Hyper-Kähler metrics from isomonodromy

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joint work with Maciej Dunajski,
`arXiv:2402.14352`

Glasgow
16 April 2024

Motivation

Slides available at `damtp.cam.ac.uk/tjahm2`

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Lax distributions

In what follows let X be a complex manifold of dimension $4n$ and TX the holomorphic tangent bundle. For our purposes:

Definition (Lax distribution)

A (hyper-Hermitian) Lax distribution is a subbundle of TX

$$L(\lambda) = \text{span}\left\{v_i + \lambda h_i\right\}_{i=1}^{2n}. \tag{1}$$

depending on a spectral parameter $\lambda \in \mathbb{C}$, where v_i, h_i are vector fields on X such that $TX = \text{span}\{v_i, h_i\}_{i=1}^{2n}$

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Letting $M = X/span\{v_i\}$ we can think of $L(\lambda)$ as a family of (not necessarily linear) Ehresmann connections on the bundle $X \to M$ depending 'linearly' on λ.

Quaternionic structure

$L(\lambda) = \text{span}\left\{v_i + \lambda h_i\right\}_{i=1}^{2n}$ determines a quaternionic structure:

$$I(v_i) = iv_i, \quad J(v_i) = -h_i, \quad K(v_i) = ih_i \tag{2}$$

$L(\lambda)$ Frobenius integrable for each λ is equivalent to integrability of the complex structures I, J, K and the existence of the twistor space $\mathcal{Z}$.

Associated family of hyper-Hermitian metrics

$L(\lambda) = \text{span}\left\{v_i + \lambda h_i\right\}_{i=1}^{2n}$ also determines a family of holomorphic metrics satisfying the *hyper-Hermitian condition* $I^*g = J^*g = K^*g = g$:

$$g = \sum_{i,j=1}^{2n} e_{ij} h^i \odot v^i \tag{3}$$

each corresponding to a non-degenerate skew matrix e_{ij} of holomorphic functions.

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$n > 1$ these are almost Grassmannian (or paraconformal) geometries. See (Bailey-Eastwood (1991))

Complex hyper-Kähler metrics

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A complex hyper-Kähler structure is a holomorphic metric g and triple of holomorphic endomorphisms I, J, K of TX satisfying the quaternion relations such that g is Hermitian for each and $\nabla I = \nabla J = \nabla K = 0$.

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General case (sufficient condition): Integrability of $L(\lambda)$ $\forall\lambda$ and the existence of ω, a symplectic form on $M := X/\mathrm{span}\{v_i\}$ with

$$\mathcal{L}_{h_i}(\pi^*\omega) = 0, \quad i = 1, ..., 2n.$$

Plebaśnki's heavenly equation

There is a hyperkähler metric in the class if and only if there exists coordinates (x^i, y^i) such that

$$v_i = \frac{\partial}{\partial y^i} \tag{4}$$

$$h_i = \frac{\partial}{\partial x_i} + \sum_{j,k=1}^{2n} \eta^{jk} \frac{\partial^2 \Theta}{\partial y^i \partial y^j} \frac{\partial}{\partial y^k} \tag{5}$$

where $\Theta(x_i, y_i)$ satisfies (a higher dimensional version of) Plebański's second heavenly equation

$$\frac{\partial^2 \Theta}{\partial y^i \partial x^j} - \frac{\partial^2 \Theta}{\partial y^j \partial x^i} - \sum_{k,l=1}^{2n} \eta^{kl} \frac{\partial^2 \Theta}{\partial y^j \partial y^k} \frac{\partial^2 \Theta}{\partial y^j \partial y^l} = 0. \tag{6}$$

Rough idea of isomonodromy

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The general solutions of linear ODE with meromorphic coefficients may have branching behaviour near poles $x_0, \ldots, x_M$ of the coefficients. The fundamental group of the punctured space $\mathbb{CP}^1 \setminus \{x_0, \ldots, x_M\}$ then has a linear representation on the space of solutions called the *monodromy*.

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Given a family of ODE depending on some parameters, a *isomonodromic flow* is a family of deformations of the parameters continuous with the identity which preserves the monodromy.

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The situation for irregular singularities is more complicated... (want to also preserve Stokes' data)

Equation for isomonodromic flows of 2nd order linear ODE

For the purposes of our problem the output of the general theory is the following theorem:

Theorem (Schlesinger / Jimbo-Miwa-Ueno(1981))

A one-parameter family $Q(t,x)$ of deformations of the potential for the ODE

$$\frac{d^2y}{dx^2} = Q(x)y$$

varying holomorphically with the parameter t has constant (generalised) monodromy if and only if

$$\frac{\partial Q(x,t)}{\partial t} = 2Q(x)\frac{\partial A(t,x)}{\partial x} + \frac{\partial Q(t,x)}{\partial x}A(t,x) - \frac{1}{2}\frac{\partial^3 A(t,x)}{\partial x^3}$$

for some $A(t,x)$.

