AN INTRODUCTION TO THE PENROSE TRANSFORM: *THE COHOMOLOGICAL VIEWPOINT*

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Abstract. We give an introduction to the Penrose transform from the cohomological viewpoint. We introduce the necessary notions of the Čech cohomology of sheaves and culminate with the proof that solutions to the holomorphic positive-helicity massless field equations correspond to particular cohomology classes on projective twistor space.

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1. Introduction

We begin with some motivating remarks from physics. In the language of *spacetime spinors* (see [20], [21]), the zero-rest-mass equations of helicity ± 1:

$$\nabla^{A}{}_{A'}\phi_{AB} = 0 \tag{1.1}$$

$$\nabla_{A}{}^{A'}\tilde{\phi}_{A'B'} = 0 \tag{1.2}$$

are Maxwell's equations without sources.

There are integral formulae for solutions of these equations. For holomorphic solutions on complexified Minkowski space (1.2) Penrose gave a solution [18] in 1968. A solution is given by

$$\tilde{\phi}_{A'B'}(x) = \oint \pi_{A'}\pi_{B'} f(iX^{AA'}\pi_{A'}, \pi_{A'})\pi^{D'} d\pi_{D'} \tag{1.3}$$

Here, f is a holomorphic function on an appropriate open subset of $\mathbb{C}^4 \setminus \{0\}$ which is homogeneous of degree -4. That is $f(\lambda z) = \lambda^{-4} f(z)$. Such functions are the same as sections of a particular canonical line bundle over the corresponding open subset of $\mathbb{CP}^3$. We will make sense of the rest of the notation in the formula in due course, but the main point is that this is a remarkable formula in that it "integrates away" the zero-rest-mass equations by producing a solution (physical data) from unconstrained geometric data defined on an open subset of $\mathbb{CP}^3$. This is the contour integral formulae version of the *Penrose transform*.

It is clear that the Penrose transform is non-local. Similar integral formulae for solutions of partial differential equations were not new, even when Penrose wrote down his formula in 1968. For example Bateman [3] in 1904 and John [17] in 1938 had written down integral formulae for solutions of Laplace's equation and the ultrahyperbolic wave equation respectively. The fields of *integral geometry* and *tomography* (related to medical imaging) are often concerned with such formulae and the existence and construction of inverse transforms.

Primarily following [9] and [22] the aim of this essay is to obtain a one-to-one correspondence between a class of "functions" on $\mathbb{CP}^3$ and solutions to the zero-rest-mass equations. Our first task is to introduce *twistor space* and the notation appearing in the above formula (1.3) so that we can interpret it and explain why it produces a solution to (1.2). We will then investigate the inherent degeneracy in (1.3). Many f produce the same $\phi_{A'B'}$. It will turn out the correct language to remove this degeneracy is the language of *sheaf cohomology*. Finally we will explain how solutions on Minkowski space correspond to cohomology classes valued in certain canonical line bundles over $\mathbb{CP}^3$.

2. Twistor space

Define *twistor space* to be the four-dimensional complex vector space with elements

$$\begin{bmatrix} \omega^A \\ \pi_{A'} \end{bmatrix} \in \mathbb{T},$$

where

$$\omega^A = \begin{bmatrix} \omega^0 \\ \omega^1 \end{bmatrix}, \quad \pi_{A'} = \begin{bmatrix} \pi_{0'} \\ \pi_{1'} \end{bmatrix}$$

are called *spinors*.

There is an identification of points in $\mathbb{C}^4$ with 2×2 complex matrices given by

$$X = (X^0, X^1, X^2, X^3) \mapsto \frac{1}{\sqrt{2}} \begin{bmatrix} X^0 + X^3 & X^1 + iX^2 \\ X^1 - iX^2 & X^0 - X^3 \end{bmatrix} := X^{AA'}. \tag{2.1}$$

Rather than explicitly writing the map (2.1), we will abuse notation and simply take it to identify $X = X^{AA'}$.

A note on notation: Throughout the essay we will employ Penrose's *abstract index notation* [20, 21] whereby upper case Latin indices are simply markers denoting the type of object and repeated indices denotes the natural pairing between a bundle and its dual. Then $X^{AA'}\pi_{A'}$ denotes the usual matrix multiplication. Indices enclosed in parentheses are symmetrised over and indices enclosed in square brackets are antisymmetrised over. The convention we adopt, in line with [9], is that when we need to refer to explicit numerical indices we will use upper case sans-serif Latin indices. For example $X^{\mathsf{AA'}}$ refers to the $\mathsf{AA'}$ component of $X^{AA'}$ where A ranges over $0, 1$ and $\mathsf{A'}$ ranges over $0, 1$.

Let V be a complex vector space. Define the *flag manifolds*:

$$F_{k_1,...,k_m}(V) := \{(A_1, ..., A_m) \mid A_1 \subset ... \subset A_m,\ A_i \text{ a subspace of dimension } k_i\}.$$

These are complex manifolds in a canonical way. $F_k(V)$ is usually referred to as the *Grassmannian of k-planes* in V.

We will write $\mathbb{P} := F_1(\mathbb{T})$ for *projective twistor space*, the space of the lines in $\mathbb{T}$, a space that is isomorphic to $\mathbb{CP}^3$. Next we will write $\mathbb{M} := F_2(\mathbb{T})$, for *compactified complexified Minkowski space* which is the space of planes in $\mathbb{T}$. Lastly the *correspondence space* is $\mathbb{F} := F_{1,2}(\mathbb{T})$, the space of lines inside planes in $\mathbb{T}$. By realising these as quotients of the general linear group or otherwise, one can see $\mathbb{P}, \mathbb{M}, \mathbb{F}$ have complex dimensions three, four and five respectively.

Consider the map $\mathbb{C}^4 \to \mathbb{M}$ given by

$$\begin{aligned} &\phi : \mathbb{C}^4 \to \mathbb{M} \\ &\phi(X^{AA'}) \mapsto \mathrm{col}\begin{bmatrix} iX^{AA'} \\ \delta^{A'}_{A'} \end{bmatrix}, \end{aligned} \tag{2.2}$$

where $\delta^{A'}_{A'}$ is the 2×2 identity matrix. This is a biholomorphism onto its image since the 4×2 matrix with $X^{AA'}$ as the top 2×2 submatrix has linearly independent columns. Permuting the rows we can obtain a set of parametrisations such that the images cover $\mathbb{M}$. The inverses give a complex atlas for $\mathbb{M}$.

We define *affine complexified Minkowski space* $\mathbb{M}^I := \mathrm{im}\,\phi \cong \mathbb{C}^4$. $\mathbb{M}^I$ is dense in $\mathbb{M}$. In fact, one can make sense of $\mathbb{M}$ as the *conformal compactification* of $\mathbb{M}^I$. That is, $\mathbb{M}$ is $\mathbb{M}^I$ with points added “at infinity” so that the inversions, a set of sensible conformal transformations, are defined everywhere [22].

We have the double fibration:

$$\begin{array}{ccccc} & & \mathbb{F} & & \\ & \overset{\mu}{\swarrow} & & \overset{\nu}{\searrow} & \\ \mathbb{P} & & & & \mathbb{M}. \end{array} \tag{2.3}$$

The fibres of ν are isomorphic to $\mathbb{CP}^1$ since a fibre is precisely the set of lines lying in a two-dimensional complex subspace and we claim that the fibres of μ are isomorphic to $\mathbb{CP}^2$. To see this note that an arbitrary two-dimensional subspace containing a line $\mathrm{span}\{W\}$ takes the form

$$\mathrm{span}\{W, x_1X + x_2Y + x_3Z\}$$

for $x, y, z \in \mathbb{C}$ where X, Y, Z, W are chosen to form a basis for $\mathbb{T}$. Map this subspace to $[x : y : z] \in \mathbb{CP}^2$ and this map is a biholomorphism.

There is a useful way of viewing $\mathbb{F}^I := \nu^{-1}(\mathbb{M}^I)$ as a product. In particular, in terms of the preferred coordinates for $\mathbb{M}^I$ define a biholomorphism $\mathbb{M}^I \times \mathbb{CP}^1 \to \mathbb{F}^I$ by

$$(X^{AA'}, [\pi_{A'}]) \mapsto \left(\mathrm{col}\begin{bmatrix} iX^{AA'}\pi_{A'} \\ \pi_{A'} \end{bmatrix}, \mathrm{col}\begin{bmatrix} iX^{AA'} \\ \delta^{A'}_{A'} \end{bmatrix} \right). \tag{2.4}$$

In terms of this trivialisation the projection $\mu : \mathbb{F}^I \to \mathbb{P}^I$ where $\mathbb{P}^I := \mu(\mathbb{F}^I)$ is just:

$$(X^{AA'}, [\pi_{A'}]) \mapsto \mathrm{col}\begin{bmatrix} iX^{AA'}\pi_{A'} \\ \pi_{A'} \end{bmatrix}. \tag{2.5}$$

There is a canonical rank-2 holomorphic vector bundle on $\mathbb{M}$ which we will denote S'^* that has as its fibre at $x \in \mathbb{M}$ the vector subspace that x represents. For a general Grassmannian this construction is called the *universal bundle* but motivated by physics, we will call S' the *primed spin bundle*. Now each fibre of S'^* is by definition a subspace of $\mathbb{T}$ so we get an injective map of vector bundles $S'^* \hookrightarrow \mathbb{T}$ (we have abused notation slightly: the target is the trivial bundle $\mathbb{T} \times \mathbb{M}$ over $\mathbb{M}$). We can then form the exact sequence of vector bundles

$$0 \to S'^* \to \mathbb{T} \to S \to 0$$

where we have defined $S := \mathbb{T}/S'^*$. Note also that $\mathbb{P}S'^* = \mathbb{F}$ as a bundle over $\mathbb{M}$ by definition, where $\mathbb{P}S'^*$ denotes the projectivisation of the vector bundle S'^* (the manifold obtained by quotienting out by scalar multiplication in the fibres of S'^*).

Proposition 2.6 (Tangent bundle to $\mathbb{M}$)**.** *There is a canonical identification:*

$$S' \otimes S \cong T\mathbb{M}.$$

where $T\mathbb{M}$ is the (complexified) tangent bundle of $\mathbb{M}$.

Proof. Using our preferred coordinates for $\mathbb{M}^I$ we may represent the tangent vectors $T_x\mathbb{M}$ for $x := X^{AA'} \in \mathbb{M}^I$ by 2×2 complex matrices $V^{AA'}$. The subspace of twistor space

$$C := \mathrm{col}\begin{bmatrix} \delta^{AA'} \\ 0 \end{bmatrix} \subseteq \mathbb{T}$$

is complementary to all $x \in \mathbb{M}^I$. Each tangent vector $V^{AA'} \in T_x\mathbb{M}$ defines a linear map $x \to C$ by

$$x \ni \begin{bmatrix} iX^{AA'}\pi_{A'} \\ \pi_{A'} \end{bmatrix} \mapsto \begin{bmatrix} V^{AA'}\pi_{A'} \\ 0 \end{bmatrix} \in C$$

and it is clear we have an isomorphism $T_x\mathbb{M}^I \cong \mathrm{Hom}(x, C)$ since $V^{AA'}$ is an arbitrary 2×2 matrix. Using the fact C is complementary, this gives an isomorphism $T\mathbb{M}^I \cong \mathrm{Hom}(S'^*|_{\mathbb{M}^I}, S|_{\mathbb{M}^I})$. It remains to check that if one defines the analogous map in the other charts by permuting the rows, one gets a well-defined isomorphism $T_V\mathbb{M} \cong \mathrm{Hom}(S'^*, S)$. This is done by computing the transition functions and the algebra is messy and unenlightening so omitted. □

A more abstract proof of the general fact that $TF_k(V) \cong \mathrm{Hom}(U, V/U)$ for U the universal bundle and V the trivial bundle is found in [13].

Note that our preferred chart for $\mathbb{M}^I$ induces preferred trivialisations for S'^* and S over $\mathbb{M}^I$: specifically, the trivialisation for S'^* maps

$$\begin{bmatrix} iX^{AA'}\pi_{A'} \\ \pi_{A'} \end{bmatrix} \in x = S'^*_x$$

to the spinor $\pi_{A'}$ while the trivialisation for S maps:

$$\begin{bmatrix} \omega^A \\ 0 \end{bmatrix} \in C \cong \mathbb{T}/x = S_x$$

to the spinor ω^A.

3. Conformal invariance

We denote the kth-exterior power of a vector bundle E by $\bigwedge^k E$ and the kth-symmetric power by $\odot^k E$. For brevity it is convenient to further introduce specific notation for the line bundles

$$\mathcal{O}[-1] := \textstyle\bigwedge^2 S^*$$
$$\mathcal{O}[-1]' := \textstyle\bigwedge^2 S'^*.$$

Define $\mathcal{O}[n] = \otimes^n(\mathcal{O}[-1])^*$ and $\mathcal{O}[-n] = \otimes^n\mathcal{O}[-1]$ for n a positive integer and similarly for primed spinors.

Given a choice of non-vanishing sections $\epsilon_{AB}, \epsilon_{A'B'}$ over $\mathbb{M}^I$ of $\mathcal{O}[-1]$ and $\mathcal{O}[-1]'$ respectively there is an induced complex metric on $T\mathbb{M}^I \cong S'|_{\mathbb{M}^I} \otimes S|_{\mathbb{M}^I}$ with components

$$g_{AA'BB'} = \epsilon_{AB}\epsilon_{A'B'}.$$

Over $\mathbb{M}^I$, the trivialisations for the spin-bundles induce *flat connections* $\nabla_{AA'}$ simply given by exterior differentiation of components. Up to multiplication by a constant there are *unique* such non-vanishing sections for which:

$$\begin{aligned}\nabla_{AA'}\epsilon_{BC} &= 0\\ \nabla_{AA'}\epsilon_{B'C'} &= 0.\end{aligned}$$

(These are just the sections with constant components with respect to the preferred trivialisations.) So given our coordinates for $\mathbb{M}^I$, we have specified a complex metric up to multiplication by a constant. We can do the same on other coordinate patches. On the overlap of coordinate patches, let the flat connection associated to the new coordinate patch be denoted $\hat{\nabla}_{AA'}$. We will have

$$\begin{aligned}\hat{\nabla}_{AA'}\hat{\epsilon}_{BC} &= 0\\ \hat{\nabla}_{AA'}\hat{\epsilon}_{B'C'} &= 0\end{aligned}$$

for $\hat{\epsilon}_{BC} = \Omega\epsilon_{BC}$ and $\hat{\epsilon}_{B'C'} = \tilde{\Omega}\epsilon_{B'C'}$ for some functions $\Omega, \tilde{\Omega}$ (using the fact $\epsilon_{BC}, \epsilon_{B'C'}$ are sections of line bundles). Then the new metric $\hat{g}_{AA'BB'} = \Omega\tilde{\Omega}g_{AA'BB'}$ is *conformally related* to the original one.

In order for our calculations to hold in generality, and not just in our trivialisation for $\mathbb{M}^I$ we would therefore like to calculate without reference to specific non-vanishing sections of $\mathcal{O}[-1]$, and $\mathcal{O}[-1]'$. What is true is that there is a canonical non-vanishing section of $\mathcal{O}[-1]\otimes\mathcal{O}[1]$ (the preimage of the constant function 2 under the natural pairing between $\mathcal{O}[-1]$ and its dual). Denoting this by ϵ_{AB} or ϵ^{AB} we will use this to raise and lower unprimed indices. For a vector bundle V over $\mathbb{M}^I$ define $V[n] := V \otimes \mathcal{O}[n]$ and so on. Sections of $V[n]$ are called *sections of V of weight n*. So given a spinor $\phi_A \in S^*$ we get a spinor $\phi^A := \phi_B\epsilon^{AB} \in S^*[-1]$. We treat the primed indices analogously.

