Fluid Dynamics I and II: Questions on Viscous Flows

1. By considering the forces acting on a small rectangular slab of incompressible, viscous fluid undergoing an unsteady parallel shear flow $\mathbf{u} = (u(y,t), 0, 0)$, including a body force $\mathbf{F} = (f_x, f_y, 0)$, show that

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} + f_x \quad \text{and} \quad 0 = -\frac{\partial p}{\partial y} + f_y.$$

2. A film of viscous fluid of uniform thickness h flows steadily under the influence of gravity down a rigid vertical wall. Assume that the surrounding air exerts only a constant pressure on the fluid. Calculate the velocity profile and find the volume flux (per unit width) of fluid down the wall.

3. A long, horizontal, two-dimensional container of uniform depth h, filled with viscous fluid, has stationary, rigid bottom and end walls, and a rigid top that moves with constant velocity $(U, 0)$ in Cartesian coordinates (x, y). Assume that the flow far from the end walls is parallel and steady with components $(u(y), 0)$. Determine $u(y)$ and hence determine the tangential stress exerted by the fluid on each of the top and bottom boundaries. Describe the overall force balance on a section $x_0 < x < x_1$ of the flow.

[*Hint: No penetration through the end walls requires the volume flux across any vertical cross-section of the flow to be zero, which determines the horizontal pressure gradient.*]

4. A two-dimensional, semi-infinite layer of viscous fluid lies above a rigid boundary at $y = 0$ that oscillates in its own plane with velocity $(U_0 \cos \omega t, 0)$. There is no applied pressure gradient and the fluid flows parallel to the boundary with velocity $(u(y,t), 0)$. By writing $u(y,t) = \mathrm{Re}\,[U_0 f(y) e^{i\omega t}]$ (where Re means "real part" not "Reynolds number"!), show that

$$f(y) = \exp\left[-(1+i)\sqrt{\frac{\omega}{2\nu}}\, y\right]$$

and hence that the velocity decays away from the boundary over a characteristic length-scale $\sqrt{\nu/\omega}$. Sketch the velocity profile at $t = 0$ and $\omega t = \pi/2$.

Calculate the shear stress on the boundary and hence calculate the mean rate of doing work (per unit area) by the boundary.

5. An infinite layer of viscous fluid of depth h is initially stationary and has a stationary rigid upper boundary, while its rigid lower boundary is set into parallel motion with constant speed U at time $t = 0$. Write down the equation, the initial condition and the boundary conditions satisfied by the subsequent flow $(u(y,t), 0)$. What is the steady flow $u_\infty(y)$ that is established after a long time? By writing $u(y,t) = u_\infty(y) - \hat{u}(y,t)$ and using separation of variables, determine a series solution for the transient flow $\hat{u}$. Show that the shear stress exerted by the fluid on the boundary at $y = 0$ is divergent as $t \to 0^+$ but that it is subsequently finite and tends to $-\mu U/h$ as $t \to \infty$.

6. For a steady shear flow $\mathbf{u} = (\gamma y, 0, 0)$, where γ is constant, show that the Navier-Stokes equations are satisfied if the pressure is uniform and the body force vanishes. If this shear is maintained in a fluid of dynamic viscosity μ flowing between two plates at $y = 0$ and $y = h$, write down the full stress tensor and use it explicitly to find the forces exerted by the fluid on each of the plates. Calculate the rate of work per unit area needed to be exerted on the top plate to maintain the flow and show that it is equal to the rate of dissipation per unit area internal to the flow.

[*Pay close attention to the direction of the normal vectors on each plate.*]

7. Show that, for the flow $\mathbf{u}$ of an incompressible, viscous fluid in a region V enclosed by a stationary rigid boundary,

$$\int_V \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\, dV = 0.$$

Hence show that the total rate of dissipation of energy $D = 2\mu \int_V E_{ij}E_{ij}\, dV$ can be written as

$$D = \mu \int_V \omega^2\, dV, \quad \text{where} \quad \boldsymbol{\omega} = \nabla \times \mathbf{u}.$$

Why does it follow that if the flow is irrotational then there is no dissipation?

