

Cosmology: Example Sheet 2

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1. In special and general relativity, the energy density ρ and pressure P of a perfect fluid combine to form the *energy-momentum* tensor. In the rest frame of the fluid, this is given by

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

After a Lorentz boost by velocity v in the x -direction, the energy-momentum tensor transforms to

$$T'^{\mu\nu} = \Lambda^\mu_\lambda \Lambda^\nu_\sigma T^{\lambda\sigma} \quad \text{with} \quad \Lambda^\mu_\lambda = \left(\begin{array}{cc|cc} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

with $\gamma = (1 - v^2/c^2)^{-1/2}$. What equation of state $P = w\rho$ gives rise to a fluid that looks the same to all inertial observers?

2. The *deceleration parameter* is defined by

$$q(t) = -\frac{\ddot{a}a}{\dot{a}^2}$$

Compute q for a flat universe dominated by a single fluid with equation of state $P = w\rho$.

For a universe with arbitrary curvature, filled with matter, radiation and a cosmological constant, show that today $q_0 = \frac{1}{2}\Omega_m + \Omega_r - \Omega_\Lambda$. What is the deceleration parameter for our universe?

Show that the Taylor expansion of the scale factor about the present day (with $a(t_0) = 1$) can be written as

$$\frac{1}{1+z} = a(t) = 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2 (t - t_0)^2 + \dots$$

Hence show that the light from a nearby galaxy at redshift $z \ll 1$ was emitted at time

$$t_0 - t = H_0^{-1}z - \frac{1}{2}H_0^{-1}(2 + q_0)z^2 + \dots$$

3. A flat universe, containing matter and a cosmological constant, expands according to the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \left(\rho_\Lambda + \frac{\rho_0}{a^3}\right)$$

Solve this to find

$$a(t) = \beta(\sinh \alpha t)^{2/3} \tag{1}$$

with α and β constant. [Hint: you may find the substitution $b = a^{3/2}$ helpful.] Verify the expected asymptotic behaviour for early and late times.

Our universe today is flat, with $\Omega_m \approx 0.3$ and $\Omega_\Lambda \approx 0.7$ and $H_0^{-1} \approx 14$ Gyr. Determine:

- The age of the universe.
- The age at which the expansion of the universe first started to accelerate.
- The age at which the energy density in matter and the cosmological constant were equal.

A cosmologically ignorant civilisation measures $H_0^{-1} \approx 14$ Gyr but refuses to countenance the possibility of a cosmological constant. How old would they believe their universe to be if they assumed it was flat with $\Omega_m = 1$?

4. The *jerk* is defined as

$$J(t) = \frac{\ddot{a} a^2}{\dot{a}^3}$$

14 billion years of cosmic expansion can be roughly characterised as: the jerk is one. To see this, consider the Friedmann equation with matter, a cosmological constant and curvature. Show that the curvature can be written as

$$\frac{kc^2}{R^2} = a^2 H^2 (J - 1)$$

Confirm that the solution (1) indeed has unit jerk.

[Fun fact: the 4th, 5th and 6th derivatives of the scale factor are called *snap*, *crackle* and *pop* respectively.]

5. A flat universe, containing radiation and a cosmological constant, expands according to the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \left(\rho_\Lambda + \frac{\rho_0}{a^4}\right)$$

Solve for $a(t)$. Verify the expected asymptotic behaviour for early and late times.

6. Consider a collection of particles, each of which interacts through the potential

$$V(\mathbf{x}_i - \mathbf{x}_j) = \alpha |\mathbf{x}_i - \mathbf{x}_j|^n$$

Prove the virial theorem,

$$2\bar{T} = n\bar{V}$$

where \bar{T} is the time-averaged kinetic energy and \bar{V} the time-averaged total potential energy.

7. The equations of motion for inflation can be written as

$$H^2 = \frac{1}{3M_{\text{pl}}^2} \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) \right] \quad (2)$$

and

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \quad (3)$$

where $M_{\text{pl}}^2 = c^2/8\pi G$ is related to the *reduced Planck mass*. Consider the potential

$$V = V_0 e^{-\lambda\phi/M_{\text{pl}}}$$

Show that the inflationary equations have the exact solution

$$a(t) = \left(\frac{t}{t_0}\right)^{2/\lambda^2} \quad \text{and} \quad \phi(t) = \phi_0 + \alpha \log t$$

for some choice of ϕ_0 and α .

For what values of λ does inflation occur? Define the pressure-energy density ratio $w_\phi = P_\phi/\rho_\phi$. For what range of w_ϕ does inflation occur?

8. Assume that an inflationary solution $\phi(t)$ is monotonic and view the Hubble parameter $H(t)$ as a function $H(\phi(t))$. Show that the equations of inflation (2) and (3) can be written in “Hamilton-Jacobi” form

$$\dot{\phi} = -2M_{\text{pl}}^2 \frac{\partial H}{\partial \phi}$$

and

$$V(\phi) = 3M_{\text{pl}}^2 H^2 - 2M_{\text{pl}}^4 \left(\frac{\partial H}{\partial \phi} \right)^2$$

[Hint: you might start by differentiating the Friedmann equation (2) with respect to time.]

The Hamilton-Jacobi formalism can be used to construct exact solutions in which one specifies a choice of $H(\phi)$ and then works backwards to read off the corresponding potential $V(\phi)$, the scalar evolution $\phi(t)$ and, hence the scale factor $a(t)$. Find these three quantities for

$$H(\phi) = \exp \left(-\sqrt{\frac{\alpha}{2}} \frac{\phi}{M_{\text{pl}}} \right)$$

9. Under the slow roll conditions, the inflationary equations become

$$H^2 \approx \frac{1}{3M_{\text{pl}}^2} V(\phi) \quad \text{and} \quad 3H\dot{\phi} \approx -\frac{\partial V}{\partial \phi}$$

Solve these equations to find the scale factor $a(t)$ for the choice of potentials

$$\begin{aligned} \text{i)} \quad & V(\phi) = \frac{1}{2} m^2 \phi^2 \\ \text{ii)} \quad & V(\phi) = \frac{1}{4} \lambda \phi^4 \end{aligned}$$