Going further, the zero-rest-mass free field equations can be seen to be invariant under general conformal transformations of the metric $g_{AA'BB'}$ by writing it in terms of spinors with appropriate weight: Fixing a connection $\nabla_{AA'}$ on S'^* the general connection is given by

$$\hat{\nabla}_{AA'}\phi_{B'} = \nabla_{AA'}\phi_{B'} + \Gamma_{AA'B'}{}^{C'}\phi_{C'}$$

where $\Gamma_{AA'B'}{}^{C'}$ is a section of $T^*\mathbb{M} \otimes \mathrm{End}(S')$. As usual we can calculate the induced connection on tensor powers via the Leibniz rule, and the induced connection on $\bigwedge^2 S'^*$ applied to a fixed choice of *scale*, a non-vanishing section $\epsilon_{A'B'}$ of $\bigwedge^2 S'^*$ defined over some open $U \subseteq \mathbb{M}$ is:

$$\begin{aligned}\hat{\nabla}_{AA'}\epsilon_{B'C'} &= \nabla_{AA'}\epsilon_{B'C'} + \Gamma_{AA'B'}{}^{D'}\epsilon_{D'C'} + \Gamma_{AA'C'}{}^{D'}\epsilon_{B'D'}\\ &= \nabla_{AA'}\epsilon_{B'C'} + 2\Gamma_{AA'[B'C']}\end{aligned}$$

where we are (for the moment) using the fixed scale $\epsilon_{A'B'}$ to raise and lower indices. We may set $2\Gamma_{AA'[B'C']} = -\nabla_{AA'}\epsilon_{B'C'}$ to insist that our new connection (defined over U) annihilates the scale. The remaining freedom lies in the bundle $T^*\mathbb{M} \otimes \odot^2 S'^*$. Similarly we may find a connection on S^* which annihilates a choice of scale ϵ_{AB}.

Proposition 3.1 (Change of conformal scale)**.** *Given choices of scale $\epsilon_{AB}, \epsilon_{A'B'}$ defined over $U \subseteq \mathbb{M}$ there are unique connections on S, S'^* defined over U such that $\nabla_{AA'}\epsilon_{BC} = 0$ and $\nabla_{AA'}\epsilon_{B'C'} = 0$ and such that the induced connection on $T^*\mathbb{M}$ is torsion-free (so corresponds to the Levi-Civita connection). Furthermore given $\hat{\epsilon}_{BC} = \Omega\epsilon_{BC}$, $\hat{\epsilon}_{B'C'} =$*

$\tilde{\Omega}\epsilon_{B'C'}$ *the connections associated with the new scales are:*

$$\hat{\nabla}_{AA'}\phi_B = \nabla_{AA'}\phi_B - \frac{1}{2}\Upsilon_{BA'}\phi_A - \frac{1}{2}\tilde{\Upsilon}_{BA'}\phi_A - \frac{1}{4}\Upsilon_{AA'}\phi_B + \frac{1}{4}\tilde{\Upsilon}_{AA'}\phi_B \tag{3.2}$$

$$\hat{\nabla}_{AA'}\phi_{B'} = \nabla_{AA'}\phi_{B'} - \frac{1}{2}\Upsilon_{AB'}\phi_{A'} - \frac{1}{2}\tilde{\Upsilon}_{AB'}\phi_{A'} + \frac{1}{4}\Upsilon_{AA'}\phi_{B'} - \frac{1}{4}\tilde{\Upsilon}_{AA'}\phi_{B'} \tag{3.3}$$

where $\Upsilon_{AA'} := \nabla_{AA'} \ln \Omega$, $\tilde{\Upsilon}_{AA'} := \nabla_{AA'} \ln \tilde{\Omega}$.

Proof. (See appendix A)

We can write any section of $\mathcal{O}[-1]'$ over U as $f\epsilon_{B'C'}$ for some smooth function f. The change in connection on $\mathcal{O}[-1]'$ is given by:

$$\hat{\nabla}_{AA'}(f\epsilon_{B'C'}) = (\nabla_{AA'}f)\epsilon_{B'C'} + f\hat{\nabla}_{AA'}\left(\frac{1}{\tilde{\Omega}}\hat{\epsilon}_{B'C'}\right) = \nabla_{AA'}(f\epsilon_{B'C'}) - \tilde{\Upsilon}_{AA'}f\epsilon_{B'C'}.$$

The Leibniz rule then implies the connection on $S'^{*}[-k]'$ is given by:

$$\hat{\nabla}_{AA'}\phi_{B'} = \nabla_{AA'}\phi_{B'} - \frac{1}{2}\Upsilon_{AB'}\phi_{A'} - \frac{1}{2}\tilde{\Upsilon}_{AB'}\phi_{A'} + \frac{1}{4}\Upsilon_{AA'}\phi_{B'} - \left(\frac{1}{4} + k\right)\tilde{\Upsilon}_{AA'}\phi_{B'}.$$

Now a general conformal transformation of the metric is given by $\hat{g}_{AA'BB'} = \lambda^2 g_{AA'BB'}$, and this can be achieved by rescaling ϵ_{BC} and $\epsilon_{B'C'}$ by setting $\Omega = \tilde{\Omega} = \lambda$ so that $\Upsilon_{AA'} = \tilde{\Upsilon}_{AA'}$. Consider the following differential equation:

$$\nabla_A{}^{A'}\phi_{A'\ldots B'C'} = 0, \tag{3.4}$$

where $\phi_{A'\ldots B'C'}$ is a section of $\odot^n S'^{*}[-1]'$. Setting $\hat{\epsilon}_{BC} = \lambda\epsilon_{BC}$, $\hat{\epsilon}_{B'C'} = \lambda\epsilon_{B'C'}$. From the Leibniz rule we get:

$$\hat{\nabla}_{AA'}\underbrace{\phi_{B'\ldots C'D'}}_{n \text{ indices}} = \nabla_A{}^{A'}\phi_{A'\ldots B'C'} - n\Upsilon_{A(B'}\phi_{C'\ldots D')A'} + \Upsilon_{AA'}\phi_{B'\ldots C'D'}$$

$$\implies \hat{\nabla}_A{}^{A'}\phi_{A'\ldots B'C'} = \nabla_A{}^{A'}\phi_{A'\ldots B'C'} + \Upsilon_A{}^{A'}\phi_{A'\ldots B'C'} - \Upsilon_A{}^{A'}\phi_{A'\ldots B'C'} = 0,$$

so that (3.4), also known as the *zero-rest-mass field equations of helicity* $n/2$ are seen to be invariant under conformal rescalings of the metric.

4. Complex projective space

We need to define the so-called tautological bundle and its tensor powers on $\mathbb{CP}^n$. For this we take the standard open cover $\{U_i\}_{i=0}^n$ for $\mathbb{CP}^n$, where

$$U_i = \{[z^0 : \ldots : z^n] \in \mathbb{CP}^n \mid z^i \neq 0\}$$

with coordinates

$$\varphi_i([z^0 : \ldots : z^n]) = (z^0/z^i, \ldots, z^n/z^i) \in \mathbb{C}^n.$$

We then define a holomorphic line bundle on $\mathbb{CP}^n$ by

$$\mathcal{O}_{\mathbb{CP}^n}(m) = \bigsqcup_{i=0,\ldots,n} U_i \times \mathbb{C} \Big/ \sim$$

where $U_i \times \mathbb{C} \ni ([z^0 : \ldots : z^n], w) \sim ([z^0 : \ldots : z^n], (z^i/z^j)^m w) \in U_j \times \mathbb{C}$.

We say a function $g : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$ is of homogeneity m if $g(\lambda z^0, \ldots, \lambda z^n) = \lambda^m g(z^0, \ldots, z^n)$. In practice it is simpler to think of local sections of the above bundles as functions on open subsets of $\mathbb{C}^{n+1}$ with particular homogeneity.

Proposition 4.1. *Let $U \subseteq \mathbb{CP}^n$ be an open set. There is an isomorphism*

$$\Gamma(U, \mathcal{O}_{\mathbb{CP}^n}(m)) \cong \{\textit{holomorphic functions on } \pi^{-1}(U) \textit{ of homogeneity } m\}$$

where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is the map sending a point to the subspace it lies in.

Proof. Over $U \cap U_i$ a section has a representation f_i which is a holomorphic function of n complex variables. On the overlap of charts the transition functions imply these functions satisfy

$$(z^i)^m f_i\Big(\frac{z^0}{z^i}, \dots \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \dots, \frac{z^n}{z^i}\Big) = (z^j)^m f_j\Big(\frac{z^0}{z^j}, \dots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \dots, \frac{z^n}{z^j}\Big)$$

which shows we can define a function of homogeneity m on $\pi^{-1}(U)$ by setting, for $(z^0, \dots, z^n) \in \pi^{-1}(U \cap U_i)$

$$f(z^0, \dots, z^n) := (z^i)^m f_i\Big(\frac{z^0}{z^i}, \dots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \dots, \frac{z^n}{z^i}\Big),$$

and this is holomorphic. We therefore have a map:

$$\Gamma(U, \mathcal{O}_{\mathbb{CP}^n}(m)) \to \{\text{holomorphic functions on } \pi^{-1}(U) \text{ of homogeneity } m\}.$$

Since a section is determined by the collection $\{f_i\}_{i=0}^n$ this map is injective. To see surjectivity let $f : \pi^{-1}(U) \to \mathbb{C}$ be a function of homogeneity m. On $\pi^{-1}(U_i \cap U)$ we have $z^i \neq 0$. Then

$$f(z^0, \dots, z^n) = (z^i)^m f\Big(\frac{z^1}{z^i}, \dots, \frac{z^{i-1}}{z^i}, 1, \frac{z^{i+1}}{z^i}, \dots, \frac{z^n}{z^i}\Big).$$

So we see that if we define a section of $\mathcal{O}_{\mathbb{CP}^n}(m)$ by giving it the local representation

$$f_i(Z^1, \dots, Z^n) = f\big(Z^1, \dots, Z^{i-1}, 1, Z^i, \dots, Z^n\big),$$

in coordinates over U_i, this produces the required function f under the map. □

Corollary 4.2. *Global sections of $\mathcal{O}_{\mathbb{CP}^n}(m)$ correspond to homogeneous polynomials of degree m in n-variables for $m \geq 0$ while there are no global sections for $m < 0$.*

Proof. A holomorphic function f on $\mathbb{C}^{n+1} \setminus \{0\}$ is holomorphic if and only if it is holomorphic separately in each variable (this is Hartog's theorem [12]). From single variable complex analysis we know that we can expand f as a Laurent series in z^i about $z^i = 0$ for each i.

$$f(z) = \sum_{k > -K} c_k(z^1, \dots, z^{i-1}, z^{i+1}, \dots, z^n)(z^i)^k$$

for some $K > 0$. Now if $c_k \neq 0$ for $k < 0$ then we obtain a contradiction since then f is clearly not holomorphic in z_i away from the origin. So we see that in fact f extends to a holomorphic function on the complex plane and hence has a Taylor series about 0. The Taylor series must only contain non-vanishing terms of order m otherwise we have a contradiction with the homogeneity. □

Another useful interpretation of $\mathcal{O}_{\mathbb{CP}^n}(-1)$ is that the fibre of $\mathcal{O}_{\mathbb{CP}^n}(-1)$ at $[v] \in \mathbb{CP}^n$ is $\operatorname{span}\{v\}$. That is, it is the universal bundle of $\mathbb{CP}^n$.

5. Contour integral formulae

We now interpret the formula (1.3) in terms of the notation we have set up and show that this formula indeed produces solutions to the zero-rest-mass field equations. In this section we will work with unweighted spinors and compute with respect to the fixed basis for $\mathbb{T}$ introduced in §2 and the associated trivialisation for the spin bundles over $\mathbb{M}^I$. The scale $\epsilon_{A'B'}$ which we use to raised and lower indices will also be fixed with $\epsilon_{0'1'} = 1$. Let f be a holomorphic function of homogeneity -4 on an open region $\mathbb{T} \setminus \{0\}$ (so this is an arbitrary holomorphic section of $\mathcal{O}_{\mathbb{P}}(-4)$ defined over an open set in $\mathbb{P}$). Recall that coordinates on $\mathbb{T}$ are given by a pair of spinors $(\omega^A, \pi_{A'})$. Now consider the 1-forms

$$\pi_{\mathsf{A}'}\pi_{\mathsf{B}'} f \pi^{\mathsf{D}'} d\pi_{\mathsf{D}'}.$$

for $\mathsf{A}' = 0, 1, \mathsf{B}' = 0, 1$. A choice of $x = X^{AA'} \in \mathbb{M}^I$ and $\pi_{A'}$ determines a point $(iX^{AA'}\pi_{A'}, \pi_{A'}) \in \mathbb{T}$. For fixed $X^{AA'}$ consider:

$$\omega_{\mathsf{A}'\mathsf{B}'} = \pi_{\mathsf{A}'}\pi_{\mathsf{B}'} f(iX^{CC'}\pi_{C'}, \pi_{C'})\pi^{\mathsf{D}'} d\pi_{\mathsf{D}'}$$

which depends only on $\pi_{A'}$. We claim that it does not depend on the scale of $\pi_{A'}$, that is, it is a pull-back of a 1-form defined on an open subset of $\mathbb{CP}^1$. More specifically we have the canonical projection $\mathrm{pr} : (\pi_{0'}, \pi_{1'}) \mapsto [\pi_{0'} : \pi_{1'}] \in \mathbb{CP}^1$ and using the usual open cover for $\mathbb{CP}^1$ with coordinates $Z^0 = \frac{\pi_{1'}}{\pi_{0'}}$ on U_0 and $Z^1 = \frac{\pi_{0'}}{\pi_{1'}}$ on U_1 we see that:

$$\mathrm{pr}^* \, dZ^0 = d(\pi^* Z^0) = d\Big(\frac{\pi_{1'}}{\pi_{0'}}\Big) = \frac{\pi_{0'} d\pi_{1'} - \pi_{1'} d\pi_{0'}}{\pi_{0'}\pi_{0'}}$$

so that $\pi^{\mathsf{D}'} d\pi_{\mathsf{D}'} = \pi_{0'}\pi_{0'} \, \mathrm{pr}^* \, dZ^0$ over $\mathrm{pr}^{-1}(U_0)$. Now, $\pi_{\mathsf{A}'}\pi_{\mathsf{B}'} f(iX^{CC'}\pi_{C'}, \pi_{C'})\pi_0\pi_0$ is a genuine holomorphic function on an open subset of U_0 since it is of homogeneity 0 in the coordinates. On the overlaps we have

$$Z^0 = \frac{1}{Z_1} \implies dZ^0 = -\frac{dZ^1}{Z^1 Z^1} \implies \pi^* dZ^0 = -\frac{\pi_1\pi_1}{\pi_0\pi_0}\pi^* dZ^1$$

That is, we see $\omega_{\mathsf{A}'\mathsf{B}'}$ is the pull-back of the 1-form $I_{\mathsf{A}'\mathsf{B}'}$ on an open subset of $\mathbb{CP}^1$ defined by

$$I_{\mathsf{A}'\mathsf{B}'} = \begin{cases} -\pi_{\mathsf{A}'}\pi_{\mathsf{B}'} f(iX^{CC'}\pi_{C'}, \pi_{C'})\pi_{0'}\pi_{0'} \; dZ^0 & \text{on } U_0 \cap \mathrm{pr}(V_x) \\ \pi_{\mathsf{A}'}\pi_{\mathsf{B}'} f(iX^{CC'}\pi_{C'}, \pi_{C'})\pi_{1'}\pi_{1'} \; dZ^1 & \text{on } U_1 \cap \mathrm{pr}(V_x) \end{cases} \tag{5.1}$$

where V_x is the set of $(\pi_{0'}, \pi_{1'})$ for which $f(iX^{CC'}\pi_{C'}, \pi_{C'})$ makes sense.