8. A layer of incompressible fluid of density ρ and dynamic viscosity μ flows steadily down a plane inclined at an angle θ to the horizontal, forming a uniform layer of thickness h parallel to the plane. A second layer of fluid, of uniform thickness αh, viscosity $\beta\mu$ and density ρ flows steadily on top of the first layer. Using Cartesian coordinates perpendicular and parallel to the plane, write down the equations of motion in each layer, the boundary conditions on the plane and on the top free surface, and the boundary conditions at the interface between the two layers.

Find the pressure, shear-stress and velocity fields in each layer. Why does the velocity profile in the bottom layer depend on α but not β?

Show that the volume flux (per unit cross-slope width) is

$$\frac{\rho g h^3 \sin\theta}{3\mu}\left(1+\frac{3\alpha}{2}\right) \qquad \text{and} \qquad \frac{\rho g(\alpha h)^3 \sin\theta}{3\beta\mu}\left(1+\frac{3\beta}{2}\frac{1+2\alpha}{\alpha^2}\right)$$

in the lower and upper layers respectively. Discuss the limits (a) $\alpha \ll 1$, (b) $\beta \ll 1$. [*Recall that the flux for a single layer of thickness h is $\rho g h^3 \sin\theta/3\mu$.*]

9. An incompressible fluid of dynamic viscosity μ flows steadily through a cylindrical tube parallel to the z-axis with velocity $\mathbf{u} = (0, 0, w(x, y))$, under a uniform pressure gradient $G = -dP/dz$. Show that the Navier-Stokes equations with no body force are satisfied provided

$$\nabla^2 w = -G/\mu,$$

and state the appropriate boundary conditions.

Find w for a tube with an elliptical cross-section with semi-axes a and b. [Hint: consider the function $f(x, y) = (1 - x^2/a^2 - y^2/b^2)$ and recall the uniqueness of the solution to Poisson's equation with Dirichlet boundary conditions.] Show that the volume flux (i.e. the volume of fluid passing through any section of the tube per unit time) is given by

$$Q = \frac{\pi a^3 b^3 G}{4(a^2 + b^2)\mu}.$$

Now consider the circular case with $a = b$ (so-called Poiseuille flow). Show that the viscous stress on the boundary, $\sigma_{rz} = \mu \partial w/\partial r$, exerts a force that exactly balances the pressure difference exerted across the ends. Further, calculate the dissipation within the tube and show that it is equal to the rate of working against the pressure difference across the ends.

10. Viscous fluid flows with steady velocity $\mathbf{u} = (0, v(r), 0)$ between two infinitely-long, coaxial cylinders $r = a$ and b ($> a$). The inner cylinder rotates with steady angular velocity Ω about its axis, while the outer cylinder is at rest. The pressure varies only in the radial direction. Using the Navier-Stokes equations in cylindrical polar coordinates, show that

$$v(r) = Ar + B/r,$$

where the constants A and B are to be determined. Calculate the torque per unit length that must be applied to the inner cylinder to maintain the motion; check the dimensions and the sign of your result. [In polar coordinates (r, ϕ), the component $E_{r\phi}$ of the strain-rate tensor is given by $2E_{r\phi} = r\partial(v/r)/\partial r$ for this flow.]

11. The plane rigid boundary of a semi-infinite domain of a viscous fluid oscillates in its own plane with velocity $U_0 \cos\omega t$. The fluid is at rest at infinity. Find the velocity field. [Hint: use complex notation by writing $\cos\omega t$ as the real part of $e^{i\omega t}$.] Show that the time-averaged rate of dissipation of energy in the fluid is

$$\frac{1}{2}\rho U_0^2 \left(\frac{1}{2}\nu\omega\right)^{1/2}$$

per unit area of the boundary. Verify that this is equal to the time average of the rate of work of the boundary on the fluid (per unit area).

