of this equation are usually given names. The integral of the electric field around the curve C is called the *electromotive force*, $\mathcal{E}$, or *emf* for short,

$$\mathcal{E} = \int_C \mathbf{E} \cdot d\mathbf{r}$$

It's not a great name because the electromotive force is not really a force. Instead it's the tangential component of the force per unit charge, integrated along the wire. Another way to think about it is as the work done on a unit charge moving around the curve C. If there is a non-zero emf present then the charges will be accelerated around the wire, giving rise to a current.

The integral of the magnetic field over the surface S is called the magnetic *flux* Φ through S,

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

The Maxwell equation (4.1) can be written as

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{4.2}$$

In this form, the equation is usually called *Faraday's Law*. Sometimes it is called the flux rule.

Faraday's law tells us that if you change the magnetic flux through S then a current will flow. There are a number of ways to change the magnetic field. You could simply move a bar magnet in the presence of circuit, passing it through the surface S; or you could replace the bar magnet with some other current density, restricted to a second wire C', and move that; or you could keep the second wire C' fixed and vary the current in it, perhaps turning it on and off. All of these will induce a current in C.

However, there is then a secondary effect. When a current flows in C, it will create its own magnetic field. We've seen how this works for steady currents in Section 3. This induced magnetic field will always be in the direction that opposes the change. This is called *Lenz's law*. If you like, "Lenz's law" is really just the minus sign in Faraday's law (4.2).

We can illustrate this with a simple example. Consider the case where C is a circle, lying in a plane. We'll place it in a uniform B field and then make B smaller over time, so $\dot{\Phi} < 0$. By Faraday's law, $\mathcal{E} > 0$ and the current will flow in the right-handed direction around C as shown. But now you can wrap your right-hand in a different way: point your thumb in the direction of the current and let your fingers curl to show you the direction of the induced magnetic field. These are the circles drawn in the figure. You see that the induced current causes $\mathbf{B}$ to increase inside the loop, counteracting the original decrease.

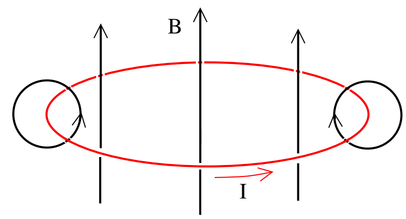

Figure 35: Lenz's law

Lenz's law is rather like a law of inertia for magnetic fields. It is necessary that it works this way simply to ensure energy conservation: if the induced magnetic field aided the process, we'd get an unstable runaway situation in which both currents and magnetic fields were increasing forever.

4.1.1 Faraday's Law for Moving Wires

There is another, related way to induce currents in the presence of a magnetic field: you can keep the field fixed, but move the wire. Perhaps the simplest example is shown in the figure: it's a rectangular circuit, but where one of the wires is a metal bar that can slide backwards and forwards. This whole set-up is then placed in a magnetic field, which passes up, perpendicular through the circuit.

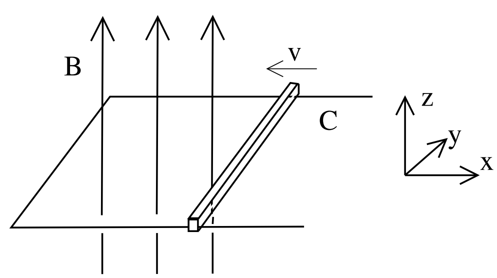

Figure 36: Moving circuit

Slide the bar to the left with speed v. Each charge q in the bar experiences a Lorentz force qvB, pushing it in the $\mathbf{y}$ direction. This results in an emf which, now, is defined as the integrated force per charge. In this case, the resulting emf is

$$\mathcal{E} = vBd$$

where d is the length of the moving bar. But, because the area inside the circuit is getting smaller, the flux through C is also decreasing. In this case, it's simple to compute the change of flux: it is

$$\frac{d\Phi}{dt} = -vBd$$

We see that once again the change of flux is related to the emf through the flux rule

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Note that this is the same formula (4.2) that we derived previously, but the physics behind it looks somewhat different. In particular, we used the Lorentz force law and didn't need the Maxwell equations.

