Electromagnetism: Example Sheet 2

1. A constant magnetic field points along the z-axis: $\mathbf{B} = B\hat{\mathbf{z}}$. Verify that each of the following vector potentials satisfies $\mathbf{B} = \nabla \times \mathbf{A}$:

- $\mathbf{A} = xB\hat{\mathbf{y}}$
- $\mathbf{A} = \frac{1}{2}(xB\hat{\mathbf{y}} - yB\hat{\mathbf{x}})$
- In cylindrical polar coordinates, $\mathbf{A} = \frac{1}{2}rB\hat{\boldsymbol{\phi}}$, with $r^2 = x^2 + y^2$
- In spherical polar coordinates, $\mathbf{A} = \frac{1}{2}r\sin\theta\, B\hat{\boldsymbol{\phi}}$, with $r^2 = x^2 + y^2 + z^2$.

2. A cylindrical conductor of radius a, with axis along the z-axis, carries a uniform current density $\mathbf{J} = J\hat{\mathbf{z}}$. Use Ampère's law to show that the magnetic field within the conductor is given, in cylindrical polar coordinates, by

$$\mathbf{B} = \frac{1}{2}\mu_0 J r\hat{\boldsymbol{\phi}}$$

with $r^2 = x^2 + y^2$. [In this question, and the following question, you may assume that the magnetic field inside a conductor is the same as in a vacuum.]

3. A steady current I flows in the z-direction uniformly in the the region between the cylinders $x^2 + y^2 = a^2$ and $(x + d)^2 + y^2 = b^2$, where $0 < d < (b - a)$. Show that the associated magnetic field $\mathbf{B}$ throughout the region $x^2 + y^2 < a^2$ is given by

$$\mathbf{B} = \frac{\mu_0 I d}{2\pi(b^2 - a^2)}\hat{\mathbf{y}}$$

4. Use the Biot-Savart law to determine the magnetic field:

- Around an infinite, straight wire carrying current I.
- At the centre of a square loop of wire, with sides of length a, carrying current I.
- At the point $(0, 0, z)$ above a loop of wire of radius a, lying in the (x, y) plane, with centre at the origin, carrying current I.

5. Explain why the force $\mathbf{F}$ and torque $\boldsymbol{\tau}$ experienced by a loop of wire C carrying current I are given by

$$\mathbf{F} = I \oint_C d\mathbf{r} \times \mathbf{B} \quad \text{and} \quad \boldsymbol{\tau} = I \oint_C \mathbf{r} \times (d\mathbf{r} \times \mathbf{B})$$

A loop of wire lies in a plane whose normal makes an angle θ with a uniform magnetic field. The loop of wire encloses a planar area A and carries current I. Compute the torque.

6. What boundary conditions apply on either side of a surface current $\mathbf{K}$?

A surface current experiences a Lorentz force from the *average* magnetic field on either side of the surface. A wire carrying current I winds N times per unit length around a cylindrical solenoid. Show that there is a force per unit area on the cylinder given by

$$\mathbf{f} = \frac{\mu_0 I^2 N^2}{2} \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is the outward normal.

7 A steady current I_1 flows around a closed loop C_1. Use the Biot-Savart law to show that this exerts a force on a second loop C_2 carrying current I_2, given by

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} d\mathbf{r}_1 \times \left(d\mathbf{r}_2 \times \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \right)$$

Write this in a form which exhibits anti-symmetry, $\mathbf{F}_{12} = -\mathbf{F}_{21}$, in agreement with Newton's third law.