12. A viscous fluid of kinematic viscosity ν and density ρ is confined between a fixed plate at $y = h$ and a plate at $y = 0$ whose velocity is $(U_0 \cos\omega t, 0, 0)$, where U_0 is a constant. There is no body force and the pressure is independent of x. Explain the physical significance of the dimensionless number $S = \omega h^2/\nu$.

Assuming that the flow remains time-periodic and unidirectional, find expressions for the flow profile and the time-average rate of working Φ per unit area by the plates on the fluid. [Hint: use complex notation and the functions sinh and cosh].

Sketch the velocity profile and evaluate Φ in the limits $S \ll 1$ and $S \to \infty$, and explain why in these limits Φ becomes independent of ω and h respectively.

13. Suppose that the tube in Question 9 has as its cross-section the sector of a circle $r < a$, $|\theta| < \beta$ in plane polar coordinates (r, θ). Show that the momentum equation has solution

$$w(r,\theta) = \frac{Gr^2}{4\mu}\left(\frac{\cos 2\theta}{\cos 2\beta} - 1\right) + \sum_{n=0}^{\infty} A_n r^{\lambda_n} \cos\lambda_n\theta,$$

where $\lambda_n = (2n+1)\pi/2\beta$ and the coefficients A_n are to be found. Determine the asymptotic behaviour of the flow near $r = 0$ [*Hint: distinguish the cases $\beta < \pi/4$ and $\beta > \pi/4$.*] Under what circumstances is the flow near $r = 0$ independent of the boundary $r = a$?

14. Starting from the Navier-Stokes equations for incompressible viscous flow with conservative forces, obtain the vorticity equation

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}.$$

Interpret the terms in the equation.

At time $t = 0$ a line vortex is created along the z-axis, with the same circulation Γ around the axis at each z. The fluid is viscous and incompressible, and for $t > 0$ has only an azimuthal velocity denoted v. Show that there is a similarity solution of the form $vr/\Gamma = f(\eta)$, where $r = (x^2 + y^2)^{1/2}$ and η is a suitable similarity variable.

Furthermore, show that all conditions are satisfied by

$$f(\eta) = \frac{1}{2\pi}(1 - e^{-\eta^2}) \quad \text{and} \quad \eta = r/2\sqrt{\nu t}.$$

Show also that the flux of vorticity across any plane $z = \textit{constant}$ remains constant at Γ for all $t > 0$. Sketch v as a function of r.

15. Calculate the vorticity $\boldsymbol{\omega}$ associated with the velocity field

$$\mathbf{u} = (-\alpha x - yf(r,t),\, -\alpha y + xf(r,t),\, 2\alpha z)\,,$$

where α is a positive constant, and $f(r,t)$ depends on $r = (x^2 + y^2)^{1/2}$ and time t. Show that the velocity field represents a dynamically possible motion if $f(r,t)$ satisfies

$$2f + r\frac{\partial f}{\partial r} = A\gamma(t)e^{-\gamma(t)r^2},$$

where

$$\gamma(t) = \frac{\alpha}{2\nu}\left(1 \pm e^{-2\alpha(t-t_0)}\right)^{-1},$$

and A and t_0 are constants.

Show that, in the case where the minus sign is taken, γ is approximately $1/4\nu(t - t_0)$ when t only just exceeds t_0. Which terms in the vorticity equation dominate when this approximation holds?

16. From the vorticity equation, derive the equation satisfied by the streamfunction $\psi(r,\theta)$ for a steady two-dimensional flow in polar coordinates. Show further that this equation has solutions of the form $\psi = Qf(\theta)$, with Q constant if f satisfies an ordinary differential equation which you should determine.

