Electromagnetism: Example Sheet 1

Professor David Tong, January 2015

1. A current density $J(\mathbf{r},t)$ has the form

$$\mathbf{J} = C\,\mathbf{r}\,e^{-atr^2}$$

where C and a are constants. Show that the equation of conservation of charge can be satisfied by writing the charge density in the form

$$\rho \;=\; (f + t\,g)\;e^{-atr^2}$$

where f and g are functions of position, to be determined.

2. In a fluid environment, charge undergoes *diffusion*. This is empirically described by *Fick's law*, which relates the current to the charge density,

$$\mathbf{J} = -D\nabla\rho$$

Here, D is called the diffusion coefficient. Show that ρ obeys the heat equation. Show that this is solved by a spreading Gaussian of the form,

$$\rho(\mathbf{r},t) = \frac{\rho_0 a^3}{(4D(t-t_0)+a^2)^{3/2}}\exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right)$$

with a, t_0 and ρ_0 constants.

3. A charge density is given by $\rho = \rho_0\;e^{-k|z|}$ with ρ_0 and k positive constants. Use Gauss' law to show that the electric field is given by $\mathbf{E} = E(z)\hat{\mathbf{z}}$ with $E(z) = -E(-z)$ and, for $z > 0$,

$$E(z) = \frac{\rho_0}{\epsilon_0 k}\left(1 - e^{-kz}\right)$$

4. Use Gauss's law to obtain the electric field due to a uniform charge density ρ occupying the region $a < r < b$, with r the radial distance from the origin.

Show that in the limit $b \to a$, $\rho \to \infty$ with $(b-a)\,\rho = \sigma$ remaining finite, the electric field suffers the expected discontinuity due to surface charge.

5. Roughly sketch the field lines (including arrows to denote sense) and equipotentials for the following systems of point charges:

- A single charge $+q$;
- Two charges $+q$ separated by a distance $2a$;
- Two charges $\pm q$ separated by a distance $2a$.

6. Compute the electric field due to an infinite line charge by integrating the expression obtained from the inverse square law.

7. A circular disk of radius a has uniform surface charge density σ. Compute the potential at a point on the axis of symmetry at distance z from the centre. Compute the electric field at this point. Find the discontinuity in the normal electric field at the centre of the disk. Show that, far along the axis of symmetry, the electric field looks approximately like that of a charged point particle.

8. Show that, far from a charge distribution $\rho(\mathbf{r})$ localised in region V, the potential takes the form

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} + \frac{1}{2}\frac{\mathbb{Q}_{ij}r_ir_j}{r^5} + \dots\right)$$

where Q is the total charge, $\mathbf{p}$ is the dipole moment and $\mathbb{Q}_{ij}$ is the quadrupole moment, defined by

$$Q = \int_V d^3r\ \rho(\mathbf{r}) \quad \text{and} \quad \mathbf{p} = \int_V d^3r\ \mathbf{r}\rho(\mathbf{r}) \quad \text{and} \quad \mathbb{Q}_{ij} = \int_V d^3r\ (3r_ir_j - \delta_{ij}r^2)\rho(\mathbf{r})$$

Compute the charge, dipole and quadrupole for:

- Two charges, $+q$ and $-q$, at points $(0,0,0)$ and $(d,0,0)$ respectively.
- Two charges $+q$ and two charges $-q$ placed on the corners of a square, with sides of length d, such that every charge has an opposite charge for each of its neighbours.
- Four charges $+q$ and four charges $-q$ placed on the corners of a cube, with sides of length d, such that every charge has an opposite charge for each of its neighbours.

9. For a charge density $\rho(\mathbf{x}, t)$ and current $\mathbf{J}(\mathbf{x}, t)$, both localised within a region V, show that

$$\int_V d^3x \; \mathbf{J} = \frac{d\mathbf{p}}{dt}$$

where $\mathbf{p}$ is the electric dipole moment.

[*Hint*: You may wish to first show that $\partial_j(x_i J_j) = J_i + x_i \nabla \cdot \mathbf{J}$]

10. A charge density $\rho(\mathbf{r})$ gives rise to a potential $\phi(\mathbf{r})$. In the lectures we derived the expression for the energy stored in the field,

$$U = \frac{1}{2} \int_{\mathbf{R}^3} d^3r \; \rho\phi$$

Show that this is equivalent to the formula

$$U = \frac{\epsilon_0}{2} \int_{\mathbf{R}^3} d^3r \; \mathbf{E} \cdot \mathbf{E}$$

Consider a total charge Q, distributed uniformly inside a sphere of radius R. Find ϕ and $\mathbf{E}$ both inside and outside the sphere. Calculate the two expressions for the energy and show that they agree.

11*. A spherical conducting shell of radius R is grounded (i.e. has potential $\phi = 0$). A charge q is placed inside the shell at at point $\mathbf{r} = (0, 0, d)$ from the centre, with $d < R$. Show that the potential inside the shell can be determined by placing an appropriate image charge outside the shell at $\mathbf{r}' = (0, 0, R^2/d)$. Show that the induced surface charge on the conductor is

$$\sigma = -\frac{q}{4\pi} \frac{R^2 - d^2}{R(R^2 - 2dR\cos\theta + d^2)^{3/2}}$$

where θ is the angle between the point on the shell and the z-axis.