It therefore makes sense to integrate $\omega_{\mathsf{A}'\mathsf{B}'}$ over a closed contour in $\mathbb{CP}^1$. Now define a symmetric spinor with the following components in our preferred trivialisation over $\mathbb{M}^I$:

$$\tilde{\phi}_{\mathsf{A}'\mathsf{B}'}(X^{CC'}) = \oint \pi_{\mathsf{A}'}\pi_{\mathsf{B}'} f(iX^{CC'}\pi_{C'}, \pi_{C'})\pi^{\mathsf{D}'} d\pi_{\mathsf{D}'}$$

where we are pushing down to $\mathbb{CP}^1$ then integrating over a closed contour γ_x. If we vary $X^{AA'}$, we may need to adjust the contour γ_x since the set V_x will change, but by Cauchy's theorem if we do this by smooth homotopy the integral will be independent of how we modify γ_x and furthermore $\tilde{\phi}_{\mathsf{A}'\mathsf{B}'}(X^{CC'})$ will be holomorphic as we vary the point $X^{CC'} \in \mathbb{M}$. The holomorphicity means it is safe to take the derivative with respect to $X^{\mathsf{CC}'}$ under the integral sign (recall the flat connection over $\mathbb{M}^I$ induced by the trivialisation just corresponds to differentiating the components with respect to $X^{\mathsf{CC}'}$). The chain rule gives:

$$\nabla_{\mathsf{AA}'} f(iX^{CC'}\pi_{C'}, \pi_{C'}) = i\pi_{\mathsf{A}'} f_{\mathsf{A}}(iX^{CC'}\pi_{C'}, \pi_{C'})$$

where f_A is the derivative of f with respect to the coordinate ω_A for $\mathbb{T}$. In particular

$$\nabla_{\mathsf{A}}{}^{\mathsf{A}'}\tilde{\phi}_{\mathsf{A}'\mathsf{B}'}(X^{CC'}) = \oint i\pi_{\mathsf{A}'}\pi_{\mathsf{B}'}\pi^{\mathsf{A}'}f_{\mathsf{A}}(iX^{CC'}\pi_{C'},\pi_{C'})\pi^{\mathsf{D}'}d\pi_{\mathsf{D}'} = 0 \tag{5.2}$$

since $\pi^{\mathsf{A}'}\pi_{\mathsf{A}'} = 0$ (we are contracting over a pair of symmetric indices). So the spinor field with components $\tilde{\phi}_{\mathsf{A}'\mathsf{B}'}$ over $\mathbb{M}^I$ is a solution to the zero-rest-mass field equations. The only input data was an unconstrained section of $\mathcal{O}_\mathbb{P}(-4)$ defined over an open set.

An identical argument shows that if we start with f of homogeneity $-n-2$ then the spinor with components

$$\phi_{\mathsf{A}'\mathsf{B}'\ldots\mathsf{C}'}(X^{DD'}) = \oint \pi_{\mathsf{A}'}\pi_{\mathsf{B}'}\ldots\pi_{\mathsf{C}'}f(iX^{DD'}\pi_{D'},\pi_{D'})\pi^{\mathsf{E}'}d\pi_{\mathsf{E}'} \tag{5.3}$$

is a solution to (3.4) with helicity $n/2$.

The effect of the choice of contour γ_x is hard to get a grip on in full generality. Vague too is the relationship between the domain of f and the domain of $\tilde{\phi}$. We can be more precise if we calculate the solution generated by a specific twistor function:

Example 5.4 (Elementary states)**.** *An explicit solution to (1.2) is given by:*

$$\tilde{\phi}_{A'B'} = 2\frac{o_A X^A{}_{(A'}X^B{}_{B')}\iota_B}{(\det(X^{CC'}))^3} \tag{5.5}$$

on a suitable region of $\mathbb{M}^I$*, where* o_A *has components* $o_0 = 1, o_1 = 0$ *and* ι_A *has components* $\iota_0 = 0, \iota_1 = 1$ *with respect to the preferred trivialisation for* S^* *over* $\mathbb{M}^I$*.*

Proof. Perhaps the simplest homogeneity -4 twistor function we can write down is:

$$f(\omega^C,\pi_{C'}) = \frac{1}{2\pi i(\omega^0)^2(\omega^1)^2}$$

This defines a section of $\mathcal{O}_\mathbb{P}(-4)$ away from the image under the projection $\mathbb{T}\to\mathbb{P}$ of the closed set $\{\omega^0 = 0\}\cup\{\omega^1 = 0\}$. We will consider

$$f(X^{CC'}\pi_{C'},\pi_{C'}) = \frac{1}{2\pi i(X^{00'}\pi_{0'} + X^{01'}\pi_{1'})^2(X^{10'}\pi_{0'} + X^{11'}\pi_{1'})^2} \tag{5.6}$$

for $X^{CC'}$ non-degenerate. The problem with letting this matrix be degenerate is if we vary $X^{CC'}$ through a degenerate point then the two poles of the integrand (5.1) will pass through each other. This is a problem since we want to avoid contours that enclose all of the poles; such contours are homotopic to a point on a region where the integrand is holomorphic, as illustrated in Figure 1. The integral therefore vanishes for such contours. Accordingly, let N denote the subset of $\mathbb{M}^I$ consisting of degenerate $X^{CC'}$ and we will calculate a solution $\tilde{\phi}_{A'B'}$ on $\mathbb{M}^I\setminus N$.

We will calculate the contour integral by pushing down to $\mathbb{CP}^1$ and working in U_0. Substitute (5.6) into (5.1) to get:

$$\begin{aligned}
\tilde{\phi}_{0'0'} &= \oint -\frac{dZ^0}{2\pi i(X^{00'} + X^{01'}Z^0)^2(X^{10'} + X^{11'}Z^0)^2}\\
\tilde{\phi}_{0'1'} &= \oint -\frac{Z^0\,dZ^0}{2\pi i(X^{00'} + X^{01'}Z^0)^2(X^{10'} + X^{11'}Z^0)^2}\\
\tilde{\phi}_{1'1'} &= \oint -\frac{(Z^0)^2\,dZ^0}{2\pi i(X^{00'} + X^{01'}Z^0)^2(X^{10'} + X^{11'}Z^0)^2}.
\end{aligned}$$

We calculate these contour integrals by the usual residue formula at the second order pole $Z^0 = -X^{00'}/X^{01'}$. The orientation and winding number of γ_x have the effect of

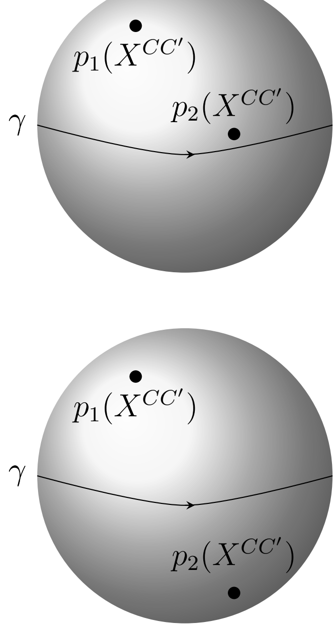

FIGURE 1. An example of a trivial contour for the integrand on $\mathbb{CP}^1 \cong S^2$ (above) and a non-trivial contour (below). Here f is holomorphic on the Riemann sphere except at the poles $p^1(X^{CC'}), p^2(X^{CC'})$ which depend on the choice of point $X^{CC'} \in \mathbb{M}^I$.

multiplying the solution by a constant integer. Take the winding number to be 1. If $X^{01'} = 0$, then the expression for the integrand in U_0 has a pole "at infinity" so we instead calculate the residue at the other pole for which the winding number -1. Since the poles are second order we need to use the formula for the residue of a meromorphic function f at a second order pole z_0:

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{d}{dz}\Big((z - z_0)^2 f(z)\Big).$$

The explicit calculations of the residues:

$$\frac{d}{dZ^0} - \frac{(Z^0)^n}{(X^{01'})^2(X^{10'} + X^{11'}Z^0)^2} = -\frac{n(Z^0)^{n-1}}{2\pi i(X^{01'})^2(X^{10'} + X^{11'}Z^0)^2} + \frac{2X^{11'}(Z^0)^n}{2\pi i(X^{01'})^2(X^{10'} + X^{11'}Z^0)^3}$$

Then the limit of the above as $Z^0 \to -X^{00'}/X^{01'}$ is:

$$(-1)^{n-1}\left(\frac{n(X^{00'})^{n-1}}{2\pi i(X^{01'})^{n-1}(X^{10'}X^{01'} - X^{11'}X^{00'})^2} + \frac{2X^{11'}(X^{00'})^n}{2\pi i(X^{01'})^{n-1}(X^{10'}X^{01'} - X^{11'}X^{00'})^3}\right)$$

and taking $n = 0, 1, 2$ give the $0'0', 0'1', 1'1'$ components of (5.5) respectively, remembering that $X^{\mathsf{A}}{}_{0'} = -X^{\mathsf{A}1'}$ and $X^{\mathsf{A}}{}_{1'} = X^{\mathsf{A}0'}$. □

The solution here is an example of what is called an *elementary state* [21].

Cauchy's theorem tells us that if we modify f by adding a function g such that $g(iX^{CC'}\pi_{C'}, \pi_{C'})$ is holomorphic on a contractible region containing the contour then the value of the integral will remain unchanged. The contour integral formulae (5.3) therefore *does not* exhibit a bijective correspondence between sections of $\mathcal{O}_{\mathbb{P}}(-n-2)$ and solutions to (3.4). For instance, if we replaced f in the previous example with:

$$\tilde{f}(\omega^C, \pi_C) = \frac{1}{2\pi i(\omega^0)^2(\omega^1)^2} + \frac{1}{2\pi i(\omega^0)^4} + \frac{1}{2\pi i(\omega^1)^4}$$

we obtain the same solution. The contour integrals resulting from the additional terms vanish since the contour is homotopic to a point on regions "each side" of the contour where the additional terms are each holomorphic. The reader familiar with Čech cohomology will find this reminiscent of a cohomology representative being defined up to a coboundary. We will need this machinery of sheaf cohomology to obtain a true correspondence.

6. Sheaf cohomology

We will give a brief introduction to sheaf cohomology.

Definition 6.1 (Sheaf). *Let M be a smooth manifold. A sheaf S on M is a map from the open sets $\tau = \{U_i\}_{i\in I}$ of M to $\mathcal{R}$-modules $\{S(U)\}$ together with, for each U, V such that $V \subseteq U$, associated restriction homomorphisms $r^U_V : S(U) \to S(V)$ so that if $W \subseteq V \subseteq U$ then*

$$r^V_W \circ r^U_V = r^U_W$$

and with $r^U_U = \mathrm{id}_{S(U)}$. We also insist on the ability to patch together local sections. That is $\{U_\alpha\}$ is an open cover for $U \subseteq M$ then if there exists $\{s_\alpha\}$ such that $s_\alpha \in S(U_\alpha)$ and

$$r^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = r^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta) \forall \alpha, \beta$$

then there exists $s \in S(U)$ with $r^U_{U_\alpha} s = s_\alpha$ for each α. On the other hand we insist that a section is determined by its local restrictions, that is, we require that if

$$r^U_{U_\alpha}(s) = r^U_{U_\alpha}(s') \ \forall \alpha$$

then $s = s'$.

Example 6.2 (Sheaf of a sections of a vector bundle). *Let $E \to M$ be a smooth (resp. holomorphic) vector bundle. For each open $U \subseteq M$ define $S(U)$ to be the $\mathbb{C}$-module of smooth (resp. holomorphic) sections of $E|_U$, equipped with, for each pair $U, V \in \tau$ such that $V \subseteq U \subseteq M$ the natural restriction maps*

$$r^U_V : S(U) \to S(V)$$

given by $s \mapsto s|_V$.

We will abuse notation slightly and sometimes not distinguish notationally between a holomorphic vector bundle and its sheaf of holomorphic sections, with the context making the distinction clear, as is standard practice [9]. We will however need notation to distinguish between the sheaves of smooth and holomorphic sections. We will write $\mathcal{E}(V)$ for the sheaf of smooth sections of a vector bundle V.

An important special case is the sheaf of holomorphic functions on a complex manifold M, denoted $\mathcal{O}_M$. This is the sheaf of holomorphic sections of the trivial bundle $\mathbb{C} \times M$ over M. Recall these are the functions that are annihilated by the Dolbeault operator $\bar{\partial}$.

An element $s \in S(U)$ is called a *section of the sheaf S over U*.

Example 6.3 (Constant sheaf). *Let M be a smooth manifold. The constant sheaf $\bar{\mathcal{R}}$ with values in $\mathcal{R}$ on M is the sheaf where $\bar{\mathcal{R}}(U)$ is defined to be the set of constant functions on U in the ring $\mathcal{R}$.*

An important example for our purposes will be the following:

Example 6.4 (Topological pull-back)**.** *Let M, N be smooth manifolds and $f : M \to N$ a smooth map. Let E be a smooth (resp. holomorphic) vector bundle over N. We get a sheaf on M by the assignment, for each $U \subseteq M$ open:*

$U \mapsto$ {*smooth (resp. holomorphic) sections of $(f^*E)|_U$ locally constant in the fibres of f*}

equipped with the usual restriction maps.

Of course being constant ordinarily has no meaning for sections of a vector bundle, but here makes sense since we may identify all of the fibres of f^*E in a fibre of f with $E_{f(x)}$. By locally constant we then mean constant on connected components. Sections are then *locally* pull-backs of sections of E.

The assumption that the sections are *locally* constant in the fibres is important. The sections of the pull-back bundle which are constant in the fibres do not form a sheaf as it is possible to imagine sections s_α, s_β over U_α, U_β respectively, constant in the fibres and agreeing on $U_\alpha \cap U_\beta$ but s_α and s_β taking different values in $U \cap f^{-1}(x)$ and $V \cap f^{-1}(x)$, respectively if $U \cap V \cap f^{-1}(x) = 0$, for example.

We will denote the topological pull-back by the same symbol as the pulled-back sheaf, with the base space making the distinction clear.

To discuss the cohomology of sheaves we need a notion of morphisms between them.

Definition 6.5 (Morphism of sheaves)**.** *A morphism f of sheaves S and T of $\mathcal{R}$-modules is a collection of $\mathcal{R}$-module homomorphisms $\{f|_U\}_{U\in\tau}$*

$$f|_U : S(U) \to T(U)$$

such that the following diagram commutes for all $U, V \in \tau$ with $V \subseteq U$

$$\begin{array}{ccc} S(U) & \xrightarrow{f|_U} & T(U) \\ \downarrow{\scriptstyle r^U_V} & & \downarrow{\scriptstyle r^U_V} \\ S(V) & \xrightarrow{f|_V} & T(V). \end{array}$$

For example, the exterior derivative gives a homomorphism of sheaves $d : \Omega^p \to \Omega^{p+1}$ of p-forms by its restriction to open sets on M.

A morphism of sheaves is called an isomorphism if each map of $\mathcal{R}$-modules is an isomorphism.

Example 6.6. *There is an isomorphism of sheaves on $\mathbb{CP}^n$*

$$\mathcal{O}(m) \cong \{\textit{holomorphic functions on } \mathbb{C}^{n+1} \setminus \{0\} \textit{ with homogeneity } m\}.$$

Proof. The isomorphisms between $\mathbb{C}$-modules of local sections constructed in proposition 4.1 obviously commute with the restriction maps. □

Definition 6.7 (Stalks of a sheaf)**.** *Let S be the sheaf. Then the stalk of S at x is*

$$S_x := \bigcup_{U,\, x\in U} S(U) \Big/ \sim$$

where $S(U) \ni s_1 \sim s_2 \in S(V)$ if and only if there is an open set $W \subseteq (U \cap V)$ such that $r^U_W(s_1) = r^V_W(s_2)$.

Note that S_x inherits a $\mathcal{R}$-module structure and furthermore the definitions of stalk and sheaf morphism are set up so that a morphism of sheaves $T \to S$ gives an induced $\mathcal{R}$-module morphism $S_x \to T_x$ for each $x \in M$. We say that a sequence of morphism of sheaves

$$S \to T \to V$$

is exact at T if

$$S_x \to T_x \to V_x$$

is exact for each $x \in M$. To check exactness of this sequence of morphisms of stalks it is sufficient to check that one can find arbitrarily small neighbourhoods W of x such that:

$$S(W) \to T(W) \to V(W)$$

is exact.

It is important to note that a sequence of homomorphisms of *global* sections of a vector bundle may not be exact even if the underlying sequence of homomorphisms of the corresponding sheaves is. For example, the topology of M may mean that the de Rham complex given by the exterior derivative $d : \Gamma(\wedge^p T^* M) \to \Gamma(\wedge^{p+1} T^* M)$ is not exact as a sequence of homomorphisms of $\mathbb{R}$-modules $\Gamma(\wedge^p T^* M)$ but the sequence of homomorphisms of sheaves $d : \Omega^p \to \Omega^{p+1}$ is always exact as a consequence of the Poincaré lemma. One might call the exactness of the de Rham complex at the level of sheaves *local exactness.*

Example 6.8 (Euler exact sequence)**.** *Let V be an arbitrary complex vector space of dimension n and $\mathbb{P}(V)$ the corresponding projective space. There is an exact sequence of sheaves of sections (the Euler sequence)*

$$0 \to \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(1) \otimes V \to T\mathbb{P}(V) \to 0. \tag{6.9}$$

Proof. From (4.1) the middle sheaf is the sheaf of holomorphic vectors fields on V which are homogeneous of degree 1. There is a canonical such vector field on V, the Euler vector field, defined by mapping to itself (that is, $V \ni v \mapsto v \in V$) under the canonical identification $T_v V \cong V$. A function on an open subset of $\mathbb{P}(V)$ pulls back to a (homogeneity 0) function on an open subset of $V \setminus \{0\}$ simply by declaring the function be independent of the radial coordinate on $V \setminus \{0\}$. Since the Euler vector field is radial, it annihilates such a pull-back.

