Classical Dynamics: Example Sheet 1

1. Consider a system with n dynamical degrees of freedom q^a, $a = 1, \ldots, n$. The most general form for a purely kinetic Lagrangian is

$$L = \tfrac{1}{2}\, g_{ab}(q_c)\, \dot{q}^a \dot{q}^b \tag{1}$$

where we use the summation convention in which all repeated indices are summed over. The functions $g_{ab} = g_{ba}$ depend on all the generalised coordinates. Assume that $\det(g_{ab}) \neq 0$ so that the inverse matrix g^{ab} exists ($g^{ab} g_{bc} = \delta^a{}_c$). Show that Lagrange's equations for this system are given by,

$$\ddot{q}^a + \Gamma^a_{bc} \dot{q}^b \dot{q}^c = 0 \tag{2}$$

where

$$\Gamma^a_{bc} = \tfrac{1}{2}\, g^{ad} \left(\frac{\partial g_{bd}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^d} \right)$$

Side Remark: The functions g_{ab} define a *metric* on the configuration space, and the equations (2) are known as the *geodesic* equations. They appear naturally in general relativity where they describe a particle moving in curved spacetime. Lagrangians of the form (1) also appear in many other areas of physics, including the study of solids, the theory of nuclear forces and string theory. In these contexts, the systems are referred to as *sigma models*.

2. A particle moves in one-dimension with position x and potential $V(x)$, governed by the Lagrangian,

$$L = \frac{1}{12} m^2 \dot{x}^4 + m \dot{x}^2 V - V^2$$

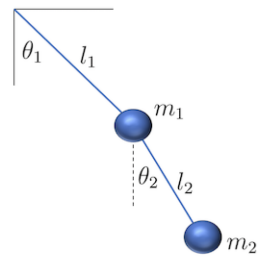
Show that the resulting equation of motion is identical to that arising from the more traditional $L = \frac{1}{2} m \dot{x}^2 - V$.

3. The Lagrangian for a relativistic point particle of mass m is,

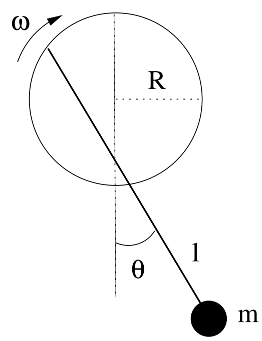
$$L = -mc^2 \sqrt{1 - (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})/c^2} - V(\mathbf{r})$$

where c is the speed of light. Derive the equation of motion. Show that it reduces to Newton's equation in the limit $|\dot{\mathbf{r}}| \ll c$.

4. A double pendulum is drawn below. Two light rods of lengths l_1 and l_2 oscillate in the same plane. Attached to them are masses m_1 and m_2. How many degrees of freedom does the system have? Write down the Lagrangian describing its dynamics.



5. The pivot of a simple pendulum is attached to a disc of radius R, which rotates in the plane of the pendulum with angular velocity ω. (See the diagram below). Write down the Lagrangian and derive the equations of motion for dynamical variable θ.



6. The motion of an electron of mass m and charge $(-e)$ moving in a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{r})$ is described by the Lagrangian

$$L = \tfrac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - e \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r})$$

Show that Lagrange's equation reproduces the Lorentz force law on the electron.

i) Work in cylindrical polar coordinates (r, θ, z) and consider the vector potential

$$\mathbf{A} = (0, f(r)/r, 0)$$

At some initial time the electron is a distance r_0 from the z-axis and has velocity in the (r, z)-plane. Show that its angular velocity about the z-axis is given by,

$$\dot{\theta} = \frac{e}{mr^2} \left[f(r) - f(r_0) \right]$$

ii) Again working in cylindrical polar coordinates, consider the vector potential

$$\mathbf{A} = (0, rg(z), 0)$$

where $g(z) > 0$. Obtain two constants of motion. Show that if the electron is projected from a point (r_0, θ_0, z_0) with velocity $\dot{r} = \dot{z} = 0$ and $\dot{\theta} = 2eg(z_0)/m$, then it will describe a circular orbit provided that $g'(z_0) = 0$. Show that these orbits are stable to shifts along the z axis if t$g'' > 0$.

