Electrodynamics: Example Sheet 2

1. An infinite straight wire lies along the z-axis, and for $t < 0$ there is no current or field. For $t > 0$ a uniform current I flows in the wire. Show that for $t > 0$ the vector potential $\mathbf{A}(t, x, y) = A\hat{\mathbf{z}}$ in the Lorentz gauge is

$$A = \begin{cases} \frac{\mu_0 I}{2\pi} \log(\theta + \sqrt{\theta^2 - 1}) & \text{for} \quad \theta > 1, \\ 0 & \text{for} \quad \theta \leq 1 \end{cases}$$

where $\theta = ct/r$ and $r = \sqrt{x^2 + y^2}$. Obtain $\mathbf{E}$ and $\mathbf{B}$ and discuss the behaviour of the fields as $t \to \infty$.

2. For a localised charge density $\rho(\mathbf{x})e^{-i\omega t}$ and current density $\mathbf{J}(\mathbf{x})e^{-i\omega t}$, use current conservation to show that

$$\int d^3x \; x_i J_j(\mathbf{x}) = \epsilon_{ijk} m_k - \frac{i\omega}{6} Q'_{ij},$$

where

$$m = \frac{1}{2} \int d^3x \; \mathbf{x} \times \mathbf{J}(\mathbf{x}) \quad \text{and} \quad Q'_{ij} = 3 \int x_i x_j \rho(\mathbf{x}) \, d^3x.$$

Hence show that if

$$\int d^3x \; \rho(\mathbf{x}) = \int d^3x \; \mathbf{x}\rho(\mathbf{x}) = 0$$

then at distances $r \gg c/\omega \gg a$, where a is the extent of the charge and current distribution, the leading contributions to the scalar $\phi(\mathbf{x})e^{-i\omega t}$ and vector potentials $\mathbf{A}(\mathbf{x})e^{-i\omega t}$ are

$$\phi(\mathbf{x}) \approx -\frac{1}{6} \frac{1}{4\pi\varepsilon_0 r} e^{ikr} k^2 \hat{x}_i \hat{x}_j Q'_{ij}$$

and

$$A_i(\mathbf{x}) \approx \frac{\mu_0 k}{4\pi r} e^{ikr} \left[i(\hat{\mathbf{x}} \times \mathbf{m})_i - \frac{\omega}{6} \hat{x}_j Q'_{ij} \right].$$

where $r = |\mathbf{x}|$ and $\hat{\mathbf{x}} = \mathbf{x}/r$ and $k = \omega/c$.

Writing $Q'_{ij} = Q_{ij} + P\delta_{ij}$, where $Q_{ii} = 0$, show that the terms involving P may be removed by a gauge transformation at large distances. These results represent magnetic dipole and electric quadrupole radiation.

3. A small loop of wire lies in a plane with unit normal $\hat{\mathbf{N}}$, and encloses an area S. A current $I_0 \cos(\omega t)$ flows around the loop, with c/ω much larger than the size of the loop. Using results from Question 2, show that in the far-field at displacement $\mathbf{x}$ from the centre of the loop, the magnetic vector potential is

$$\mathbf{A}(t, \mathbf{x}) = \hat{\mathbf{x}} \times \hat{\mathbf{N}} \, \frac{\mu_0 I_0 S \omega}{4\pi r c} \sin(\omega t - kr) + \mathcal{O}\left(\frac{\infty}{\nabla^{\in}}\right),$$

where $k = \omega/c$ and $r = |\mathbf{x}|$.

[*You may use the result* $\oint x_i dx_j = S\epsilon_{ijk} N_k$.]

Find the leading-order magnetic field in the far-field and show that the average radiated power dE/dt is

$$\frac{dE}{dt} = \frac{\mu_0}{12\pi} \frac{S^2 I_0^2 \omega^4}{c^3}.$$

4. Let ϕ be the retarded scalar potential given by

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3y \; \frac{\rho(t_{\mathrm{ret}}, \mathbf{y})}{R}$$

where $R = |\mathbf{x} - \mathbf{y}|$, $t_{\mathrm{ret}} = t - R/c$, and set $\hat{\mathbf{R}} = (\mathbf{x} - \mathbf{y})/R$. Show that

$$\frac{\partial}{\partial t}\phi(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3y \; \frac{\dot{\rho}(t_{\mathrm{ret}}, \mathbf{y})}{R}$$

where $\dot{\rho}(t_{\mathrm{ret}}, \mathbf{y})$ is $\partial\rho(t, \mathbf{y})/\partial t$ evaluated at t_{ret}. Show further that

$$\nabla\phi(t, \mathbf{x}) = -\frac{1}{4\pi\varepsilon_0} \int d^3y \; \hat{\mathbf{R}} \left(\frac{1}{R^2}\rho(t_{\mathrm{ret}}, \mathbf{y}) + \frac{1}{cR}\dot{\rho}(t_{\mathrm{ret}}, \mathbf{y}) \right) .$$

Hence verify, using $\nabla^2(1/R) = -4\pi\delta^{(3)}(\mathbf{x} - \mathbf{y})$, that ϕ satisfies the wave equation:

$$\Box\phi(t, \mathbf{x}) = -\frac{1}{\varepsilon_0}\rho(t, \mathbf{x}).$$

Write down a similar retarded solution for the vector potential $\mathbf{A}$ in terms of the current density $\mathbf{J}$.