The first map in the above sequence is defined by sending a holomorphic function on $\mathbb{P}(V)$ to the Euler vector field multiplied by the holomorphic function pulled back to V. That the pull-back is not defined at the origin is not a problem because the Euler vector field vanishes here. The next thing to investigate is how a vector field of homogeneity 1 on V naturally gives rise to a vector field on $\mathbb{P}(V)$. A holomorphic vector field on $\mathbb{P}(V)$ will be a derivation of the holomorphic functions on $\mathbb{P}(V)$, which is the same as a derivation of homogeneity 0 functions on $V \setminus \{0\}$. We can apply a vector field $X = X^i \partial_i$ of homogeneity 1 to a function f of homogeneity 0 to obtain $X^i \partial_i f$. If we define a function l on $V \setminus \{0\}$ by $l(z) = \lambda z$ then$f \circ l = f$ and differentiating both sides of this equation using the chain rule we see that $\partial_i f$ is of homogeneity -1 and hence that $X^i \partial_i f$ is a homogeneity 0. Therefore $X^i \partial_i$ defines a derivation of homogeneity 0 functions on $V \setminus \{0\}$. We can obtain $n-1$ linearly independent derivations from vector fields of homogeneity 1 and so surjectivity is immediate. A holomorphic vector field acts trivially on functions independent of the radial coordinate if and only if it is equal to a holomorphic function multiplied by the Euler vector field and so we have exactness of (6.9). □

We can dualise (6.9) to obtain

$$0 \to \Omega^1 \to \mathcal{O}_{\mathbb{P}(V)}(-1) \otimes V^* \to \mathcal{O}_{\mathbb{P}(V)} \to 0. \tag{6.10}$$

and thinking about transposes of linear maps, the map $\mathcal{O}_{\mathbb{P}(V)}(-1) \otimes V^* \to \mathcal{O}_{\mathbb{P}(V)}$ must be given by contraction with the Euler vector field. This dualised exact sequence will be a prototype for important exact sequences in §10.

Next, we will formulate sheaf cohomology in terms of Čech cohomology.

Definition 6.11 (p-cochains)**.** *Let M be a smooth manifold and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover. Let S be a sheaf. Denote the set of $p+1$ tuples $(i_0, ..., i_p) \in I^{p+1}$ such that $\cap_{k=0}^{p} U_{i_k} \neq \emptyset$ by N^p . A p-cochain c with respect to $\mathcal{U}$ with values in S is a function*

$$c : N^k \to S$$
$$c(i_0, ..., i_p) =: c_{i_0 \dots i_p}$$

such that $c_{i_0\dots i_p} \in S(\cap_{k=0}^{p} U_{i_k})$ and $c_{[i_0\dots i_p]} = c_{i_0\dots i_p}$. We write

$$C^p(\mathcal{U}, S) := \{p\text{-cochains with respect to } \mathcal{U} \text{ with values in } S\}.$$

We may add cochains and multiply them by elements of the ring so there is a $\mathcal{R}$-module structure on $C^p(\mathcal{U}, S)$.

Definition 6.12 (Coboundary operator)**.** *Define an operator $\delta_p : C^p(\mathcal{U}, S) \to C^{p+1}(\mathcal{U}, S)$ by*

$$(\delta c)_{i_0 i_1 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j c_{i_0 i_1 \dots \hat{i_j} \dots i_{p+1}} \tag{6.13}$$

where the $\hat{i_j}$ denotes omission of the index and we are implicitly applying the restriction map to restrict $c_{i_0 i_1 \dots \hat{i_j} \dots i_{p+1}}$ from its domain of definition to $\cap_{k=0}^{p+1} U_{i_k}$.

Lemma 6.14. *$\delta_{p+1} \circ \delta_p = 0$ and so*

$$0 \longrightarrow C^1(\mathcal{U}, S) \xrightarrow{\delta_1} \dots \xrightarrow{\delta_{p-1}} C^p(\mathcal{U}, S) \xrightarrow{\delta_p} \dots$$

is a complex of $\mathcal{R}$-modules.

Proof. In the expansion for $(\delta\delta c)_{i_0 i_1 \dots i_{p+2}}$, the term $c_{i_0 i_1 \dots \hat{i_j} \dots \hat{i_k} \dots i_{p+2}}$ will appear exactly twice, with opposite signs. □

Definition 6.15 (pth Čech cohomology with respect to $\mathcal{U}$)**.** *Define $Z^p(\mathcal{U}, S) := \ker \delta_p$ and $B^p(\mathcal{U}, S) := \mathrm{im}\ \delta_{p-1}$ which we call the set of p-cocyles and the set of p-coboundaries respectively. Define*

$$\check{H}^p(\mathcal{U}, S) = Z^p(\mathcal{U}, S)/B^p(\mathcal{U}, S) \tag{6.16}$$

which we call the Čech cohomology of M with respect to $\mathcal{U}$ with values in the sheaf S.

For smooth sections of a vector bundle, the existence of partitions of unity means one can "chop-up" a p-cocycle to exhibit it as a coboundary:

Proposition 6.17. *Let S be the sheaf of smooth sections of a vector bundle over M. Then given an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ for M*

$$\check{H}^p(\mathcal{U}, S) = 0, \ p > 0.$$

Proof. Take a (locally finite) smooth partition of unity $\{\varphi_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$. Given a 1-cocycle c define a 0-cochain b by:

$$b_i = -\sum_k \varphi_k c_{ik}.$$

Then

$$(\delta b)_{ij} = \sum_k \varphi_k c_{ik} - \sum_k \varphi_k c_{jk} = \sum_k \varphi_k (c_{ik} - c_{jk})$$

but since c is a cocycle $c_{ik} - c_{jk} = c_{ij}$ so in fact $(\delta b)_{ij} = c_{ij}$. The proof is the same for higher cohomology, but with more indices. □

Since analytic partitions of unity do not exist in general, the Čech cohomology with values in holomorphic sections of a vector bundle is more interesting.

Given complexes of $\mathcal{R}$-modules $A = \{A_i\}_{i=1}^{\infty}$, $B = \{B_i\}_{i=1}^{\infty}$ with differentials $\delta_i^A : A_i \to A_{i+1}$, $\delta_i^B : B_i \to B_{i+1}$, a map of complexes $f : A \to B$ is a collection of homomorphisms $f_i : A_i \to B_i$ which commute with the differentials. These maps descend to cohomology.

Note there is a dependence of the Čech cohomology groups constructed above on the open cover $\mathcal{U}$. We want to somehow construct an invariant that is independent of the open cover $\mathcal{U} = \{U_i\}_{i\in I}$. We will outline this procedure here and for full details see, for example [11]. The first thing to note is that there is a partial ordering on open covers of M given by $\mathcal{V} < \mathcal{U}$ if $\mathcal{V} = \{V_j\}_{j\in J}$ is a refinement of $\mathcal{U}$. By refinement we mean that $\forall V \in \mathcal{V}$, $V \subseteq U$ for some $U \in \mathcal{U}$. Pick a map $\tau : J \to I$ such that $V_j \subseteq U_{\tau(j)}$ $\forall j \in J$ and define $\tau^* : C^p(\mathcal{U}, S) \to C^p(\mathcal{V}, S)$ by

$$(\tau^* c)_{\alpha_0\alpha_1\ldots\alpha_p} = c_{\tau(\alpha_0)\tau(\alpha_1)\ldots\tau(\alpha_p)}. \tag{6.18}$$

We get a map of complexes of $\mathcal{R}$-modules and hence we get an induced map on cohomology $h_{\mathcal{V}}^{\mathcal{U}} : \check{H}^p(\mathcal{U}, S) \to \check{H}^p(\mathcal{V}, S)$, but a priori this depends on τ. Given another $\tau' : J \to I$, a simple combinatorial argument shows that one can construct a *cochain homotopy* between τ^* and $(\tau')^*$, that is, the induced maps on cohomology are the same as for τ.

Definition 6.19 (pth Čech cohomology)**.** *Define $\check{H}^p(M, S)$, the pth Čech cohomology group of M with values in S as follows:*

$$\check{H}^p(M, S) = \bigcup_{\mathcal{U}} \check{H}^p(\mathcal{U}, S)\Big/ \sim$$

taking the union across open covers $\mathcal{U}$, and where the equivalence relation is defined by $[c] \ni \check{H}^p(\mathcal{U}, S) \sim [c'] \ni \check{H}^p(\mathcal{V}, S)$ if and only if there is a common refinement $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{V}$ such that $h_{\mathcal{W}}^{\mathcal{U}}[c] = h_{\mathcal{W}}^{\mathcal{V}}[c']$.

The good news is that we will never need to construct a Čech cohomology group this way. We have the following theorems:

Theorem 6.20 (Leray)**.** *If $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover such that for all non-empty finite intersections $\sigma = U_{i_0} \cap \ldots \cap U_{i_q}$ we have*

$$\check{H}^p(\sigma, S) = 0, \ \forall p > 0$$

then

$$\check{H}^p(M, S) \cong \check{H}^p(\mathcal{U}, S), \ \forall p > 0.$$

Such a cover is called a Leray cover.

For a proof see Hirzebruch [14, p. 26].

This is particularly useful in combination with:

Theorem 6.21 (Cartan's Theorem B)**.** *Let S be the sheaf of holomorphic sections of a holomorphic vector bundle over M where M is a domain of holomorphy in $\mathbb{C}^n$, in the sense M is an open set and there exists a holomorphic function on M which cannot be extended to a larger domain. Then*

$$\check{H}^p(M, S) = 0, \ \forall p > 0$$

For a proof see Gunning and Rossi [12, p. 243]. In particular, if we can find a cover of a complex manifold with non-empty intersections which are domains of holomorphy (we can do this, as polydiscs and their intersections are domains of holomorphy [15, p. 39,40]), then this is a Leray cover and we may compute Čech cohomology cover.

Example 6.22 (Vanishing theorem for $\mathbb{CP}^1$)**.**

$$\check{H}^0(\mathbb{CP}^1, \mathcal{O}(-1)) = 0$$
$$\check{H}^1(\mathbb{CP}^1, \mathcal{O}(-1)) = 0$$

Proof. We have $\check{H}^0(\mathbb{CP}^1, \mathcal{O}(-1)) = \Gamma(\mathbb{CP}^1, \mathcal{O}(-1))$ since a 0-cocycle is precisely a global section since it agrees on overlaps. $\Gamma(\mathbb{CP}^1, \mathcal{O}(-1)) = 0$ is precisely corollary 4.2.

Next, consider the usual cover $\{U_0, U_1\}$. We have $U_i \cong \mathbb{C}$ and $U_0 \cap U_1 \cong \mathbb{C} \setminus \{0\}$ biholomorphically so this is a Leray cover. With respect to this cover every 1-cochain is a cocycle since there are no triple intersections. We therefore need to show that a holomorphic section of $\mathcal{O}(-1)$ defined over $U_0 \cap U_1$ is a coboundary. Now by proposition 4.1 a holomorphic section of $\mathcal{O}(-1)$ over $U_0 \cap U_1$ corresponds to a holomorphic function of homogeneity -1 on $\mathbb{C}^2 \setminus (\{z_0 = 0\} \cup \{z_1 = 0\})$. We can write f as a Laurent series about $z^0 = 0$ with coefficients that are holomorphic functions on $\mathbb{C} \setminus \{z^1 = 0\}$ and then expand these holomorphic functions as Laurent series about $z^1 = 0$ to obtain:

$$f = \sum_{i,j > -K} f_{ij} (z^0)^i (z^1)^j$$

where K is some positive integer. The homogeneity condition on f forces $i + j = -1$ and so it follows

$$f = \underbrace{\sum_{n=1}^{K+2} f_{-n-1,n} (z^0)^{-n-1} (z^1)^n}_{=:F_0} + \underbrace{\sum_{n=1}^{K+2} f_{n,-n-1} (z^0)^n (z^1)^{-n-1}}_{=:-F_1}.$$

Now F_0 extends holomorphically to $\pi^{-1}(U_0)$ and F_1 extends holomorphically to $\pi^{-1}(U_1)$ so that $f = F_0|_{\pi^{-1}(U_0)} - F_1|_{\pi^{-1}(U_1)}$ which exhibits f as a coboundary. □

We will need a standard fact from homological algebra. A sequence of maps of complexes $A \xrightarrow{f} B \xrightarrow{g} C$ is called exact at B if each $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$ is exact at B_i. The following can be proved from the famous snake lemma:

Lemma 6.23 (Long exact sequence of cohomology)**.** *Let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence of complexes of $\mathcal{R}$-modules, then there is an exact sequence:

$$\begin{array}{c} 0 \longrightarrow H^0(A) \longrightarrow H^0(B) \longrightarrow H^0(C) \xrightarrow{\delta^*} \\ H^1(A) \longrightarrow H^1(B) \longrightarrow H^1(C) \xrightarrow{\delta^*} \\ H^2(A) \longrightarrow H^2(B) \longrightarrow H^2(C) \longrightarrow \cdots \end{array}$$

Furthermore the map $\delta^* : H^k(C) \to H^{k+1}(A)$ is given by

$$[\phi] \mapsto [\delta^B_k \tilde{\phi}]$$

where $\tilde{\phi} \in B_k$ satisfies $g_k(\tilde{\phi}) = \phi$.

To see that $\delta^B_k \tilde{\phi}$ can be identified as a cocycle in A^{k+1} use the fact the homomorphisms commute with the coboundary operator so $(\delta^C_k \circ g_k)(\tilde{\phi}) = (g_{k+1} \circ \delta^B_k)(\tilde{\phi}) = 0$ and use exactness of

$$0 \to A_{k+1} \xrightarrow{f_{k+1}} B_{k+1} \xrightarrow{g_{k+1}} C_{k+1} \to 0.$$

Let V be a holomorphic vector bundle and $\mathcal{E}^{p,q}(V)$ denote the bundle of (smooth) (p,q)-forms with values in V. Take local trivialising sections $\{s_i\}$ for V. We may locally write any section of $\mathcal{E}^{p,q}(V)$ as $s_i \otimes \omega^i$ for (p,q)-forms ω_i. There is a canonical operator $\bar{\partial} : \mathcal{E}^{p,q}(V) \to \mathcal{E}^{p,q+1}(V)$ given by

$$s_i \otimes \omega^i \mapsto s_i \otimes \bar{\partial}\omega^i$$

This is well-defined as the transition functions are holomorphic and so annihilated by $\bar{\partial}$.

The long exact sequence lemma combined with the $\bar{\partial}$-Poincaré lemma (see [12]) can be used to prove the following standard theorem:

Theorem 6.24 (Dolbeault's theorem). *The following complex*

$$0 \longrightarrow \mathcal{O}(V) \xrightarrow{\bar{\partial}} \mathcal{E}(V) \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1}(V) \xrightarrow{\bar{\partial}} \cdots$$

is exact on the level of sheaves and for U an open set

$$\check{H}^p(U, \mathcal{O}(V)) \cong H^p(\Gamma(U, \mathcal{E}(V)) \ \forall p$$

where $H^p(\Gamma(U, \mathcal{E}(V))$ is the pth cohomology of the complex

$$0 \longrightarrow \Gamma(U, \mathcal{E}(V)) \xrightarrow{\bar{\partial}} \Gamma(U, \mathcal{E}^{0,1}(V)) \xrightarrow{\bar{\partial}} \cdots$$

Details of the proof are omitted as we will see similar calculations using long exact sequences in the next sections.

7. The relative de Rham sequence

Let $\Omega^1_{\mathbb{F}}$ and $\Omega^1_{\mathbb{P}}$ denote the sheaves of holomorphic forms of degree p on $\mathbb{F}$ and $\mathbb{P}$ respectively.

