Vector Calculus: Example Sheet 1

1. Sketch the curve in the plane given parametrically by

$$\mathbf{x}(t) = (a\cos^3 t, a\sin^3 t), \quad 0 \le t \le 2\pi.$$

Calculate $\dot{\mathbf{x}}(t)$ at each point and hence find its total length.

2. A circular helix is given by

$$\mathbf{x}(t) = (a\cos t, a\sin t, ct), \quad t \in \mathbb{R}.$$

Calculate the tangent $\mathbf{t}$, curvature κ, principal normal $\mathbf{n}$, binormal $\mathbf{b}$ and torsion τ. Give a sketch of the curve indicating the directions of the vectors $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$.

3. Show that a planar curve $\mathbf{x}(t) = (x(t), y(t), 0)$ has curvature

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

Find the minimum and maximum values of the curvature on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

4. Evaluate explicitly each of the line integrals

$$\int (x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z), \quad \int (y\,\mathrm{d}x + x\,\mathrm{d}y + \mathrm{d}z), \quad \int \left(y\,\mathrm{d}x - x\,\mathrm{d}y + e^{x+y}\,\mathrm{d}z\right)$$

along (i) the straight line path from the origin to $(1,1,1)$, and (ii) the parabolic path given parametrically by $(x,y,z) = (t,t,t^2)$ from $t=0$ to $t=1$. For which of these integrals do the two paths give the same results, and why?

5. Consider the vector fields $\mathbf{F}(\mathbf{x}) = (3x^2yz^2, 2x^3yz, x^3z^2)$ and $\mathbf{G}(\mathbf{x}) = (3x^2y^2z, 2x^3yz, x^3y^2)$. Compute the line integrals $\int \mathbf{F} \cdot \mathrm{d}\mathbf{x}$ and $\int \mathbf{G} \cdot \mathrm{d}\mathbf{x}$ along the following paths, each of which consist of straight line segments joining the specified points:

(i) $(0,0,0) \to (1,1,1)$,

(ii) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$,

(iii) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$.

Are either of the differentials $\mathbf{F} \cdot \mathrm{d}\mathbf{x}$ or $\mathbf{G} \cdot \mathrm{d}\mathbf{x}$ exact?

6. A curve C is given parametrically by

$$\mathbf{x}(t) = (\cos(\sin nt)\cos t\ ,\ \cos(\sin nt)\sin t\ ,\ \sin(\sin nt)) \qquad 0 \le t \le 2\pi,$$

where n is some fixed integer. Sketch the curve. Show that

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 2\pi, \quad \text{where} \quad \mathbf{F}(\mathbf{x}) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

and C is traversed in the direction of increasing t. Can $\mathbf{F}(\mathbf{x})$ be written as the gradient of a scalar function? Comment on your results.

[Hint: when sketching the curve, you may find it helpful to use spherical polar coordinates, defined by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$ with $\theta \in [0,\pi]$ and $\phi \in [0, 2\pi)$.]

7. Use the substitution $x = r\cos\phi$, $y = \frac{1}{2}r\sin\phi$, to evaluate

$$\int_D \frac{x^2}{x^2+4y^2}\,\mathrm{d}A$$

where D is the region between the two ellipses $x^2+4y^2 = 1$, $x^2+4y^2 = 4$.

8. A closed curve C in the $z = 0$ plane consists of the arc of the parabola $y^2 = 4ax$ $(a > 0)$ between the points $(a, \pm 2a)$ and the straight line joining $(a, \mp 2a)$. The area inclosed by C is D. Show, by calculating the integrals explicitly, that

$$\oint_C \left(x^2y\,\mathrm{d}x + xy^2\,\mathrm{d}y\right) = \int_D (y^2 - x^2)\,\mathrm{d}A = \frac{104}{105}a^4$$

9. The region D is bounded by the segments $x = 0,\ 0 \leq y \leq 1; y = 0, 0 \leq x \leq 1; y = 1, 0 \leq x \leq \frac{3}{4}$, and by an arc of the parabola $y^2 = 4(1-x)$. Consider a mapping into the (x, y)-plane from the (u, v)-plane defined by the transformation $x = u^2 - v^2$, $y = 2uv$. Sketch D and also the two regions in the (u, v)-plane which are mapped into it. Hence evaluate

$$\int_D \frac{\mathrm{d}A}{(x^2 + y^2)^{1/2}}$$

10. Compute the volume of a cone of height h and radius a using (a) cylindrical polars, (b) spherical polars.

11. By using a suitable change of variables, calculate the volume within an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

12. Compute the volume of the region V defined by the intersection of the three cylinders

$$V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1,\ y^2 + z^2 \leq 1,\ z^2 + x^2 \leq 1\}.$$

[Warning: The sole purpose of this question is to show you that volume integrals can be arbitrarily hard. Only attempt if that's your thing.]