As in our previous example, the emf will drive a current around the loop C. And, just as in the previous example, this current will oppose the motion of the bar. In this case, it is because the current involves charges moving with some speed u around the circuit. These too feel a Lorentz force law, now pushing the bar back to the right. This means that if you let the bar go, it will not continue with constant speed, even if the connection is frictionless. Instead it will slow down. This is the analog of Lenz's law in the present case. We'll return to this example in Section 4.1.3 and compute the bar's subsequent motion.

The General Case

There is a nice way to include both the effects of time-dependent magnetic fields and the possibility that the circuit C changes with time. We consider the moving loop $C(t)$, as shown in the figure. Now the change in flux through a surface S has two terms: one because B may be changing, and one because C is changing. In a small time δt, we have

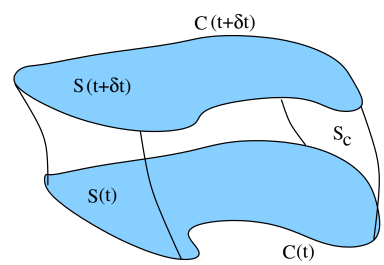

Figure 37: Moving Circuits

$$\delta\Phi = \Phi(t+\delta t) - \Phi(t) = \int_{S(t+\delta t)} \mathbf{B}(t+\delta t)\cdot d\mathbf{S} - \int_{S(t)} \mathbf{B}(t)\cdot d\mathbf{S}$$
$$= \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t}\,\delta t\cdot d\mathbf{S} + \left[\int_{S(t+\delta t)} - \int_{S(t)}\right] \mathbf{B}(t)\cdot d\mathbf{S} + \mathcal{O}(\delta t^2)$$

We can do something with the middle terms. Consider the closed surface created by $S(t)$ and $S(t+\delta t)$, together with the cylindrical region swept out by $C(t)$ which we call S_c. Because $\nabla\cdot\mathbf{B} = 0$, the integral of $\mathbf{B}(t)$ over any closed surface vanishes. But

$\int_{S(t+\delta t)} - \int_{S(t)}$ is the top and bottom part of the closed surface, with the minus sign just ensuring that the integral over the bottom part $S(t)$ is in the outward direction. This means that we must have

$$\left[\int_{S(t+\delta t)} - \int_{S(t)}\right] \mathbf{B}(t) \cdot d\mathbf{S} = -\int_{S_c} \mathbf{B}(t) \cdot d\mathbf{S}$$

For the integral over S_c, we can write the surface element as

$$d\mathbf{S} = (d\mathbf{r} \times \mathbf{v})\delta t$$

where $d\mathbf{r}$ is the line element along $C(t)$ and $\mathbf{v}$ is the velocity of a point on C. We find that the expression for the change in flux can be written as

$$\frac{d\Phi}{dt} = \lim_{\delta t \to 0} \frac{\delta \Phi}{\delta t} = \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - \int_{C(t)} (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r}$$

where we've taken the liberty of rewriting $(d\mathbf{r} \times \mathbf{v}) \cdot \mathbf{B} = d\mathbf{r} \cdot (\mathbf{v} \times \mathbf{B})$. Now we use the Maxwell equation (4.1) to rewrite the $\partial \mathbf{B}/\partial t$ in terms of the electric field. This gives us our final expression

$$\frac{d\Phi}{dt} = -\int_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r}$$

where the right-hand side now includes the force tangential to the wire from both electric fields and also from the motion of the wire in the presence of magnetic fields. The electromotive force should be defined to include both of these contributions,

$$\mathcal{E} = \int_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r}$$

and we once again get the flux rule $\mathcal{E} = -d\Phi/dt$.

4.1.2 Inductance and Magnetostatic Energy

In Section 2.3, we computed the energy stored in the electric field by considering the work done in building up a collection of charges. But we didn't repeat this calculation for the magnetic field in Section 3. The reason is that we need the concept of emf to describe the work done in building up a collection of currents.