Define $\Omega^k_\mu := \wedge^k(\ker D\mu)^*$ where $D\mu : T^{0,1}\mathbb{F} \to T^{0,1}\mathbb{P}$ is the push-forward of holomorphic tangent vectors. In other words, Ω^1_μ is dual to the vertical bundle of the fibration μ. We can canonically identify $\mu^*\Omega^1_{\mathbb{P}}$ as the sheaf of sections of the holomorphic subbundle of $\Omega^1_{\mathbb{F}}$ generated by pull-backs of holomorphic 1-forms on $\mathbb{P}$. There is a surjective mapping $\Omega^k_{\mathbb{F}} \to \Omega^k_\mu$ given by restriction to vertical tangent vectors. Define $\mu^*\Omega^1_{\mathbb{P}} \wedge \Omega^1_{\mathbb{F}}$ to be the sheaf of holomorphic 2-forms on $\mathbb{F}$ generated by elements of the form $\psi \wedge \omega$ for $\psi \in \mu^*\Omega^1_{\mathbb{P}}$ and $\omega \in \Omega^1_{\mathbb{F}}$. Consider the diagram of holomorphic sheaves:

$$\begin{array}{ccccccc} & & \Omega^1_\mu & & \Omega^2_\mu & & \\ & & \uparrow & & \uparrow & & \\ \mathcal{O}_{\mathbb{F}} & \longrightarrow & \Omega^1_{\mathbb{F}} & \longrightarrow & \Omega^2_{\mathbb{F}} & \longrightarrow & \ldots \\ & & \uparrow & & \uparrow & & \\ & & \mu^*\Omega^1_{\mathbb{P}} & & \mu^*\Omega^1_{\mathbb{P}} \wedge \Omega^1_{\mathbb{F}} & & \end{array}$$

The horizontal mappings give the usual holomorphic de Rham complex on $\mathbb{F}$ and the columns are exact.

This gives rise to the *relative de Rham sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{F}} \xrightarrow{d_\mu} \Omega^1_\mu \xrightarrow{d_\mu} \Omega^2_\mu \longrightarrow 0 \tag{7.1}$$

as follows: There is a canonical inclusion of $\mathcal{O}_{\mathbb{P}}$ (the topological pull-back of holomorphic functions on $\mathbb{P}$) as a sheaf on $\mathbb{F}$ into $\mathcal{O}_{\mathbb{F}}$ as holomorphic functions on $\mathbb{F}$ locally constant in fibres. Define $d_\mu : \mathcal{O}_{\mathbb{F}} \to \Omega^1_\mu$ simply as the composition $\mathcal{O}_{\mathbb{F}} \to \Omega^1_{\mathbb{F}} \to \Omega^1_\mu$. One should think of this as differentiating holomorphic functions on $\mathbb{F}$ in the fibres of $\mu : \mathbb{F} \to \mathbb{P}$. Next define $d_\mu : \Omega^1_\mu \to \Omega^2_\mu$ on some element of Ω^1_μ by picking an element in its preimage under the surjection $\Omega^1_{\mathbb{F}} \to \Omega^1_\mu$, applying the exterior derivative, and restricting to get an element of Ω^2_μ. This is well-defined since we claim the composition $\mu^*\Omega^1_{\mathbb{P}} \to \Omega^1_{\mathbb{F}} \to \Omega^2_{\mathbb{F}} \to \Omega^2_\mu$ vanishes and hence this is independent of the choice of preimage, since any two preimages will differ by an element in $\mu^*\Omega^1_{\mathbb{P}}$. To see that the composition vanishes note that $\mu^*\Omega^1_{\mathbb{P}}$ is generated by elements of the form $f\mu^*\omega$. Then:

$$d(f\mu^*\omega) = f d(\mu^*\omega) + df \wedge \mu^*\omega = f\mu^* d\omega + df \wedge \mu^*\omega$$

and the right hand side vanishes when restricted to vertical tangent vectors.

That (7.1) is a complex can be deduced from the fact the de Rham sequence is a complex. In particular

$$(d_\mu \circ d_\mu) f = (d \circ d) f|_{\ker D\mu} = 0.$$

Proposition 7.2 (Local exactness of the relative de Rham complex)**.** *The complex (7.1) is an exact sequence of sheaves.*

Proof. Exactness at $\mathcal{O}_{\mathbb{F}}$: $d_\mu f = 0$ is precisely to say f is locally constant in the fibres of $\mu : \mathbb{F} \to \mathbb{P}$.

Exactness at Ω^1_μ: $d_\mu \omega = 0$ implies there is a $\tilde{\omega} \in \Omega^1_{\mathbb{F}}$ such that $d\tilde{\omega}|_{\ker D\mu} = 0$. Since μ is a holomorphic submersion, about any point $p \in \mathbb{F}$ and complex coordinates (z^1, z^2, z^3) for $\mathbb{P}$ about $\mu(p)$ we may take complex coordinates $(z^1, z^2, z^3, w^1, w^2)$ for $\mathbb{F}$ about p such that the coordinate representation of μ is simply

$$(z^1, z^2, z^3, w^1, w^2) \mapsto (z^1, z^2, z^3). \tag{7.3}$$

We will show there exists a holomorphic function η such that $d_\mu \eta = \omega$ on a polydisc centred at $(z^1_0, z^2_0, z^3_0, w^1_0, w^2_0)$. In our preferred coordinates the vertical bundle is spanned by the w^1 and w^2 coordinate fields so we can identify Ω^1_μ with the subbundle span$\{dw^1, dw^2\}$ and write

$$\omega = f_i(z^1, z^3, z^3, w^1, w^2) dw^i$$

where f_i is a holomorphic function of $(z^1, z^3, z^3, w^1, w^2)$. With this identification d_μ simply corresponds to taking the exterior derivative and projecting onto span$\{dw^1 \wedge dw^2\}$. Taking the exterior derivative and restricting we see that the d_μ closed condition is equivalent to

$$\frac{\partial f_1}{\partial w^2} = \frac{\partial f_2}{\partial w^1}.$$

On a sufficiently small polydisc about $(z^1_0, z^2_0, z^3_0, w^1_0, w^2_0)$ we get a well-defined holomorphic function

$$F_1(z^1, z^3, z^3, w^1, w^2) = \int_{\gamma_1} f_1(z^1, z^2, z^3, \tilde{w}^1, w^2) d\tilde{w}^1$$

where γ_1 is any contour from w^1_0 to w^1. This satisfies

$$\frac{\partial F_1}{\partial w^1}(z) = f_1(z)$$

where $z := (z^1, z^2, z^3, w^1, w^2)$ and differentiating under the integral

$$\frac{\partial F_1}{\partial w^2}(z) = \int_{\gamma_1} \frac{\partial f_1}{\partial w^2}(z^1, z^2, z^3, \tilde{w}^1, w^2) d\tilde{w}^1 = \int_{\gamma_1} \frac{\partial f_2}{\partial \tilde{w}^1}(z^1, z^2, z^3, \tilde{w}^1, w^2) d\tilde{w}^1$$
$$= f_2(z) - f_2(z_0)$$

where $z_0 := (z^1, z^2, z^3, w_0^1, w^2)$. Now

$$d_\mu F_1(z) = f_1(z) dw^1 + (f_2(z) + f_2(z_0)) dw^2 = \omega(z) + f_2(z_0) dw^2.$$

Lastly the function

$$F_2(z^1, z^2, z^3, w^1, w^2) = \int_{\gamma_2} f_2(z^1, z^2, z^3, w_0^1, \tilde{w}^2) d\tilde{w}^2$$

where γ_2 is any contour from w_0^2 to w^2, satisfies $d_\mu F_2 = f_2(z_0) dw^2$ so that $d_\mu(F_1 - F_2) = \omega$ as required for exactness.

Exactness at Ω_μ^2*:* Every section of Ω_μ^2 is automatically d_μ closed and takes the form $f(z^1, z^2, z^3, w^1, w^2) dw^1 \wedge dw^2$ in our preferred coordinates. Similarly to before define a holomorphic function

$$F = \int_{\gamma_1} f(z_1, z_2, z_3, \tilde{w}_1, w_2) d\tilde{w}_1$$

then $d_\mu(F dw^2) = f dw^1 \wedge dw^2$ by construction. □

Again we should stress that this is exact *on the level of sheaves* and the corresponding complex of global sections may possess cohomology.

The complex (7.1) is only a special case of the general complex we will need. We will need to couple this complex to powers of the tautological bundle $\mathcal{O}(n)$ pulled back over $\mathbb{F}$. Specifically we will make use of:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(n) \longrightarrow \mu^*\mathcal{O}_{\mathbb{P}}(n) \xrightarrow{d_\mu} \Omega_\mu^1 \otimes \mu^*\mathcal{O}_{\mathbb{P}}(n) \xrightarrow{d_\mu} \Omega_\mu^2 \otimes \mu^*\mathcal{O}_{\mathbb{P}}(n) \longrightarrow 0$$

We can write any section of $\Omega_\mu^k \otimes \mu^*\mathcal{O}_{\mathbb{P}}(n)$ like $\omega \otimes \mu^*\psi$ for ψ a section of $\mathcal{O}_{\mathbb{P}}(n)$ over $\mathbb{P}$. We define

$$d_\mu(\omega \otimes \mu^*\psi) := d_\mu\omega \otimes \mu^*\psi$$

from which we see the exactness is preserved.

We can repeat the whole argument in the smooth category to instead obtain the locally exact sequence of sheaves

$$0 \longrightarrow \mathcal{E}_{\mathbb{P}} \longrightarrow \mathcal{E}_{\mathbb{F}} \xrightarrow{d_\mu} \mathcal{E}_\mu^1 \xrightarrow{d_\mu} \mathcal{E}_\mu^2 \longrightarrow \cdots \tag{7.4}$$

noting that the sequence does not terminate after forms of degree two since the vertical bundle is a real vector bundle of rank four. For V a smooth vector bundle on $\mathbb{P}$ the coupled version is:

$$0 \longrightarrow \mathcal{E}(V) \longrightarrow \mathcal{E}(\mu^*V) \xrightarrow{d_\mu} \mathcal{E}_\mu^1 \otimes \mathcal{E}(\mu^*V) \xrightarrow{d_\mu} \mathcal{E}_\mu^2 \otimes \mathcal{E}(\mu^*V) \longrightarrow \cdots \tag{7.5}$$

again locally exact. We need to think about the cohomology of (7.5) on the level of sections over an open set $U \subseteq \mathbb{F}$. Define $H^p(\Gamma(U, \mathcal{E}(\mu^*V)))$ to be the pth cohomology of the complex:

$$0 \longrightarrow \Gamma(U, \mathcal{E}(\mu^*V)) \xrightarrow{d_\mu} \Gamma(U, \mathcal{E}_\mu^1 \otimes \mathcal{E}(\mu^*V)) \xrightarrow{d_\mu} \Gamma(U, \mathcal{E}_\mu^2 \otimes \mathcal{E}(\mu^*V)) \longrightarrow \cdots$$

.

Theorem 7.6 (Abstract de Rham theorem for the relative de Rham sequence)**.** *Let U be a connected open set. There is an isomorphism of cohomology groups*

$$H^p(\Gamma(U, \mathcal{E}(\mu^*V))) \cong \check{H}^p(U, \mathcal{E}(V)) \ \forall p$$

Proof. We will only need the result in the case of first cohomology groups so we will only prove that here. We calculate Čech cohomology with respect to a Leray cover. Consider the following diagram:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & C^1(U, \mathcal{E}(V)) & \hookrightarrow & C^1(U, \mathcal{E}(\mu^*V)) & \xrightarrow{d_\mu} & C^1(U, \mathcal{Z}^1_\mu(V)) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & C^0(U, \mathcal{E}(V)) & \hookrightarrow & C^0(U, \mathcal{E}(\mu^*V)) & \xrightarrow{d_\mu} & C^0(U, \mathcal{Z}^1_\mu(V)) & \longrightarrow & 0 \end{array}$$

where $\mathcal{Z}^1_\mu(V) := \ker d_\mu : \mathcal{E}^1_\mu \otimes \mathcal{E}(\mu^*V) \to \mathcal{E}^2_\mu \otimes \mathcal{E}(\mu^*V)$ and the horizontal maps are given by applying d_μ to each cochain in the natural way. The horizontal maps give short exact sequences by the local exactness of the relative de Rham complex. This induces a long exact sequence on Čech cohomology:

$$0 \to \Gamma(U, \mathcal{E}(V)) \xrightarrow{d_\mu} \Gamma(U, \mathcal{E}(\mu^*V)) \xrightarrow{d_\mu} \Gamma(U, \mathcal{Z}^1_\mu(V)) \to \check{H}^1(U, \mathcal{E}(V)) \to 0$$

where the Čech cohomology in degree zero is just sections over U since U is connected and we have used the fact $\check{H}^1(U, \mathcal{E}(\mu^*V)) = 0$ since $\mathcal{E}(\mu^*V)$ admits partitions of unity. The map with codomain $\check{H}^1(U, \mathcal{E}(V))$ must be a surjection by exactness and so the first isomorphism theorem says $\check{H}^1(U, \mathcal{E}(V))$ must be the quotient of $\Gamma(U, \mathcal{Z}^1_\mu(V))$ by the image of $\Gamma(U, \mathcal{E}(\mu^*V))$ which is precisely $H^1(\Gamma(U, \mathcal{E}(\mu^*V))$. The general case can be obtained by breaking the double complex up into more exact sequences. □

Later we will need a technical lemma that depends on this theorem:

Lemma 7.7. *Suppose that the fibres of $U \to \mu(U)$ are all simply connected. Then*

$$\check{H}^1(U, \mathcal{E}(V)) = 0.$$

Proof. By theorem 7.6 we need only check the sequence

$$\Gamma(U, \mathcal{E}(\mu^*V)) \xrightarrow{d_\mu} \Gamma(U, \mathcal{E}^1_\mu \otimes \mathcal{E}(\mu^*V)) \xrightarrow{d_\mu} \Gamma(U, \mathcal{E}^2_\mu \otimes \mathcal{E}(\mu^*V)) \tag{7.8}$$

is exact.

By (2.3) we can think of $\mathbb{F}$ as a $\mathbb{CP}^2$ bundle over $\mathbb{P}$ so given any $p \in \mu(U)$ (and using the fact μ is an open mapping) we can find a neighbourhood $N \subseteq \mu(U)$ of p such that $\mu^{-1}(N) \cong N \times \mathbb{CP}^2$. Shrinking N further if necessary we can also assume V is trivial over N. Let $k = \text{rank}(V)$ and $N' = \mu^{-1}(N) \cap U$. Since the vector bundle is trivial we can think of sections as $\mathbb{R}^k$-valued functions and hence

$$\check{H}^1(N', \mathcal{E}(\mu^*V)) \cong \check{H}^1(N', \mathcal{E}_{\mathbb{F}})^k. \tag{7.9}$$

So to calculate the cohomology on the left we only need to calculate the first Čech cohomology with values in pull-backs of smooth *functions* on N. Again by theorem 7.6 the cohomology on the right vanishes if and only if

$$\Gamma(N', \mathcal{E}_{\mathbb{F}}) \xrightarrow{d_\mu} \Gamma(N', \mathcal{E}^1_\mu) \xrightarrow{d_\mu} \Gamma(N', \mathcal{E}^2_\mu) \tag{7.10}$$

is exact. We can interpret any relative 1-form $\omega \in \Gamma(N', \mathcal{E}^1_\mu)$ as a family of genuine one forms on open subsets $\mathbb{CP}^2$ varying as we move around the base. Write $\omega(q)$ for the 1-form defined on the submanifold $(\{q\} \times \mathbb{CP}^2) \cap U$ this way. Suppose that $d_\mu \omega = 0$. Then $\omega(q)$ corresponds to a *closed* 1-form on $\mathbb{CP}^2$.

