begin{bmatrix} \phi_D \\ \mu_{DE} + \Upsilon_{DEF}\phi^F \\ \rho_D + \Upsilon_{DEF}\mu^{EF} + \frac{1}{2}\Upsilon_{DEF}\Upsilon^{EF}{}_G\phi^G + \frac{1}{4}\nabla_{EFG}\Upsilon^{EFG}\phi_D \end{bmatrix}.$$

Geometric structure on the prolongation bundle

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- An invariant signature-(3,4) metric for which the invariant rank-2 subbundle is null:

$$\begin{bmatrix} \phi_D \\ \mu_{DE} \\ \sigma_D \end{bmatrix} \otimes \begin{bmatrix} \psi_D \\ \nu_{DE} \\ \lambda_D \end{bmatrix} \mapsto \phi_D \lambda^D + \mu_{DE} \nu^{DE} - \sigma_D \psi^D$$

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- An invariant 3-form into which the invariant rank-2 subbundle inserts trivially:

$$\begin{bmatrix} \phi_D \\ \mu_{DE} \\ \rho_D \end{bmatrix} \otimes \begin{bmatrix} \psi_D \\ \nu_{DE} \\ \lambda_D \end{bmatrix} \otimes \begin{bmatrix} \gamma_D \\ \kappa_{DE} \\ \zeta_D \end{bmatrix} \mapsto$$

$$\rho_D \nu^{DE} \gamma_E - \lambda_D \mu^{DE} \gamma_E + \phi_D \kappa^{DE} \lambda_E - \psi_D \kappa^{DE} \rho_E - \phi_E \nu^{DE} \zeta_D + \psi_D \mu^{DE} \zeta_E + \mu_{DE} \nu^{DF} \kappa^E_F$$

G_2 contact geometries from Legendrean data

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Take the bundle $S = E \oplus F$ over M where E and F are bundles of appropriate weights. Then define $\odot^3 S^2 \cong E^3 \oplus E^2 F \oplus EF^2 \oplus F^3 \to P^* \oplus V^*$ by:

$$(x, y, z, w) \mapsto (x\Phi_{\bar{\alpha}} - \frac{y}{\sqrt{3}} J(\Psi_\alpha), -\frac{z}{\sqrt{3}} J(\Phi_{\bar{\alpha}}) + w\Psi_\alpha)$$

which defines a G_2 contact structure (basically the 'flying saucers' construction in [Eastwood, Nurowski, 2020]).

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See [Eastwood, M., 2022] for computation of the G_2 contact torsion (the Cartan curvature) in terms of Legendrean contact data.

Legendrean geometries from G_2 contact data

Conversely every G_2 contact structure and a choice of $\phi_A, \psi_B \in S$ gives rise to a Legendrean contact structure via:

$$P^* = \text{span}\{\phi_A\phi_B\phi_C, \phi_{(A}\phi_B\psi_{C)}\}$$
$$V^* = \text{span}\{\psi_A\psi_B\psi_C, \psi_{(A}\psi_B\phi_{C)}\}$$

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and this Legendrean contract geometry gives rise to the original G_2 contact geometry via the previous construction. So every G_2 contact structure (locally) arises by the previous slide's construction. Details in [M. 2021].

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