Suppose that a constant current I flows along some curve C. From the results of Section 3 we know that this gives rise to a magnetic field and hence a flux $\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$ through the surface S bounded by C. Now increase the current I. This will increase the flux Φ. But we've just learned that the increase in flux will, in turn, induce an emf around the curve C. The minus sign of Lenz's law ensures that this acts to resist the change of current. The work needed to build up a current is what's needed to overcome this emf.

Inductance

If a current I flowing around a curve C gives rise to a flux $\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$ then the *inductance* L of the circuit is defined to be

$$L = \frac{\Phi}{I}$$

The inductance is a property only of our choice of curve C.

An Example: The Solenoid

A solenoid consists of a cylinder of length l and cross-sectional area A. We take $l \gg \sqrt{A}$ so that any end-effects can be neglected. A wire wrapped around the cylinder carries current I and winds N times per unit length. We previously computed the magnetic field through the centre of the solenoid to be (3.7)

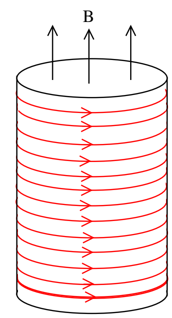

Figure 38:

$$B = \mu_0 I N$$

This means that a flux through a single turn is $\Phi_0 = \mu_0 I N A$. The solenoid consists of Nl turns of wire, so the total flux is

$$\Phi = \mu_0 I N^2 A l = \mu_0 I N^2 V$$

with $V = Al$ the volume inside the solenoid. The inductance of the solenoid is therefore

$$L = \mu_0 N^2 V$$

Magnetostatic Energy

The definition of inductance is useful to derive the energy stored in the magnetic field. Let's take our circuit C with current I. We'll try to increase the current. The induced emf is

$$\mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$

As we mentioned above, the induced emf can be thought of as the work done in moving a unit charge around the circuit. But we have current I flowing which means that, in time δt, a charge $I\delta t$ moves around the circuit and the amount of work done is

$$\delta W = \mathcal{E} I \delta t = -LI\frac{dI}{dt}\delta t \quad \Rightarrow \quad \frac{dW}{dt} = -LI\frac{dI}{dt} = -\frac{L}{2}\frac{dI^2}{dt}$$

The work needed to build up the current is just the opposite of this. Integrating over time, we learn that the total work necessary to build up a current I along a curve with inductance L is

$$W = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi$$

Following our discussion for electric energy in (2.3), we identify this with the energy U stored in the system. We can write it as

$$U = \frac{1}{2}I\int_S \mathbf{B}\cdot d\mathbf{S} = \frac{1}{2}I\int_S \nabla\times\mathbf{A}\cdot d\mathbf{S} = \frac{1}{2}I\oint_C \mathbf{A}\cdot d\mathbf{r} = \frac{1}{2}\int d^3x\ \mathbf{J}\cdot\mathbf{A}$$

where, in the last step, we've used the fact that the current density $\mathbf{J}$ is localised on the curve C to turn the integral into one over all of space. At this point we turn to the Maxwell equation $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$ to write the energy as

$$U = \frac{1}{2\mu_0}\int d^3x\ (\nabla\times\mathbf{B})\cdot\mathbf{A} = \frac{1}{2\mu_0}\int d^3x\ \left[\nabla\cdot(\mathbf{B}\times\mathbf{A}) + \mathbf{B}\cdot(\nabla\times\mathbf{A})\right]$$

We assume that $\mathbf{B}$ and $\mathbf{A}$ fall off fast enough at infinity so that the first term vanishes. We're left with the simple expression

$$U = \frac{1}{2\mu_0}\int d^3x\ \mathbf{B}\cdot\mathbf{B}$$

Combining this with our previous result (2.27) for the electric field, we have the energy stored in the electric and magnetic fields,

$$U = \int d^3x\ \left(\frac{\epsilon_0}{2}\,\mathbf{E}\cdot\mathbf{E} + \frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}\right) \qquad (4.3)$$

This is a nice result. But there's something a little unsatisfactory behind our derivation of (4.3). First, we reiterate a complaint from Section 2.3: we had to approach the energy in both the electric and magnetic fields in a rather indirect manner, by focussing not on the fields but on the work done to assemble the necessary charges and currents. There's nothing wrong with this, but it's not a very elegant approach and it would be nice to understand the energy directly from the fields themselves. One can do better by using the Lagrangian approach to Maxwell's equations.