Define a smooth function on N' by

$$f(q,s) = \int_{(q,s_0(q))}^{(q,s)} w(q)$$

where we are identifying a point in $N' \subseteq \mu^{-1}(N)$ with $(q,s) \in N \times \mathbb{CP}^2$. Here the integral is taken along any oriented curve with endpoints $(q, q(s_0))$ and (q,s) lying in N' with fixed q (so this makes sense as a line integral in $\mathbb{CP}^2$) and s_0 varies with q smoothly in such a way that $(q, s_0(q))$ lies in N'. This last fact depends on $\mu : N' \to N$ admitting a smooth section but this is a surjective submersion and so we can find one, shrinking N if necessary. Lastly, this function is well-defined since the preimage of q under $U \to \mu(U)$ is simply connected by assumption and $\omega(p)$ is a closed 1-form; the integral is path independent.

By construction (noting that d_μ differentiates only in the fibre) we have

$$d_\mu f = \omega$$

and hence (7.11) is exact and both sides of (7.9) vanish. This time applying theorem 7.6 to the left hand side of (7.9) we see that

$$\Gamma(N', \mathcal{E}(\mu^* V)) \xrightarrow{d_\mu} \Gamma(N', \mathcal{E}^1_\mu \otimes \mathcal{E}(\mu^* V)) \xrightarrow{d_\mu} \Gamma(N', \mathcal{E}^2_\mu \otimes \mathcal{E}(\mu^* V)) \tag{7.11}$$

is exact, which is reminiscent of what we are looking for in the exactness of (7.8) except only locally.

We have just shown there exists an open covering $\{N_\alpha\}$ of $\mu(U)$ such that (7.11) is exact with $N' = N'_\alpha$ for each α. Take $\omega \in \Gamma(U, \mathcal{E}^1_\mu \otimes \mathcal{E}(\mu^* V))$ such that $d_\mu \omega = 0$. Exactness means we can find $f_\alpha \in \Gamma(N'_\alpha, \mathcal{E}^1_\mu \otimes \mathcal{E}(\mu^* V))$ satisfying $d_\mu f_\alpha = \omega$ in N'_α. We have $\bigcup_\alpha N'_\alpha = U$. Define a partition of unity $\{\rho_\alpha\}$ with respect to the open cover $\{N_\alpha\}$ then $\{\mu^* \varphi_\alpha\}$ is a partition of with respect to the open cover $\{N'_\alpha\}$ for U. Note that $d_\mu \mu^* \varphi_\alpha = 0$ since the pull-back is constant in the fibres. Then

$$d_\mu\Big(\sum_\alpha (\mu^* \varphi_\alpha) f_\alpha\Big) = \sum_\alpha \mu^* \varphi_\alpha d_\mu f_\alpha = \omega.$$

and so (7.8) is exact as desired. □

Note that this was really a general statement about surjective submersions between smooth manifolds rather than the particular complex manifolds we are concerned with. [5] has a generalisation for higher cohomology groups, given higher cohomology groups of the fibres vanish.

8. Pulling back cohomology

Retain our hypothesis on $U \subseteq \mathbb{F}$, namely that the mapping $\mu : U \to \mu(U)$ has simply connected fibres.

Let us pull the (exact) Dolbeault sequence of sheaves on $\mu(U)$ with values in a holomorphic vector bundle $V \to \mathbb{P}$ back over U by taking the topological pull-backs of the sheaves. We claim we get the sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(V) \xrightarrow{\bar\partial} \mathcal{E}^{0,0}(V) \xrightarrow{\bar\partial} \mathcal{E}^{0,1}(V) \xrightarrow{\bar\partial} \cdots \tag{8.1}$$

over U. Here $\bar{\partial}$ makes sense for sections of the topological-bull pack since the sections are locally pull-backs of sections of V, which agree on overlaps; $\bar{\partial}$ is defined by applying it to these local sections and then pulling-back. For any open $W \subseteq U$, there is a canonical homomorphism of $\mathcal{O}_{\mathbb{P}}$-modules

$$\mu^* : \Gamma(\mu(W), \mathcal{E}^{0,k}(V)) \to \Gamma(W, \mathcal{E}^{0,k}(V))$$

given by pulling back sections. From the definition of topological pull-back, if the fibres of $\mu : W \to \mu(W)$ are connected, then this gives an isomorphism. In particular since we can find an arbitrarily small neighbourhood of any point in U with connected fibres we see that (8.1) inherits the local exactness of the Dolbeault complex on $\mu(U)$.

Now since $\mu : U \to \mu(U)$ has connected fibres, there is an induced isomorphism on cohomology at the level of sections:

$$\mu^* : H^p(\Gamma(\mu(U), \mathcal{E}(V))) \to H^p(\Gamma(U, \mathcal{E}(V))). \tag{8.2}$$

Similar to the proof of the abstract de Rham theorem we consider the diagram:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & C^1(U, \mathcal{O}_{\mathbb{P}}(V)) & \hookrightarrow & C^1(U, \mathcal{E}(V)) & \xrightarrow{\bar{\partial}} & C^1(U, \mathcal{Z}^{0,1}(V)) & \xrightarrow{\bar{\partial}} & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & C^0(U, \mathcal{O}_{\mathbb{P}}(V)) & \hookrightarrow & C^0(U, \mathcal{E}(V)) & \xrightarrow{\bar{\partial}} & C^0(U, \mathcal{Z}^{0,1}(V)) & \xrightarrow{\bar{\partial}} & 0 \end{array}$$

where $\mathcal{Z}^{0,1}(V) := \ker \bar{\partial} : \mathcal{E}^{0,1}(V) \to \mathcal{E}^{0,2}(V)$. The exactness of the pull-back complex of sheaves (8.1) implies that the rows are exact. This induces a long exact sequence on cohomology:

$$0 \to \Gamma(U, \mathcal{O}_{\mathbb{P}}(V)) \to \Gamma(U, \mathcal{E}(V)) \to \Gamma(U, \mathcal{Z}^{0,1}(V)) \to \check{H}^1(U, \mathcal{O}_{\mathbb{P}}(V)) \to 0$$

where the Čech cohomology in degree zero is just sections over U since U is connected and we have set $\check{H}^1(U, \mathcal{E}(V)) = 0$ by lemma 7.7 since U is simply connected. We immediately obtain:

$$H^1(\Gamma(U, \mathcal{E}(V))) \cong \check{H}^1(U, \mathcal{O}_{\mathbb{P}}(V)).$$

Putting this and (8.2) together with the Dolbeault theorem 6.24 we have constructed the "cohomological pull-back" part of the Penrose transform, the following composition of isomorphisms:

$$\check{H}^1(\mu(U), \mathcal{O}_{\mathbb{P}}(V)) \to H^1(\Gamma(\mu(U), \mathcal{E}(V))) \xrightarrow{\mu^*} H^1(\Gamma(U, \mathcal{E}(V))) \to \check{H}^1(U, \mathcal{O}_{\mathbb{P}}(V))$$

We therefore have $\check{H}^1(\mu(U), \mathcal{O}_{\mathbb{P}}(V)) \cong \check{H}^1(U, \mathcal{O}_{\mathbb{P}}(V))$ and can calculate the Čech cohomology of the sheaf on the base as the Čech cohomology of the topological pull-back.

See [5] for a generalisation of this isomorphism for higher cohomology (with further hypotheses on the fibres $\mu : U \to \mu(U)$).

9. The Penrose transform

We have established a relation between the $\mathcal{O}_{\mathbb{P}}(V)$ valued Čech cohomology on $\mathbb{P}$ and the $\mathcal{O}_{\mathbb{P}}(V)$ valued Čech cohomology on $\mathbb{F}$. In some sense the final step in constructing the Penrose transform at a theoretical level (before thinking about interpretations from physics) is to relate this latter cohomology $\check{H}^p(U, \mathcal{O}_{\mathbb{P}}(V))$ to solutions of differential equations on $\mathbb{M}$. This will involve the *holomorphic* relative de Rham sequence constructed in §7. Here V will be one of the canonical-line bundles on $\mathbb{P}$ introduced in section §4.

From now on we will take $U = \mathbb{F}^I \cong \mathbb{M}^I \times \mathbb{CP}^1$. In this case the fibres of the projection μ are simply connected and so this open set meets the hypotheses of §8.

In what follows $n > 0$ is a positive integer that will have a natural interpretation in the next section. The first thing to note is that since the following diagram commutes

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
C^1(\mathbb{F}^I, \mathcal{O}_{\mathbb{P}}(-n-2)) & \hookrightarrow & C^1(\mathbb{F}^I, \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)) & \xrightarrow{d_\mu} & C^1(\mathbb{F}^I, \Omega^1_\mu \otimes \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)) & \xrightarrow{d_\mu} & \\
\uparrow & & \uparrow & & \uparrow & & \\
C^0(\mathbb{F}^I, \mathcal{O}_{\mathbb{P}}(-n-2)) & \hookrightarrow & C^0(\mathbb{F}^I, \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)) & \xrightarrow{d_\mu} & C^0(\mathbb{F}^I, \Omega^1_\mu \otimes \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)) & \xrightarrow{d_\mu} &
\end{array}$$

the relative de Rham operator

$$d_\mu : \mu^*\mathcal{O}_{\mathbb{P}}(-n-2) \to \Omega^1_\mu \otimes \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)$$

induces

$$d_\mu : \check{H}^1(\mathbb{F}^I, \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)) \to \check{H}^1(\mathbb{F}^I, \Omega^1_\mu \otimes \mu^*\mathcal{O}_{\mathbb{P}}(-n-2))$$

by applying d_μ to cocycles. Furthermore, the rows of the double complex are exact (choose a Leray cover and use local exactness of the relative de Rham complex to see this). Taking the corresponding long exact sequence of cohomology, the following is exact:

$$\cdots \longrightarrow \Gamma(\mathbb{F}^I, Z) \longrightarrow \check{H}^1(\mathbb{F}^I, \mathcal{O}_{\mathbb{P}}(-n-2)) \longrightarrow \check{H}^1(\mathbb{F}^I, \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)) \xrightarrow{d_\mu} \ldots$$

Where Z is the sheaf of d_μ closed sections of $\Omega^1_\mu \otimes \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)$. So by exactness there is a canonical homomorphism

$$\check{H}^1(\mathbb{F}^I, \mathcal{O}_{\mathbb{P}}(-n-2)) \to \ker\{d_\mu : \check{H}^1(\mathbb{F}^I, \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)) \to \check{H}^1(\mathbb{F}^I, \Omega^1_\mu \otimes \mu^*\mathcal{O}_{\mathbb{P}}(-n-2))\}.$$

In the next section we will show that $\Gamma(\mathbb{F}^I, \Omega^1_\mu \otimes \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)) = 0$ and hence $\Gamma(\mathbb{F}^I, Z) = 0$ so, this map is in fact an isomorphism. Let us take this as a given for now.

We define a vector bundle $V_n \to \mathbb{M}^I$ as follows, we define the fibre at $x \in \mathbb{M}^I$ to be

$$(V_n)_x := \check{H}^1(\nu^{-1}(x), \mu^*\mathcal{O}_{\mathbb{P}}(-n-2))$$

that this is a well-defined holomorphic vector bundle follows from the construction of direct images of sheaves [24]. Recall that each fibre $\nu^{-1}(x) \cong \mathbb{CP}^1$.

Proposition 9.1. *There is an isomorphism*

$$\check{H}^1(\mathbb{F}^I, \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)) \to \Gamma(\mathbb{M}^I, V_n)$$

Proof. Consider the cover for $\mathbb{F}^I \cong \mathbb{M}^I \times \mathbb{CP}^1$, namely $\mathbb{M}^I \times U_0$, $\mathbb{M}^- \times U_1$. Each of these sets is biholomorphic to $\mathbb{C}^4$ so this gives a Leray cover as per theorem 6.21 and we can calculate Čech cohomology using this. A 1-cocyle $c_{01} \in \check{H}^1(\mathbb{F}^1, \mu^*\mathcal{O}_{\mathbb{P}}(-n-2))$ is simply a section of $\mu^*\mathcal{O}_{\mathbb{P}}(-n-2)$ defined over $\mathbb{M}^I \times (U_0 \cap U_1)$. So the way to define a section $s \in \Gamma(\mathbb{M}^I, V_n)$ is to define the value of s at x to be the cocycle c_{01} restricted to $\nu^{-1}(x) \cong \{x\} \times (U_0 \cap U_1)$. That this map descends to cohomology comes about because a coboundary $b_0 - b_1 \in \check{H}^1(\mathbb{F}^1, \mu^*\mathcal{O}_{\mathbb{P}}(-n-2))$ defines a coboundary in each $\{x\} \times (U_0 \cap U_1)$, again by restriction. The inverse of this construction is to patch a section of $\mu^*\mathcal{O}_{\mathbb{P}}(-n-2)$ over $\mathbb{M}^I \times (U_0 \cap U_1)$ together from cocycles in each $\nu^{-1}(x)$. □

Define V_n^α by

$$(V_n^\alpha)_x = \check{H}^1(\nu^{-1}(x), \Omega^1_\mu \otimes \mu^*\mathcal{O}_{\mathbb{P}}(-n-2))$$

an analogous argument shows there is an isomorphism

$$\check{H}^1(\mathbb{F}^I, \Omega^1_\mu \otimes \mu^*\mathcal{O}_\mathbb{P}(-n-2)) \to \Gamma(\mathbb{M}^I, V_n^\alpha).$$

We put this together in a diagram

$$\begin{array}{ccccc} \check{H}^1(\mathbb{F}^I, \mathcal{O}_\mathbb{P}(-n-2)) & \longrightarrow & \check{H}^1(\mathbb{F}^I, \mu^*\mathcal{O}_\mathbb{P}(-n-2)) & \xrightarrow{d_\mu} & \check{H}^1(\mathbb{F}^I, \Omega^1_\mu \otimes \mu^*\mathcal{O}_\mathbb{P}(-n-2)) \\ & & \downarrow & & \downarrow \\ & & \Gamma(\mathbb{M}^I, V_n) & \overset{\nabla}{\dashrightarrow} & \Gamma(\mathbb{M}^I, V_n^\alpha) \end{array} \tag{9.2}$$

where the vertical maps are the isomorphisms given by restricting cocycles. The dotted line is the differential operator induced by d_μ and we will make sense of the suggestive notation in the next section. Nonetheless since we know

$$\check{H}^1(\mathbb{F}^I, \mathcal{O}_\mathbb{P}(-n-2)) = \ker d_\mu$$

we have an isomorphism

$$\check{H}^1(\mathbb{F}^I, \mathcal{O}_\mathbb{P}(-n-2)) \cong \ker \nabla : \Gamma(\mathbb{M}^I, V_n) \to \Gamma(\mathbb{M}^I, V_n^\alpha)$$

and using the results of §8 can identify the left hand side with the cohomology on $\mathbb{P}$:

$$\check{H}^1(\mathbb{P}^I, \mathcal{O}_\mathbb{P}(-n-2)) \cong \ker \nabla : \Gamma(\mathbb{M}^I, V_n) \to \Gamma(\mathbb{M}^I, V_n^\alpha).$$

We have achieved in finding a correspondence between cohomology on $\mathbb{P}$ and solutions of certain differential equations defined on vector bundles over $\mathbb{M}$. The problem is these bundles: V_n, V_n^α as of yet do not have an obvious description in terms of the geometry on $\mathbb{M}$. Their interpretation will be the subject of the next section.

10. Massless fields

It is time to remedy the conspicuous absence of spinors in our discussion about cohomology. We will interpret the "differential equations on $\mathbb{M}$" of the previous section in terms of spinors.

First let us interpret the sheaf of relative 1-forms on $\mathbb{F}$ as a pull-back of a sheaf on $\mathbb{M}$.

Proposition 10.1. *There is a canonical isomorphism of sheaves*

$$\Omega^1_\mu \cong \nu^* S^*[-1]' \otimes \mu^*\mathcal{O}_\mathbb{P}(1).$$

Proof. Recall Ω^1_μ was defined as the dual of $\ker D\mu$. Using our preferred trivialisation (2.4) the tangent bundle of $\mathbb{F}^I$ splits $T\mathbb{F}^I \cong \nu^*T\mathbb{M}^I \oplus \mathrm{pr}_2{}^* T\mathbb{CP}^1$. From (2.5) we see the kernel of $D\mu$ is contained in the $\nu^*T\mathbb{M}^I \cong \nu^*(S' \otimes S)$ summand and $(V^{AA'}, 0) \in T\mathbb{F}^I$ is annihilated by $D\mu$ if and only if $V^{AA'}\pi_{A'} = 0$ where $V^{AA'}$ is a tangent vector at $(X^{AA'}, [\pi_{A'}]) \in \mathbb{F}^I$. One may readily check a degenerate 2×2 matrix is a decomposable element of the tensor product. We therefore have an identification of $\ker D\mu$ at $(X^{AA'}, [\pi_{A'}])$ with spinors of the form $\phi^A\pi^{A'}$ where $\phi^A \in \nu^*S[1]' \otimes \mu^*\mathcal{O}_\mathbb{P}(-1)$ (having raised an index on $\pi_{A'}$ using the canonical skew-form introduced in §3). Taking the inverse transpose of $V^{AA'} \mapsto \phi^A$ gives the desired isomorphism over $\mathbb{F}^I$ (where we will need it). This isomorphism is in fact well-defined over all of $\mathbb{F}$, see [22]. □

This proposition means we can prove a statement we used without proof in the previous section.

