Notes on Spherical Bessel Functions

Spherical Bessel functions play an important role in scattering theory. They obey the equation

$$\frac{d^2y_l}{dx^2} + \frac{2}{x}\frac{dy_l}{dx} + \left(1 - \frac{l(l+1)}{x^2}\right) y_l = 0 \tag{1}$$

The solutions are denoted as $j_l(x)$ and $n_l(x)$. In this note, we derive some of their properties.

Before we proceed, I should stress that spherical Bessel functions are not the same thing as Bessel functions, which are usually denoted as $J_\nu(x)$ and $N_\nu(x)$. However, they are related; you can check that the function $\sqrt{x}\, j_l(x)$ and $\sqrt{x}\, n_l(x)$ obey the Bessel equation.

Recursive Solution

Let $y_l(x) = x^l\, Y_l(x)$. It is straightforward to show that this new function Y_l obeys

$$\frac{d^2Y_l}{dx^2} + \frac{2(l+1)}{x}\frac{dY_l}{dx} + Y_l = 0 \tag{2}$$

It's simple to solve this for low l. First notice that although we're ultimately we're interested in $l = 0, 1, 2, \ldots$ the equation also makes sense for $l = -1$ where the two solutions are simply

$$Y_{-1}(x) = \cos x \quad \text{and} \quad Y_{-1}(x) = \sin x$$

It is not very much harder to solve for $l = 0$ where the two solutions are

$$Y_0(x) = \frac{\sin x}{x} \quad \text{and} \quad Y_0(x) = -\frac{\cos x}{x}$$

where the overall minus sign is by convention. For higher l, we can solve recursively. To do this, we first differentiate (2) again to get

$$\frac{d^3Y_l}{dx^3} + \frac{2(l+1)}{x}\frac{d^2Y_l}{dx^2} - \frac{2(l+1)}{x^2}\frac{dY_l}{dx} + Y_l = 0$$

After dividing by $1/x$, we can write this as

$$\frac{d^2}{dx^2}\left(\frac{1}{x}\frac{dY_l}{dx}\right) + \frac{2(l+2)}{x}\frac{d}{dx}\left(\frac{1}{x}\frac{dY_l}{dx}\right) + \frac{1}{x}\frac{dY_l}{dx} = 0$$

which is the same equation as (2), but with $(l+1)$ replaced by $(l+2)$. This means that we can take

$$Y_l = -\frac{1}{x}\frac{dY_{l-1}}{dx} = \left(-\frac{1}{x}\frac{d}{dx}\right)^l Y_0$$

where, once again, the choice of minus sign is by convention. Putting all this together, we arrive at two recursive solutions to the spherical Bessel equation given by $y_l(x) = j_l(x)$ and $y_l(x) = n_l(x)$ where

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x} \quad \text{and} \quad n_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x}$$

Asymptotic Behaviour

For scattering problems we usually need the asymptotic behaviour of these functions, both at $x \to \infty$ and $x \to 0$. We start with large x. Here, the spherical Bessel functions are largest if the d/dx factors keep hitting the trigonometric $\sin x$ and $\cos x$ factors, leaving us with a term which scales as $1/x$ at large distances. Specifically, we have

$$j_l(x) \;\to\; \begin{cases} (-1)^{l/2}\,\sin x/x & l \text{ even} \\ -(-1)^{(l-1)/2}\,\cos x/x & l \text{ odd} \end{cases}$$

and

$$n_l(x) \;\to\; \begin{cases} -(-1)^{l/2}\,\cos x/x & l \text{ even} \\ -(-1)^{(l-1)/2}\,\sin x/x & l \text{ odd} \end{cases}$$

We can combine these to write

$$j_l(x) \to \frac{\sin(x - (l\pi/2))}{x} \quad \text{and} \quad n_l(x) \to -\frac{\cos(x - (l\pi/2))}{x} \quad \text{as} \quad x \to \infty$$

To see the small x behaviour of $j_l(x)$, we Taylor expand

$$\frac{\sin x}{x} = \frac{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots}{x}$$

After hitting this with $(\frac{1}{x}\frac{d}{dx})^l$, the leading order piece will come from the $\frac{(-1)^l}{(2l+1)!}x^{2l}$ term. The differentiation will pull down a factor $2l(2l-2)(2l-4)\ldots$. The upshot is that at small x we have

$$j_l(x) \approx \frac{x^l}{1 \cdot 3 \cdot 5 \ldots (2l+1)}$$

Meanwhile for $n_l(x)$, we have

$$\frac{\cos x}{x} = \frac{1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \dots}{x}$$

This time the leading term comes from repeatedly differentiating the $1/x$ piece. We have

$$\left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{1}{x} = (-1)^l \, \frac{1 \cdot 3 \cdot 5 \dots (2l-1)}{x^{2l+1}}$$

This means that the solution $n_l(x)$ diverges at the origin, and is given by

$$n_l(x) \to -\frac{1.3.5.\dots(2l-1)}{x^{l+1}} \quad \text{as} \quad x \to 0$$

where the numerator is simply 1 when $l = 0$.