7. A particle of mass m_1 is restricted to move on a circle of radius R_1 in the plane $z = 0$, with center at $(x, y) = (0, 0)$. A second particle of mass m_2 is restricted to move on a circle of radius R_2 in the plane $z = c$ with center at $(x, y) = (0, a)$. The two particles are connected by a spring resulting in the potential

$$V = \tfrac{1}{2}\omega^2 d^2$$

where d is the distance between the particles. Identify the two generalised coordinates and write down the Lagrangian of the system. Show that when the circles lie directly beneath each other, $a = 0$, then there is an extra conserved quantity.

8. Two particles of mass m are connected by a light rope of length l. One particle sits on a smooth horizontal table at a distance r from a hole through which the rope is threaded. The second particle hangs straight beneath the hole.

i) Assume the second particle hangs straight beneath the hole. Write down the Lagrangian of the system in terms of r and an angle ψ that the first particle makes with respect to a fixed axis. Identify the ignorable coordinate. Write down the equation of motion for the remaining coordinate assuming the rope remains taught.

ii) Now let the second particle oscillate beneath the table as a spherical pendulum. How many degrees of freedom does the system now have? Write down the Lagrangian describing the motion assuming the rope remains taught at all times. How many ignorable coordinates are there?

9. The linear triatomic molecule drawn in figure 1 consists of two identical outer atoms of mass m and a middle atom of mass M. It is a rough approximation to CO_2. The interactions between neighbouring atoms are governed by a complicated potential $V(x_i - x_{i+1})$. If we restrict attention to motion in the x direction parallel to the molecule, the Lagrangian is

$$L = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}M\dot{x}_2^2 + \tfrac{1}{2}m\dot{x}_3^2 - V(x_1 - x_2) - V(x_2 - x_3) \tag{3}$$

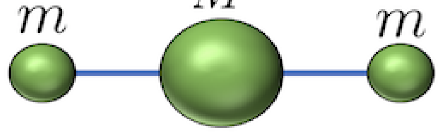


Figure 1: The linear triatomic molecule

where x_i is the position of the i^{th} particle. Define the equilibrium separation $r^0 = |x_i - x_{i+1}|$ of this system. Write down the equation describing small deviations from equilibrium in terms of the masses and the quantity

$$k = \left.\frac{\partial^2 V(r)}{\partial r^2}\right|_{r=r^0} \tag{4}$$

Show that the system has three normal modes and calculate the frequencies of oscillation of the system. One of these frequencies vanishes: what is the interpretation of this?

10. A pendulum consists of a mass m at the end of light rod of length l. The pivot of the pendulum is attached to a mass M which is free to slide without friction along a horizontal rail. Take the generalised coordinates to be the position x of the pivot and the angle θ that the pendulum makes with the vertical.

a. Write down the Lagrangian and derive the equations of motion.

b. Find the non-zero frequency of small oscillations around the stable equillibrium.

c. Now suppose a force acts on the the mass M causing it to travel with constant acceleration a in the positive x direction. Find the equilibrium angle θ of the pendulum.

11. Two equal masses m are connected to each other and to fixed points by three identical springs of force constant k as shown in figure 2. Write down the equations describing motion of the system in the direction parallel to the springs. Find the normal modes and their frequencies.

Suppose now that there are N equal masses joined by $N + 1$ springs with fixed end points. Write down the equations of motion in matrix form. Find the normal mode

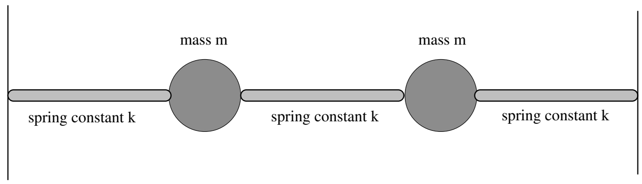


Figure 2: It's remarkably hard to draw curly springs on a computer.

frequencies. (**Hint:** To find the normal mode frequencies, you could first try the easier problem with "periodic boundary conditions" in which all masses lie on a circle with the first and last masses are identified)