17. Write down the equation satisfied by the vorticity $\omega(x,y,t)$ in a two-dimensional flow in Cartesian coordinates. Introduce a streamfunction ψ and show that $\omega = -\nabla^2\psi$. Show that the vorticity equation has a time-dependent similarity solution of the form

$$\psi = CxH(t)^{-1}\phi(\eta), \quad \omega = -CxH(t)^{-3}\phi_{\eta\eta}(\eta), \quad \text{for } \eta = yH(t)^{-1},$$

if $H(t) = (2Ct)^{1/2}$ and if ϕ satisfies an ordinary differential equation which you should determine involving an effective Reynolds number, $R = C/\nu$.

18. Show that in an unbounded Stokes flow at rest at infinity, two identical spheres, arbitrarily aligned, fall under gravity at constant separation, i.e. neither separating nor coming closer together.

19. An external force is applied at the centre of a cube in a direction normal to one flat surface. Show that in an unbounded Stokes flow, the cube moves in the direction of the applied force without rotating. Using linearity and rotational symmetry, deduce that, in all orientations, the terminal velocity of a cube of uniform density sedimenting in a fluid is vertical. Furthermore, using reflectional symmetry, show that it falls with no rotation. Using similar arguments, show that when the cube simply rotates about an axis through its centre, the resisting hydrodynamic torque is parallel to the angular velocity and the hydrodynamic force is zero.

Show that the same applies to a regular tetrahedron. How about an ellipsoid?

20. If the strain-rate tensor $E_{ij}(\mathbf{x})$ vanishes throughout a connected region, show that the flow is rigid body motion. [*Hint: take the curl of* $E_{ij}(\mathbf{x})$ *to show that the vorticity tensor is uniform (constant) throughout the region.*]

Show that if the surface stress is specified on a bounding surface then the Stokes flow in the interior is unique to within the addition of a rigid body motion. What condition(s) must the prescribed surface stress satisfy for there to be a Stokes flow in the interior?
[*Hint: in the absence of body forces the Stokes equation can be written* $\partial\sigma_{ij}/\partial x_j = 0$.]

21. If $\mathbf{A}(\mathbf{x})$ is a vector harmonic function, i.e. $\nabla^2\mathbf{A} = 0$, show that the flow

$$\mathbf{u} = 2\mathbf{A} - \nabla(\mathbf{A}\cdot\mathbf{x}) \quad \text{and} \quad P = -2\mu\nabla\cdot\mathbf{A}$$

is incompressible and satisfies the Stokes equation with no body force. Calculate the stress tensor.

For a sphere of radius a translating at velocity $\mathbf{V}$ through a fluid that is otherwise at rest, explain why the harmonic function takes the form

$$\mathbf{A} = \alpha a\mathbf{V}\frac{1}{r} + \beta a^3(\mathbf{V}\cdot\nabla)\nabla\frac{1}{r}.$$

[*Hint: How many vector harmonic functions that are linear in* $\mathbf{V}$ *can you construct using the fundamental harmonic solution* $1/r$ *and its derivatives?*]

Find the values of the constants α and β.

22. Consider an unbounded Stokes flow outside a rigid sphere of radius a rotating with angular velocity $\mathbf{\Omega}$. Show that the pressure gradient is zero. Then derive the velocity field as

$$\mathbf{u}(\mathbf{x}) = \mathbf{\Omega} \times \mathbf{x}\frac{a^3}{r^3}.$$

[*Hint: How many true vector harmonic functions,* $\mathbf{u}$*, can you construct using the fundamental harmonic solution* $1/r$ *and its derivatives that are linear in the pseudo vector* $\mathbf{\Omega}$*?*]

Show that the torque exerted on the sphere by the flow is $-8\pi\mu a^3\mathbf{\Omega}$.

23. Consider a spherical bubble of radius a in a uniform flow $\mathbf{U}$. The Stokes flow outside a sphere is of the form

$$\mathbf{u}(\mathbf{x}) = \mathbf{U}f(r) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})g(r).$$

Applying boundary conditions on $r = a$ of no normal component of velocity and no tangential component of surface traction (i.e. no tangential stress), find the flow $\mathbf{u}(\mathbf{x})$. Show that the drag force is $4\pi\mu a\mathbf{U}$.