Second, we computed the energy for the electric fields and magnetic fields alone and then simply added them. We can't be sure, at this point, that there isn't some mixed contribution to the energy such as $\mathbf{E}\cdot\mathbf{B}$. It turns out that there are no such terms. Again, we'll postpone a proof of this until the next course.

4.1.3 Resistance

You may have noticed that our discussion above has been a little qualitative. If the flux changes, we have given expressions for the induced emf $\mathcal{E}$ but we have not given an explicit expression for the resulting current. And there's a good reason for this: it's complicated.

The presence of an emf means that there is a force on the charges in the wire. And we know from Newtonian mechanics that a force will cause the charges to accelerate. This is where things start to get complicated. Accelerating charges will emit waves of electromagnetic radiation, a process that you will explore later. Relatedly, there will be an opposition to the formation of the current through the process that we've called Lenz's law.

So things are tricky. What's more, in real wires and materials there is yet another complication: friction. Throughout these lectures we have modelled our charges as if they are moving unimpeded, whether through the vacuum of space or through a conductor. But that's not the case when electrons move in real materials. Instead, there's stuff that gets in their way: various messy impurities in the material, or sound waves (usually called phonons in this context) which knock them off-course, or even other electrons. All these effects contribute to a friction force that acts on the moving electrons. The upshot of this is that the electrons do not accelerate forever. In fact, they do not accelerate for very long at all. Instead, they very quickly reach an equilibrium speed, analogous to the "terminal velocity" that particles reach when falling in gravitational field while experiencing air resistance. In many circumstances, the resulting current I is proportional to the applied emf. This relationship is called *Ohm's law*. It is

$$\mathcal{E} = IR \tag{4.4}$$

The constant of proportionality R is called the resistance. The emf is $\mathcal{E} = \int \mathbf{E} \cdot d\mathbf{x}$. If we write $\mathbf{E} = -\nabla\phi$, then $\mathcal{E} = V$, the potential difference between two ends of the wire. This gives us the version of Ohm's law that is familiar from school: $V = IR$.

The resistance R depends on the size and shape of the wire. If the wire has length L and cross-sectional area A, we define the *resistivity* as $\rho = AR/L$. (It's the same Greek letter that we earlier used to denote charge density. They're not the same thing. Sorry for any confusion!) The resistivity has the advantage that it's a property of the material only, not its dimensions. Alternatively, we talk about the conductivity $\sigma = 1/\rho$. (This is the same Greek letter that we previously used to denote surface charge density. They're not the same thing either.) The general form of Ohm's law is then

$$\mathbf{J} = \sigma \mathbf{E}$$

Unlike the Maxwell equations, Ohm's law does not represent a fundamental law of Nature. It is true in many, perhaps most, materials. But not all. There is a very simple classical model, known as the *Drude model*, which treats electrons as billiard balls experiencing linear drag which gives rise to Ohm's law.. But a proper derivation of Ohm's law needs quantum mechanics and a more microscopic understanding of what's happening in materials. Needless to say, this is (way) beyond the scope of this course. So, at least in this small section, we will take Ohm's law (4.4) as an extra input in our theory.

When Ohm's law holds, the physics is very different. Now the applied force (or, in this case, the emf) is proportional to the velocity of the particles rather than the acceleration. It's like living in the world that Aristotle envisaged rather than the one Galileo understood. But it also means that the resulting calculations typically become much simpler.

An Example

Let's return to our previous example of a sliding bar of length d and mass m which forms a circuit, sitting in a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. But now we will take into account the effect of electrical resistance. We take the resistance of the sliding bar to be R. But we'll make life easy for ourselves and assume that the resistance of the rest of the circuit is negligible.