Corollary 10.2. $\Gamma(\mathbb{F}^I, \Omega^1_\mu \otimes \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)) = 0$.

Proof. By the above proposition

$$\Gamma(\mathbb{F}^I, \Omega^1_\mu \otimes \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)) \cong \Gamma(\mathbb{F}^I, \nu^*S^*[-1]' \otimes \mu^*\mathcal{O}_{\mathbb{P}}(-n-1)).$$

Suppose there is a non-vanishing section of $\nu^*S^*[-1]' \otimes \mu^*\mathcal{O}_{\mathbb{P}}(-n-1)$ over $\mathbb{F}^I$. For some $x \in \mathbb{M}$ the restriction to the fibre $\nu^{-1}(x)$ is therefore non-vanishing. The spin bundle factor is obviously trivial when restricted $\nu^{-1}(x)$ since it is a pull-back by ν while $\mu^*\mathcal{O}_{\mathbb{P}}(-n-1)$ restricted to $\nu^{-1}(x) \cong \mathbb{CP}^1$ can be canonically identified with the bundle $\mathcal{O}_{\nu^{-1}(x)}(-n-1)$, (to see this note that $\mu^*\mathcal{O}_{\mathbb{P}}(-1)$ has fibres over $\nu^{-1}(x)$ which are the 1-dimensional subspaces of $\nu^{-1}(x)$). The spin bundle factor being trivial implies we obtain a non-vanishing section of $\mathcal{O}_{\nu^{-1}(x)}(-n-1)$. Corollary 4.2 is precisely the statement there are no global sections of this bundle (recall n is non-negative.) We therefore have a contradiction. $\square$

Theorem 10.3. *There are vector bundle isomorphisms for $n > 0$:*

$$V_n \cong \odot^n S'^*[-1]'$$

and

$$V_n^\alpha \cong S'^* \otimes \odot^{n-1} S'^*[-2]'.$$

Proof. Recall the dualised Euler exact sequence (6.10) of sheaves. Also recall from the discussion in §2 that for each $x \in \mathbb{M}$ we have $\nu^{-1}(x) = \mathbb{P}S'^*_x$. It turns out the correct thing to do is rewrite (6.10) with $V = S'^*_x$ and taking the tensor product with the trivial line bundle $\wedge^2 S'^*_x$ over $\mathbb{P}S'^*_x$:

$$0 \to \Omega^1_{\mathbb{P}S'^*_x} \otimes \wedge^2 S'^*_x \to \mathcal{O}_{\mathbb{P}S'^*_x}(-1) \otimes S'^*_x \to \wedge^2 S'^*_x \to 0. \tag{10.4}$$

Here we have paired indices to effect an isomorphism:

$$\mathcal{O}_{\mathbb{P}S'^*_x}(-1) \otimes S'_x \otimes \wedge^2 S'^*_x \to \mathcal{O}_{\mathbb{P}S'^*_x}(-1) \otimes S'^*_x.$$

With this identification the map $\mathcal{O}_{\mathbb{P}S'^*_x}(-1) \otimes S'^*_x \to \wedge^2 S'^*_x$ is given by wedging with the Euler vector field $\pi_{A'}$. Rather than try to directly interpret $\Omega^1_{\mathbb{P}S'^*_x} \otimes \wedge^2 S'^*_x$ it is best to think along similar lines to four discussion of the Euler sequence (6.9). We think of sections of the sheaf $\mathcal{O}_{\mathbb{P}S'^*_x}(-1) \otimes S'^*_x$ as vector fields on S'^*_x of homogeneity -1 and wedging with the Euler vector field $\pi_{A'}$ produces a bivector field of homogeneity 0. The kernel of this map is precisely homogeneity -1 vector fields which are holomorphic multiples of $\pi_{A'}$. We have an obvious map from functions on S'^*_x of homogeneity -2 to vector fields on S'^*_x of homogeneity -1 given by multiplication with the Euler vector field. This means we have an exact sequence of sheaves over $\mathbb{P}S'^*_x$:

$$0 \to \mathcal{O}_{\mathbb{P}S'^*_x}(-2) \to \mathcal{O}_{\mathbb{P}S'^*_x}(-1) \otimes S'^*_x \to \wedge^2 S'^*_x \to 0. \tag{10.5}$$

More generally we will need the exact sequence of sheaves over $\mathbb{P}S'^*_x$ for $n \geq 0$

$$0 \to \mathcal{O}_{\mathbb{P}S'^*_x}(-n-2) \to \mathcal{O}_{\mathbb{P}S'^*_x}(-1) \otimes \odot^{n+1} S'^*_x \to \odot^n S'^*_x[-1]' \to 0. \tag{10.6}$$

The surjective map is given by contraction with $\pi^{C'} := \epsilon^{C'B'}\pi_{B'}$. In abstract indices:

$$\phi_{A'\dots B'C'} \mapsto \phi_{A'\dots B'C'}\pi^{C'}$$

To see this is a surjective morphism of sheaves we need only to check this locally. Since $\pi^{A'}$ is non-vanishing, locally we can find $\tau_{A'}$ such that $\tau_{A'}\pi^{A'} = 1$. We claim that given the symmetric spinor $\psi_{A'_1A'_2\dots A'_n}$ there exists $\phi_{A'_1A'_2\dots A'_{n+1}}$ such that $\phi_{A'_1A'_2\dots A'_{n+1}}\pi^{A'_{n+1}} = \psi_{A'_1A'_2\dots A'_n}$. First note that this is true for $\psi_{A'_1A'_2\dots A'_n} = \tau_{A'_1}\tau_{A'_2}\dots\tau_{A'_n}$ since

$$\tau_{A'_1}\tau_{A'_2}\dots\tau_{A'_n}\tau_{A'_{n+1}}\pi^{A'_{n+1}} = \tau_{A'_1}\tau_{A'_2}\dots\tau_{A'_n}.$$

Then calculate

$$\frac{n+1}{n-k+1}\chi_{(A'_1A'_2\ldots A'_k}\tau_{A'_{k+1}}\ldots\tau_{A'_{n+1})}\pi^{A'_{n+1}}$$
$$= \chi_{(A'_1A'_2\ldots A'_k}\tau_{A'_{k+1}}\ldots\tau_{A'_n)} + \frac{k}{n-k+1}\pi^{A'_{n+1}}\chi_{A'_{n+1}(A'_1\ldots A'_{k-1}}\tau_{A'_k}\ldots\tau_{A'_n)}$$

Now the term on the right contains $n-k+1$ factors of $\tau_{A'}$ and so by induction is $\phi_{A'_1\ldots A'_{n+1}}\pi^{A'_{n+1}}$ for some symmetric spinor $\phi_{A'_1\ldots A'_{n+1}}$ from which we may write

$$\left(\frac{n+1}{n-k+1}\chi_{(A'_1A'_2\ldots A'_k}\tau_{A'_{k+1}}\ldots\tau_{A'_{n+1})} + \phi_{A'_1\ldots A'_{n+1}}\right)\pi^{A'_{n+1}} = \chi_{(A'_1A'_2\ldots A'_k}\tau_{A'_{k+1}}\ldots\tau_{A'_n)}$$

and so terms with $n-k$ factors of $\tau_{A'}$ are in the range, showing the result by descending induction on k.

The injective map from functions of homogeneity $-n-2$ is given by multiplication by the $(n+1)$th symmetric power of π_A. Exactness follows immediately because the image of this map is clearly annihilated by contracting with π_A and the kernel must be rank 1, since $\mathcal{O}_{\mathbb{P}S'^*_x}(-1)\otimes\odot^{n+1}S'^*_x$ and $\odot^n S'^*_x[-1]'$ are of rank $n+2$ and $n+1$ respectively.

We now take the long exact sequence of Čech cohomology associated with (10.6). Part of this is:

$$\cdots \longrightarrow \Gamma(\mathbb{P}S'^*_x, \mathcal{O}_{\mathbb{P}S'^*_x}(-1)\otimes\odot^{n+1}S'^*_x)) \longrightarrow \Gamma(\mathbb{P}S'^*_x, \odot^n S'^*_x[-1]') \xrightarrow{\delta^*}$$

(10.7)
$$\check{H}^1(\mathbb{P}S'^*_x, \mathcal{O}_{\mathbb{P}S'^*_x}(-n-2)) \longrightarrow \check{H}^1(\mathbb{P}S'^*_x, \mathcal{O}_{\mathbb{P}S'^*_x}(-1)\otimes\odot^{n+1}S'^*_x) \longrightarrow \cdots$$

Now since the bundle $\odot^n S'^*_x$ is trivial over $\mathbb{P}S'^*_x$ we have

$$\Gamma(\mathbb{P}S'^*_x, \mathcal{O}_{\mathbb{P}S'^*_x}(-1)\otimes\odot^{n+1}S'^*_x)) \cong \Gamma(\mathbb{P}S'^*_x, \mathcal{O}_{\mathbb{P}S'^*_x}(-1))^{n+2}$$

and similarly

$$\check{H}^1(\mathbb{P}S'^*_x, \mathcal{O}_{\mathbb{P}S'^*_x}(-1)\otimes\odot^n S'^*_x) \cong \check{H}^1(\mathbb{P}S'^*_x, \mathcal{O}_{\mathbb{P}S'^*_x}(-1))^{n+1}$$

and from example 6.22 the modules on the right vanish and so we obtain for $n \geq 0$

(10.8)
$$(\delta^*)^{-1} : \check{H}^1(\mathbb{P}S'^*_x, \mathcal{O}_{\mathbb{P}S'^*_x}(-n-2)) \cong \Gamma(\mathbb{P}S'^*_x, \odot^n S'^*_x[-1]').$$

Now global sections of the trivial bundle $\odot^n S'^*_x[-1]'$ over $\mathbb{P}S'^*_x$ are necessarily constant by Liouville's theorem and so $\Gamma(\mathbb{P}S'^*_x, \odot^n S'^*_x[-1]')$ can be canonically identified with the fibre of the bundle $\odot^n S'^*[-1]'$ over $\mathbb{M}$ at x. The left hand side of (10.8) is the fibre of V_n at x. This implies applying $(\delta^*)^{-1}$ fibrewise gives an isomorphism of vector bundles:

$$V_n \cong \odot^n S'^*[-1]'.$$

Next, proposition (10.1) shows

$$\check{H}^1(\nu^{-1}(x), \Omega^1_\mu \otimes \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)) \cong \check{H}^1(\mathbb{P}S'^*_x, S^*_x[-1]'\otimes\mathcal{O}_{\mathbb{P}S'^*_x}(-n-1))$$

and since $S^*_x[-1]'$ is a trivial bundle over $\mathbb{P}S'^*_x$ we may simply couple it with (10.8) for $n > 0$ which yields

$$(\delta^*)^{-1} : \check{H}^1(\mathbb{P}S'^*_x, S^*_x[-1]'\otimes\mathcal{O}_{\mathbb{P}S'^*_x}(-n-1)) \cong \Gamma(\mathbb{P}S'^*_x, S^*_x\otimes\odot^{n-1}S'^*_x[-2]')$$

and hence there is a vector bundle isomorphism

$$V^\alpha_n \cong S^*\otimes\odot^{n-1}S'^*[-2]'$$

which completes the proof of the theorem. □

Recall from (9.2) that d_μ induced a differential operator

$$\nabla : \Gamma(\mathbb{M}^I, V_n) \to \Gamma(\mathbb{M}^I, V_n^\alpha)$$

and hence using $(\delta^*)^{-1}$ to make the identifications as above, a differential operator on spinors

$$\nabla : \Gamma(\mathbb{M}^I, \odot^n S'^*[-1]') \to \Gamma(\mathbb{M}^I, S^* \otimes \odot^{n-1} S'^*[-2]').$$

We will interpret this in terms of the flat connection $\nabla_{AA'}$ over $\mathbb{M}^I$ induced by our preferred trivialisations for the spin bundles over $\mathbb{M}^I$ (see §2, §3). This pulls back to a differential operator on the pull-back of spin bundles to $\mathbb{F}^I$ given by differentiation in the base direction with respect to the trivialisation $\mathbb{F}^I \cong \mathbb{M}^I \times \mathbb{CP}^1$.

Proposition 10.9. *Let $f \in \Gamma(U, \mu^* \mathcal{O}_{\mathbb{P}}(-n-2))$ where $U \subseteq \mathbb{F}^I$. Then (in components with respect to our identification in proposition 10.1)*

$$(d_\mu f)_A = \pi^{A'} \nabla_{AA'} f.$$

Proof. Recall from the proof of (10.1) that $\ker D\mu$ is a subbundle of the pull-back of $\nu^* T\mathbb{M}^I$ consisting of elements of the form $\phi^A \pi^{A'}$ for $\phi^A \in \nu^* S^*[-1]' \otimes \mu^* \mathcal{O}_{\mathbb{P}}(1)$. Now $d_\mu f(v)$ for $v \in \ker D\mu$ may be calculated by taking the derivative of f in the base direction then contracting with v. So in indices this means:

$$(d_\mu f)_A \phi^A = \phi^A \pi^{A'} \nabla_{AA'} f$$

according to our identification (10.1). □

Theorem 10.10.

$$\nabla : \Gamma(\mathbb{M}^I, \odot^n S'^*[-1]') \to \Gamma(\mathbb{M}^I, S^* \otimes \odot^{n-1} S'^*[-2]').$$

is given by

$$\phi_{A'B'\ldots C'} \mapsto \nabla_A{}^{A'} \phi_{A'\ldots C'}.$$

Proof. We compute Čech cohomology using the standard Leray cover $\mathbb{M}^I \times U_0$, $\mathbb{M}^I \times U_1$. From (9.2) ∇ may be computed as $(\delta^*)^{-1} \circ d_\mu \circ \delta^*$ so we need to make sense of the connecting homomorphism δ^* that comes from the long exact sequence. Translating 6.23 into our setting, (10.7) this is given by (applying δ^* fibrewise):

$$\delta^* : \phi_{A'B'\ldots C'} \mapsto [f] \in \check{H}^1(\mathbb{F}^I, \mu^* \mathcal{O}_{\mathbb{P}}(-n-2))$$

where $f \underbrace{\pi_{A'}\pi_{B'}\ldots\pi_{C'}\pi_{D'}}_{n+1 \text{ factors}} = \psi^{(1)}_{A'B'\ldots C'D'} - \psi^{(0)}_{A'B'\ldots C'D'}$ with $\psi^{(i)}_{A'B'\ldots C'D'} \pi^{D'} = \phi_{A'B'\ldots C'}$ on $\mathbb{M}^I \times U_i$.