24. Using the minimum dissipation theorem by stating carefully the flows you are comparing and exploiting the result from Question 222, find upper and lower bounds for the hydrodynamic torque on a regular tetrahedron rotating about its centre in a viscous fluid.

[*Hint: the radius of the inscribed sphere for a regular tetrahedron of edge length a is* $a/\sqrt{24}$ *while that of the circumscribed sphere is* $\sqrt{6}a/4$.]

25. An incompressible, viscous fluid is contained in the two-dimensional region $-\alpha < \theta < \alpha$ between two rigid hinged plates rotating with equal and opposite angular velocity of magnitude ω. Therefore, in plane polar coordinates, the velocity components on the hinged plates satisfy

$$u_r = 0 \quad \text{and} \quad u_\theta = \mp\omega r \quad \text{on} \quad \theta = \pm\alpha.$$

Neglecting all inertial forces, show that a solution to the Stokes problem is of the form

$$\psi = \frac{1}{2}\omega r^2 g(\theta)$$

(why?) and find the function $g(\theta)$. Deduce the pressure field $P(r,\theta)$. Is the logarithmic divergence of the pressure an issue? What is the physical interpretation of the singularity in g?

26. An incompressible, viscous fluid occupies the region $0 < \theta < \alpha$, $0 < r < \infty$ in plane-polar coordinates (r, θ). It is bounded by a stationary, rigid plate at $\theta = \alpha$ and a rigid plate at $\theta = 0$ that translates with constant velocity U in its own plane in the negative r direction. Calculate the resulting Stokes flow of the fluid. Calculate the stresses on each of the plates and comment on the external forces required to sustain the flow.

27. A spherical annulus of incompressible viscous liquid of volume V occupies the region $R_1(t) < r < R_2(t)$ between two free surfaces on the outside of which pressures (i.e. normal stresses) $P_1(t)$ and $P_2(t)$ are applied. The resulting flow is spherically symmetric. Neglecting inertia, gravity and surface tension, show that

$$\frac{d}{dt}\left(R_1^3\right) = \frac{\pi(P_1 - P_2)}{\mu V} R_1^3 \left(R_1^3 + \frac{3V}{4\pi}\right).$$

[*Hints:* $u_r = f(t)/r^2$ *(why?) and* $\sigma_{rr} = -p + 2\mu \partial u_r/\partial r$ *in this flow. Also, be careful to distinguish between pressure and normal stresses.*]

Show that if $P_1 - P_2$ is maintained positive and constant, then R_1 becomes infinite in a finite time. What happens if $P_1 - P_2$ is maintained negative and constant?

28. The concept of a boundary layer can be illustrated by ordinary differential equations. Consider the equation satisfied on the interval $[0, 1]$ by the function $f(x)$

$$\epsilon f' + f = 1$$

with $f(0) = 0$. Find the exact solution and plot it for small values of ϵ. Formally take $\epsilon = 0$ in the differential equation and find its solution, $f_0(x)$. Is $f_0(x)$ compatible with the boundary condition? Compare the exact solution to $f_0(x)$ and explain what happens. What is the "size" of the boundary layer at $x = 0$?

29. Wind blowing over a deep reservoir exerts at the water surface a uniform tangential stress, S, which is normal to, and away from, a straight side of the reservoir. Use dimensional analysis, based on (a) balancing the inertial and viscous forces in a thin boundary layer and (b) on the imposed boundary condition, to find order-of-magnitude estimates $\delta(x)$ for the boundary-layer thickness and $U(x)$ for the surface velocity as functions of distance x from the shore. Using the boundary-layer equations, find the ordinary differential equation governing the non-dimensional function f defined by

$$\psi(x, y) = U(x)\delta(x)f(\eta) \quad \text{where} \quad \eta = y/\delta(x).$$

What are the boundary conditions on f?