Now assume that ρ and $\mathbf{J}$ are non-zero only in a finite region. Setting $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, show that the leading terms in the far-field expansion are

$$\begin{aligned} \mathbf{E}(t, \mathbf{x}) &\approx \frac{\mu_0}{4\pi|\mathbf{x}|} \int d^3y \; \left(\hat{\mathbf{x}} c \dot{\rho}(t_{\mathrm{ret}}, \mathbf{y}) - \dot{\mathbf{J}}(t_{\mathrm{ret}}, \mathbf{y}) \right) \\ &= \frac{\mu_0}{4\pi|\mathbf{x}|} \hat{\mathbf{x}} \times \left(\hat{\mathbf{x}} \times \int d^3y \; \dot{\mathbf{J}}(t_{\mathrm{ret}}, \mathbf{y}) \right) \end{aligned}$$

where conservation of current, integration by parts, and discarding of a surface integral have all been used, and

$$\mathbf{B}(t,\mathbf{x}) \approx -\frac{\mu_0}{4\pi c|\mathbf{x}|}\hat{\mathbf{x}} \times \int d^3y\ \dot{\mathbf{J}}(t_{\rm ret},\mathbf{y}) = \frac{1}{c}\hat{\mathbf{x}} \times \mathbf{E}(t,\mathbf{x}).$$

(Note that these results do not assume the dipole approximation.) Determine the Poynting vector.

[*Hint: when using current conservation, and integration by parts, be careful with the* $\mathbf{y}$*-dependence of* $t_{\rm ret}$*.*)

5. Starting from the power radiated in the electric-dipole approximation, derive Larmor's formula for the rate at which radiation is produced by a non-relativistic particle of charge q moving along a trajectory $\mathbf{x}(t)$.

A non-relativistic particle of mass m, charge q, and energy E is incident along a radial line in a central potential $V(r)$. The potential is vanishingly small for large r, but increases without bound as $r \to 0$. Show that the total amount of energy $\mathcal{E}$ radiated by the particle is

$$\mathcal{E} = \frac{\mu_0 q^2}{3\pi c m^2}\sqrt{\frac{m}{2}}\int_\infty^{r_0} \frac{1}{\sqrt{E - V(r)}}\left(\frac{dV}{dr}\right)^2 dr,$$

where $V(r_0) = E$, assuming $E \ll V$.

Suppose V is a Coulomb potential $V(r) = C/r$. Evaluate $\mathcal{E}$.

6. For a relativistic particle of charge q on a trajectory $y^\mu(\tau)$, where τ is proper time, the current density 4-vector is

$$J^\mu(x) = qc\int d\tau\ \delta^{(4)}(x - y(\tau))\dot{y}^\mu(\tau)$$

with $\dot{y}^\mu \dot{y}_\mu = -c^2$ and $\dot{y}^0 > 0$. Show that the 4-vector potential is given by

$$\begin{aligned} A^\mu(x) &= \frac{\mu_0}{2\pi}\int d^4z\ \Theta(x^0 - z^0)\delta(\eta_{\alpha\beta}(x^\alpha - z^\alpha)(x^\beta - z^\beta))J^\mu(z) \\ &= -\frac{\mu_0 qc}{4\pi}\frac{\dot{y}^\mu(\tau^*)}{R^\nu(\tau^*)\dot{y}_\nu(\tau^*)} \end{aligned}$$

where $R^\nu(\tau) = x^\nu - y^\nu(\tau)$ and τ^* is determined by $R^\mu(\tau^*)R_\mu(\tau^*) = 0$ and $R^0(\tau^*) > 0$.

Verify that the Lorenz gauge condition $\partial_\mu A^\mu = 0$ holds and show that

$$F^{\mu\nu} = -\frac{\mu_0 q c}{4\pi} \frac{1}{(R^\rho \dot{y}_\rho)^2} (R^\mu S^\nu - R^\nu S^\mu),$$

where

$$S^\nu = \ddot{y}^\nu - \frac{\dot{y}^\nu}{R^\rho \dot{y}_\rho}(c^2 + R^\tau \ddot{y}_\tau)$$

and all quantities on the right are evaluated at $\tau_\star$. Check this result for the case of a stationary charge at the origin.