Now d_μ in (9.2) is given by applying d_μ on cocycles and we know how to write d_μ in terms of spinor indices from proposition 10.9:

$$d_\mu : [f] \mapsto [\pi^{A'} \nabla_{AA'} f] \in \check{H}^1(\mathbb{F}^I, S^*[-1]' \otimes \mu^* \mathcal{O}_{\mathbb{P}}(-n-1))$$

So we want to undo the steps and obtain $(\delta^*)^{-1}([\pi^{A'} \nabla_{AA'} f])$ in terms of a differential operator acting on $\phi_{A'B'\ldots C'}$. Well since $\check{H}^1(\mathbb{F}^I, S^* \otimes \odot^n S'^*[-1]' \otimes \mathcal{O}_{\mathbb{P}}(-1))$ vanishes (example 6.22 applied fibrewise) we may write

$$(\pi^{A'} \nabla_{AA'} f) \underbrace{\pi_{B'}\ldots\pi_{C'}\pi_{D'}}_{n \text{ factors}} = \tau^{(0)}_{AB'C'\ldots D'} - \tau^{(1)}_{AB'C'\ldots D'} \tag{10.11}$$

and by the definition of the connecting homomorphism, $(\delta^*)^{-1}([\pi^{A'} \nabla_{AA'} f])$ will be given by

$$\tau^{(i)}_{AB'\ldots C'D'} \pi^{A'},$$

on $\mathbb{M}^I \times U_i$, so we need to solve for these. Start by rewriting the left hand side

$$(10.12)\qquad \begin{aligned}(\pi^{A'}\nabla_{AA'}f)\pi_{B'}...\pi_{C'}\pi_{D'} &= -\nabla_A{}^{A'}(f\pi_{A'}\pi_{B'}...\pi_{C'}\pi_D)\\ &= -\nabla_A{}^{A'}(\psi^{(1)}_{A'B'...C'D'} - \psi^{(0)}_{A'B'...C'D'})\end{aligned}$$

where we have used the fact the Euler vector field is parallel with respect to $\nabla_{AA'}$ (its components do not depend on the base coordinates in the trivialisation $\mathbb{F}^I \cong \mathbb{M}^I \times \mathbb{CP}^1$). Then comparing (10.11) and (10.12):

$$(\nabla_A{}^{A'}\psi^{(1)}_{A'B'...C'D'} - \tau^{(1)}_{AB'...C'D'}) - (\nabla_A{}^{A'}\psi^{(0)}_{A'B'...C'D'} - \tau^{(0)}_{AB'...C'D'}) = 0$$

so we have agreement on overlaps: a global section defined on $\mathbb{F}^I$ given by

$$\nabla_A{}^{A'}\psi^{(i)}_{A'B'...C'D'} - \tau^{(i)}_{AB'...C'D'}$$

on $\mathbb{M}^I \times U_i$. However as we have used before, there are no-global sections of homogeneity -1 which implies $\nabla_A{}^{A'}\psi^{(i)}_{A'B'...C'D'} - \tau^{(i)}_{AB'...C'D'} = 0$. Lastly we contract with $\pi^{A'}$:

$$\tau^{(i)}_{AB'...C'D'}\pi^{A'} = \nabla_A{}^{A'}(\psi^{(i)}_{A'B'...C'D'})\pi^{A'} = \nabla_A{}^{A'}(\psi^{(i)}_{A'B'...C'D'}\pi^{A'}) = \nabla_A{}^{A'}\phi_{A'B'...C'}.$$

on $\mathbb{M}^I \times U_i$, which shows $(\delta^*)^{-1}([\pi^{A'}\nabla_{AA'}f]) = \nabla_A{}^{A'}\phi_{A'B'...C'}$. ☐

Corollary 10.13. *There is an isomorphism:*

$$(10.14)\qquad \check{H}^1(\mathbb{P}^I, \mathcal{O}_{\mathbb{P}}(-n-2)) \cong \{\phi_{A'...B'} \in \Gamma(\mathbb{M}^I, \odot^n S'^*[-1]') \mid \nabla_A{}^{A'}\phi_{A'...B'} = 0\}$$

The module on the right is interpreted as the space of zero-rest-mass fields of helicity $n/2$ holomorphic on $\mathbb{M}^I$. Solutions to the obvious analogue:

$$\nabla^A{}_{A'}\phi_{A...B} = 0$$

(n-indices) are called zero-rest-mass fields of helicity $-n/2$. There is a similar procedure for obtaining the correspondence for these fields, with some modifications to be made: One works with *potentials* for fields instead of the fields themselves. See [22] or [16] for the construction.

There are distinguished subsets of $\mathbb{M}^I$ that are of particular interest. There is a hermitian metric Φ on $\mathbb{T}$ given by

$$\Phi : \begin{bmatrix}\omega^A\\ \pi_{A'}\end{bmatrix} \otimes \begin{bmatrix}\tau^A\\ \kappa_{A'}\end{bmatrix} \mapsto \begin{bmatrix}\bar{\omega}^{A'} & \bar{\pi}_A\end{bmatrix}\begin{bmatrix}0 & \delta_{A'}{}^{B'}\\ \delta_A{}^B & 0\end{bmatrix}\begin{bmatrix}\tau^B\\ \kappa_{B'}\end{bmatrix}$$

where $\bar{\omega}^{\mathsf{A}'} := \overline{\omega^{\mathsf{A}}}$ and $\bar{\pi}_{\mathsf{A}} := \overline{\pi_{\mathsf{A}'}}$. We define $\mathbb{M}^+, \mathbb{M}^-, M^0$ to be the space of planes on which the Hermitian metric is positive definite, negative definite, and null respectively ($x \in \mathbb{M}$ is null to say $\Phi(Z,Z) = 0\ \forall Z \in x$). That $\mathbb{M}^+, \mathbb{M}^-$ are open subsets of $\mathbb{M}^I$ can be seen by noting that $\mathbb{M}^I$ consists precisely of subspaces with no non-vanishing vectors of the form:

$$X = \begin{bmatrix}\omega^A\\ 0\end{bmatrix}$$

and every $x \notin \mathbb{M}^I$ therefore contains a non-vanishing vector of this form but $\Phi(X,X) = 0$. To interpret $M^0 \cap \mathbb{M}^I$ we compute Φ on $\mathbb{M}^I$ as follows:

$$\begin{aligned}\Phi : \begin{bmatrix}X^{AA'}\pi_{A'}\\ \pi_{A'}\end{bmatrix} \otimes \begin{bmatrix}X^{AA'}\pi_{A'}\\ \pi_{A'}\end{bmatrix} &\mapsto \begin{bmatrix}-i\bar{\pi}_A\bar{X}^{AA'} & \bar{\pi}_A\end{bmatrix}\begin{bmatrix}0 & \delta_{A'}{}^{B'}\\ \delta_A{}^B & 0\end{bmatrix}\begin{bmatrix}iX^{BB'}\pi_{B'}\\ \pi_{B'}\end{bmatrix}\\ &= \bar{\pi}_A Y^{AA'}\pi_{A'}.\end{aligned}$$

where $Y^{AA'}$ is the anti-hermitian part of $X^{AA'}$. Noting that the identification (2.1) identifies $\mathbb{R}^4 \hookrightarrow \mathbb{C}^4$ with the set of Hermitian matrices, we see that Φ vanishes on the image of $\mathbb{R}^4$ under our parametrisation (2.2) for $\mathbb{M}^I$ and so can identify $M^0 \cap \mathbb{M}^I$ as real Minkowski space.

Given a holomorphic solution on $\mathbb{M}^I$ produced by (10.13) we can restrict it to $M^0 \cap \mathbb{M}^I$ to obtain a real analytic solution of the zero-rest-mass field equations on a region of real Minkowski space. $M^0 \cap \mathbb{M}^I$ forms part of the boundary of both $\mathbb{M}^+$ and $\mathbb{M}^-$. Solutions on $M^0 \cap \mathbb{M}^I$ that are boundary values of solutions on $\mathbb{M}^+$ and $\mathbb{M}^-$ have interpretations in physics as positive and negative frequency fields respectively.

Obtaining just real analytic solutions is unsatisfactory from the point of view of physics. Certainly not all physical solutions of the zero-rest-mass equations, a hyperbolic PDE, are real analytic, let alone smooth. The equations make sense for a broad class of *generalised functions*, like distributions. In [25] the author constructs the Penrose transform for a broad class of generalised functions, and this construction agrees with the transform as constructed in §9 and §10. All this involves machinery from analysis (e.g. distribution theory) which is certainly beyond the scope of this essay.

11. A historical note and further directions

The notion that massless fields could be represented by sheaf cohomology classes on $\mathbb{P}$ with values in $\mathcal{O}_{\mathbb{P}}(n)$ first appears (in handwritten notes) by Penrose [19]. This link was further developed in Eastwood et. al. [9] in which the authors present the Penrose transform as a purely sheaf-theoretic construction without explicit reference to any contour integral formulae. Generalisations of the Penrose transform to settings in which the correspondence space $\mathbb{F} = F_{1,2}(\mathbb{C}^4)$ is replaced by a different flag or indeed, where $\mathbb{F}$ is replaced by the quotient $G/(P \cap Q)$ of a semi-simple complex Lie group G by the intersection of two *parabolic* subgroups P, Q was the subject of Baston and Eastwood [4] and this work leant heavily into representation theory. In particular, to obtain correspondences between higher cohomology on G/P (standing in for $\mathbb{P}$) and solutions to differential equations on G/Q (standing in for $\mathbb{M}$) the relative de Rham sequence (§7) is replaced by *Bernstein-Gelfand-Gelfand* sequences. These sequences are differential complexes that turn out to correspond to *Bernstein-Gelfand-Gelfand sequences* of quotients of *Verma modules* from Lie algebra representation theory. The generalisation of the various canonical vector bundles are homogeneous vector bundles on G/P and G/Q. If at times the treatment of the Penrose transform is this essay is a little ad-hoc, this text is highly systematic.

Throughout this essay we computed using the flat Levi-Civita connection on open subsets of $\mathbb{M}$. One may ask what occurs if we replaced $\mathbb{M}$ with an arbitrary complex 4-manifold M with a conformal class of holomorphic metrics (that is, a metric defined up to holomorphic rescaling). The Penrose transform can be recovered in this setting, for example in [8], with the space of null-geodesics on M playing the role of $\mathbb{P}$.

Simultaneously taking both avenues for generalising the Penrose transform leads into the field of *parabolic geometry*, surveyed in [6] which is the study of spaces which are curved analogues of homogeneous spaces G/P where G is a semi-simple Lie group and P is a parabolic subgroup.

Penrose's twistor programme had its sights set on a complete geometric reformulation of fundamental physics that would assist in unifying general relativity and quantum mechanics. Proponents of twistor theory agree it is, thus far, a long way off achieving this lofty goal [2]. Successes of twistor theory in mathematics have been found in the study of the integrability approach to solving non-linear partial differential equations. See [7] for an account. More recently, successes in physics have come from applications to calculating scattering amplitudes in string theory [1].

Appendix A. Proof of conformal change of scale formulae

Proof (of proposition 3.1). Pick connections $\nabla_{AA'}$ on S^*, S'^* over U that annihilate the scales. By using the Leibniz rule, we calculate the freedom to change the connection on $T^*\mathbb{M} \cong S'^* \otimes S^*$ by changing the two connections on S^*, S'^* in such a way that the new pair of connections on the spin bundles still annihilate the scales. The new connection is given by

$$\hat{\nabla}_{AA'}V_{BB'} = \nabla_{AA'}V_{BB'} + \Delta_{AA'B}{}^{C}V_{CB'} + \Gamma_{AA'B'}{}^{C'}V_{BC'}.$$

where $\Delta_{AA'(BC)} = \Delta_{AA'BC}$ and $\Gamma_{AA'(B'C')} = \Gamma_{AA'B'C'}$. The change in torsion is calculated by anti-symmetrising over the interchange of pairs AA', BB' (equivalent to anti-symmetrising over spatial indices). The torsion $\hat{\tau}_{AA'BB'}{}^{CC'}$ of the induced connection is given by:

$$\hat{\tau}_{AA'BB'}{}^{CC'} = \tau_{AA'BB'}{}^{CC'} + \Delta_{AA'B}{}^{C}\delta^{C'}_{B'} + \Gamma_{AA'B'}{}^{C'}\delta^{C}_{B} - \Delta_{BB'A}{}^{C}\delta^{C'}_{A'} - \Gamma_{BB'A'}{}^{C'}\delta^{C}_{A}.$$

Now consider the bundle homomorphism: $(T^*\mathbb{M}\otimes\odot^2 S^*)\oplus(T^*\mathbb{M}\otimes\odot^2 S'^*) \to \wedge^2 T^*\mathbb{M}\otimes T\mathbb{M}$ given by:

$$(\Delta_{AA'BC}, \Gamma_{AAB'C'}) \mapsto \Delta_{AA'B}{}^{C}\delta^{C'}_{B'} + \Gamma_{AA'B'}{}^{C'}\delta^{C}_{B} - \Delta_{BB'A}{}^{C}\delta^{C'}_{A'} - \Gamma_{BB'A'}{}^{C'}\delta^{C}_{A}.$$

We claim this is an isomorphism and hence there is a unique choice of $\Delta_{AA'BC}$ and $\Gamma_{AAB'C'}$ such that the torsion of the new connection vanishes. The rank of the domain and codomain are both 24 and so it is sufficient to show this is injective (the dimension of the nth-symmetric power of the spin bundle is $n+1$).

$$\begin{aligned} &\Delta_{AA'B}{}^{C}\delta^{C'}_{B'} + \Gamma_{AA'B'}{}^{C'}\delta^{C}_{B} - \Delta_{BB'A}{}^{C}\delta^{C'}_{A'} - \Gamma_{BB'A'}{}^{C'}\delta^{C}_{A} = 0 \\ \iff\ &\Delta_{AA'BC}\epsilon_{C'B'} + \Gamma_{AA'B'C'}\epsilon_{CB} - \Delta_{BB'AC}\epsilon_{C'A'} - \Gamma_{BB'A'C'}\epsilon_{CA} = 0 \end{aligned} \tag{A.1}$$

Given this, symmetrising on ABC then contracting with $\epsilon^{B'C'}$ yields:

$$\Delta_{(A|A'|BC)} = 0.$$

On the other hand symmetrising (A.1) on BC then contracting with $\epsilon^{B'C'}$ yields:

$$2\Delta_{AA'BC} - \Delta_{(B|A'A|C)} = 0$$

and anti-symmetrising on AB we obtain:

$$\Delta_{[A|A'|B]C} = 0.$$

We have shown the irreducible components of $\Delta_{AA'BC}$ are vanishing. More specifically we can decompose:

$$\Delta_{AA'BC} = \Delta_{(A|A'|BC)} + \frac{2}{3}\Delta_{[A|A'|B]C} + \frac{2}{3}\Delta_{[A|A'|C]B} = 0.$$

Similarly we can show $\Gamma_{AAB'C'} = 0$ and so the map is injective. This completes the part of the proof showing existence and uniqueness.

It is left to show that the formulae (3.2), (3.3) define connections which annihilate $\epsilon_{AB}, \epsilon_{A'B'}$ and furthermore induce a torsion-free connection on $T\mathbb{M}$. We have, from the Leibniz rule:

$$\begin{aligned} \hat{\nabla}_{AA'}\hat{\epsilon}_{BC} &= \nabla_{AA'}\hat{\epsilon}_{BC} - \Upsilon_{[B|A'}\hat{\epsilon}_{A|C]} - \tilde{\Upsilon}_{[B|A'}\hat{\epsilon}_{A|C]} - \frac{1}{2}\Upsilon_{AA'}\epsilon_{BC} + \frac{1}{2}\tilde{\Upsilon}_{AA'}\epsilon_{BC} \\ &= (\nabla_{AA'}\Omega)\epsilon_{BC} - (\nabla_{[B|A'}\Omega)\epsilon_{A|C]} - (\nabla_{[B|A'}\tilde{\Omega})\epsilon_{A|C]} - \frac{1}{2}(\nabla_{AA'}\Omega)\epsilon_{BC} + \frac{1}{2}(\nabla_{AA'}\tilde{\Omega})\epsilon_{BC} \\ &= 0 \end{aligned}$$

where in the last line we have used the fact the second last line vanishes after contracting with ϵ^{BC} and contracting with ϵ^{BC} gives an isomorphism $T^*\mathbb{M} \otimes \wedge^2 S^* \to T^*\mathbb{M}$. Similarly one can show the connection on S'^* defined by (3.3) annihilates $\epsilon_{B'C'}$.

Lastly, using the Leibniz rule we calculate the induced connection on $T^*\mathbb{M}$:

$$\hat{\nabla}_{AA'} V_{BB'} = \nabla_{AA'} V_{BB'} - \frac{1}{2}\Upsilon_{AB'} V_{BA'} - \frac{1}{2}\tilde{\Upsilon}_{AB'} V_{BA'} - \frac{1}{2}\Upsilon_{BA'} V_{AB'} - \frac{1}{2}\tilde{\Upsilon}_{BA'} V_{AB'}.$$

The change in connection is evidently symmetric under interchange of AA' and BB' and therefore $\nabla_{AA'}$ is torsion-free. □

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