Geometry of isomonodromic flows

If potentials are parametrised by a space X with coordinates w^i, then a one-parameter family of deformations corresponds to a vector field U satisfying the Schlesinger equation:

$$U(Q(x, w^i)) = 2Q\frac{\partial A}{\partial x} + \frac{\partial Q}{\partial x}A - \frac{1}{2}\frac{\partial^3 A}{\partial x^3} \tag{7}$$

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for some $A(x,w^i)$. Suppose we have another:

$$V(Q(x,w^i)) = 2Q\frac{\partial B}{\partial x} + \frac{\partial Q}{\partial x}B - \frac{1}{2}\frac{\partial^3 B}{\partial x^3}. \tag{8}$$

Proposition (Lie bracket of isomonodromic flows)

$$[U,V](Q(x,w^i)) = 2Q\frac{\partial C}{\partial x} + \frac{\partial Q}{\partial x}C - \frac{1}{2}\frac{\partial^3 C}{\partial x^3}.$$

where

$$C = U(B) - V(A) - \left(A\frac{\partial B}{\partial x} - B\frac{\partial A}{\partial x}\right)$$

Deformed cubic oscillator I

The following example is due to Bridgeland-Masoero (2022):

$$Q(x) = \frac{Q_0(x)}{\lambda^2} + \frac{Q_1(x)}{\lambda} + Q_2(x)$$

where

$$Q_0(x) = x^3 + ax + b \tag{9}$$

$$Q_1(x) = \frac{p}{x-q} + r \tag{10}$$

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where $p^2 = q^3 + aq + b$.
Loosely, we pick $Q_2(x)$ to be the simplest function so that the ODE, written as a first order system, has no singularity at q after a gauge transformation. Specifically

$$Q_2(x) = \frac{3}{4(x-q)^2} + \frac{r}{2p(x-q)} + \frac{r^2}{4p^2}. \tag{11}$$

Deformed cubic oscillator II

$$Q(x) = \frac{Q_0(x)}{\lambda^2} + \frac{Q_1(x)}{\lambda} + Q_2(x) \tag{12}$$

$$Q_0(x) = x^3 + ax + b, \quad Q_1(x) = \frac{p}{x-q} + r \tag{13}$$

$$Q_2(x) = \frac{3}{4(x-q)^2} + \frac{r}{2p(x-q)} + \frac{r^2}{4p^2}. \tag{14}$$

The ODE is therefore specified by a point on a manifold X with local coordinates (a, b, q, r). The isomonodromic flows are of the right number and have the right form to define a Lax pair:

$$U = -\frac{\partial}{\partial r} + \lambda\left(\frac{\partial}{\partial b} + \frac{r}{2p^2}\frac{\partial}{\partial r}\right) \tag{15}$$

$$V = -2p\frac{\partial}{\partial q} + \lambda\left(\frac{\partial}{\partial a} - \frac{r}{p}\frac{\partial}{\partial q} - \frac{r(3q^2r + ar - qp)}{2p^3}\frac{\partial}{\partial r}\right)$$

A_2 complex hyper-Kähler metric

There is a hyper-Kähler metric on X in the conformal class:

$$g^{\omega} = \left(\frac{r(3q^2r + ar - 2qp)}{2p^3}da - \frac{r}{2p^2}db - \frac{q}{2p}dq + dr\right) \odot da - \left(\frac{r}{2p^2}da + \frac{1}{2p}dq\right) \odot db.$$

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$$- \left(\frac{r}{2p^2} da + \frac{1}{2p} dq\right) \odot db.$$

with homothetic Killing vector

$$W = \frac{4a}{5}\frac{\partial}{\partial a} + \frac{6b}{5}\frac{\partial}{\partial b} + \frac{2q}{5}\frac{\partial}{\partial q} + \frac{r}{5}\frac{\partial}{\partial r}.$$

Identifying $TM \to X$

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So X fibres over $M = (a, b)$ a space of stability conditions.

Gauss-Manin isomorphism

To get a (local identification) $TM \to X$ we note that a point in $(a, b) \in M$ defines an elliptic curve

$$\Sigma_{(a,b)} = \{(x, y) \in \mathbb{C}^2 \mid y^2 = x^3 + ax + b\} \cup \{\infty\} \tag{16}$$

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Consider the holomorphic vector bundle E of rank two with fibre $H^1(\Sigma_{(a,b)}, \mathbb{C})$ at (a, b).

It has a canonical connection $\nabla^{GM} : T^*M \otimes \Gamma(E) \to \Gamma(E)$ the *Gauss-Manin* connection: the flat connection with parallel sections those that take values in the fundamental co-cycles.

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We also have a canonical section Z with value at (a, b) given by

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Then we get an isomorphism $TM \cong E$ given by

$$v \mapsto \nabla^{GM}_v Z.$$

Abelian holonomy map

We claim $(a, b, q, r) \in X$ defines a class in $H^1(\Sigma_{(a,b)}, \mathbb{C}^\times)$:

$$Q_1(x) = \frac{p}{x-q} + r \leftrightarrow \varpi = 2\pi i\left(\frac{y+p}{x-q} + r\right)\frac{dx}{2y} \tag{17}$$

a meromorphic one-form on $\Sigma_{(a,b)}$ on the elliptic curve $\Sigma_{(a,b)}$ with residues integer multiples of $2\pi i \implies$ integration of ϖ over homology classes is well-defined after exponentiation.

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a meromorphic one-form on $\Sigma_{(a,b)}$ on the elliptic curve $\Sigma_{(a,b)}$ with residues integer multiples of $2\pi i \implies$ integration of ϖ over homology classes is well-defined after exponentiation. We have maps over M:

$$X \longleftarrow E^\times \twoheadleftarrow_{\exp} E \xleftarrow[v \mapsto \nabla_v^{GM} Z]{} TM$$

Where E, $E^\times$ have fibres $H^1(\Sigma_{(a,b)}, \mathbb{C})$, $H^1(\Sigma_{(a,b)}, \mathbb{C}^\times)$ respectively. All this is made rigorous in Bridgeland-Masoero (2022).

Generalising the deformed cubic oscillator metric

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In `arXiv:2402.14352` we generalise this construction to produce explicit expressions for a complex hyper-Kähler metric in $4n$ dimensions from the isomonodromy of ODE with potentials having leading term a polynomial of degree $2n + 1$.

Deformed polynomial oscillator

The ODE setup is the obvious generalisation:

$$Q(x) = \frac{Q_0(x)}{\lambda^2} + \frac{Q_1(x)}{\lambda} + Q_2(x)$$

where

$$Q_0(x) = x^{2n+1} + a_n x^{2n-1} + \ldots + a_1 x^n + b_n x^{n-1} + \ldots + b_1 \quad (18)$$

$$Q_1(x) = \sum_{i=1}^{n} \frac{p_i}{x - q_i} + R(x) \quad (19)$$

where $p_i^2 = q_i^3 + aq_i + b$ and $R(x)$ is the general polynomial of degree at most $n-1$ (parametrised $(v_1, \ldots v_n)$).

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Again $Q_2(x)$ is picked so that there is no singularity at q_i after a gauge transformation.

Isomonodromy result

Proposition (Dunajski-M,(2023))

The equation with potential specified by (18) and (19) and $Q_2(x)$ chosen appropriately has $2n$ linearly independent isomonodromic flows of the form

$$L_i = v_i + \lambda h_i$$

where $TX = span\{v_i, h_i\}_{i=1}^{2n}$ and the v_i are vertical for the projection $X \to M$.

The proof proceeds by breaking down the Schlesinger equation into manageable subsystems by Laurent expanding at the various poles $\infty, q_1, ..., q_n$.

A hyper-Kähler metric in the class

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So we have the Lax distribution and hence a family of metrics. How can we distinguish a hyper-Kähler metric? Recall the family of metrics

$$g = e_{ij} h^i \odot v^j \tag{20}$$

Choose $e_{ij} = \omega_{ij}$, the pull-back of the natural symplectic form ω on M (affine symplectic fibration).

Theorem (Dunajski-M(2023))

The metric

$$g^\omega = \omega_{ij} h^i \odot v^j$$

is complex hyper-Kähler.

We call it the A_{2n} complex hyper-Kähler metric.

Symplectic structure on M

To see this symplectic structure note $p \in M$ defines a hyper-elliptic curve

$$\Sigma = \{y^2 = x^{2n+1} + a_n x^{2n-1} + \ldots + a_1 x^n + b_n x^{n-1} + \ldots + b_1\} \quad (21)$$

each with cohomology intersection form $H^1(\Sigma, \mathbb{C}) \times H^1(\Sigma, \mathbb{C}) \to \mathbb{C}$.

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each with cohomology intersection form $H^1(\Sigma, \mathbb{C}) \times H^1(\Sigma, \mathbb{C}) \to \mathbb{C}$.

Recall the Gauss-Manin connection defines an isomorphism $T_{(a,b)}M \to H^1(\Sigma, \mathbb{C})$ by

$$v \mapsto \nabla_v^{GM} Z.$$

ω is the pull-back of the intersection form by this isomorphism.

Intriguing geometry: hyper-Lagrangians

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The A_{2n} metrics g^{ω} admit foliations by submanifolds which are Lagrangian for the symplectic forms $\Omega_I, \Omega_J, \Omega_K$. We call such a foliation a hyper-Lagrangian foliation.

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Proposition (Projectable hyper-Lagrangian foliation)

Given a hyper-Lagrangian foliation which pushes down to a Lagrangian foliation of M, Θ can be taken to be at most quadratic in half of the coordinates y^i. When $n = 1$ such a foliation implies the heavenly equation linearises in an appropriate sense.

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- A_{2n} Frobenius structure on the base space M... Is this encoded by the Joyce structure? This is somehow true for the cubic oscillator. Our metrics could answer this question after taking an appropriate limit.

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- Example rather contrived without link to Joyce structures... Generally, which isomonodromy problems are set up correctly to have an analogous complex hyper-Kähler metric on the space X parametrising potentials?
- Calculations can presumably be adapted to quadratic differentials with any fixed number of poles with fixed orders... Does some confluence phenomenon manifest on the level of the metric?
- A_{2n} Frobenius structure on the base space M... Is this encoded by the Joyce structure? This is somehow true for the cubic oscillator. Our metrics could answer this question after taking an appropriate limit.

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