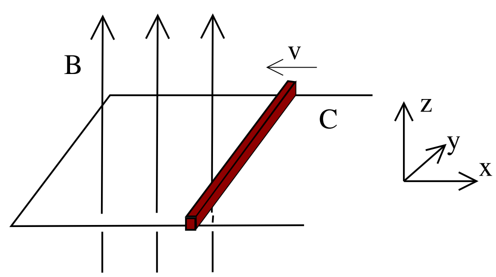

Figure 39:

There are two dynamical degrees of freedom in our problem: the position x of the sliding bar and the current I that flows around the circuit. We take $I > 0$ if the current flows along the bar in the positive $\hat{\mathbf{y}}$ direction. The Lorentz force law tells us that the force on a small volume of the bar is $\mathbf{F} = IB\,\hat{\mathbf{y}} \times \hat{\mathbf{z}}$. The force on the whole bar is therefore

$$\mathbf{F} = IBd\,\hat{\mathbf{x}}$$

The equation of motion for the position of the wire is then

$$m\ddot{x} = IBd$$

Now we need an equation that governs the current $I(t)$. If the total emf around the circuit comes from the induced emf, we have

$$\mathcal{E} = -\frac{d\Phi}{dt} = -Bd\dot{x}$$

Ohm's law tells us that $\mathcal{E} = IR$. Combining these, we get a simple differential equation for the position of the bar

$$m\ddot{x} = -\frac{B^2d^2}{R}\,\dot{x}$$

which we can solve to see that any initial velocity of the bar, v, decays exponentially:

$$\dot{x}(t) = -ve^{-B^2d^2t/mR}$$

Note that, in this calculation we neglected the magnetic field created by the current. It's simple to see the qualitative effect of this. If the bar moves to the left, so $\dot{x} < 0$, then the flux through the circuit decreases. The induced current is $I > 0$ which increases $\mathbf{B}$ inside the circuit which, in accord with Lenz's law, attempts to counteract the reduced flux.

In the above derivation, we assumed that the total emf around the circuit was provided by the induced emf. This is tantamount to saying that no current flows when the bar is stationary. But we can also relax this assumption and include in our analysis an emf $\mathcal{E}_0$ across the circuit (provided, for example, by a battery) which induces a current $I_0 = \mathcal{E}_0 d/R$. Now the total emf is

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_{\text{induced}} = \mathcal{E}_0 - Bd\dot{x}$$

The total current is again given by Ohms law $I = \mathcal{E}/R$. The position of the bar is now governed by the equation

$$m\ddot{x} = -\frac{Bd}{R}\left(\mathcal{E}_0 - Bd\dot{x}\right)$$

Again, it's simple to solve this equation.

Joule Heating

In Section 4.1.2, we computed the work done in changing the current in a circuit C. This ignored the effect of resistance. In fact, if we include the resistance of a wire then we need to do work just to keep a constant current. This should be unsurprising. It's the same statement that, in the presence of friction, we need to do work to keep an object moving at a constant speed.

Let's return to a fixed circuit C. As we mentioned above, if a battery provides an emf $\mathcal{E}_0$, the resulting current is $I = \mathcal{E}_0/R$. We can now run through arguments similar to those that we saw when computing the magnetostatic energy. The work done in moving a unit charge around C is $\mathcal{E}_0$ which means that amount of work necessary to keep a current I moving for time δt is

$$\delta W = \mathcal{E}_0 I \delta t = I^2 R \, \delta t$$

We learn that the power (work per unit time) dissipated by a current passing through a circuit of resistance R is $dW/dt = I^2 R$. This is not energy that can be usefully stored like the magnetic and electric energy (4.3); instead it is lost to friction which is what we call *heat*. (The difference between heat and other forms of energy is explained in the Thermodynamics section in the *Statistical Physics* notes). The production of heat by a current is called *Joule heating* or, sometimes, *Ohmic heating*.

4.1.4 Michael Faraday (1791-1867)

> "The word "physicist" is both to my mouth and ears so awkward that I think I shall never be able to use it. The equivalent of three separate sounds of "s" in one word is too much."
>
> *Faraday in a letter to William Whewell* [3]

Michael Faraday's route into science was far from the standard one. The son of a blacksmith, he had little schooling and, at the age of 14, was apprenticed to a bookbinder. There he remained until the age of 20 when Faraday attended a series of popular lectures at the Royal Institution by the chemist Sir Humphry Davy. Inspired, Faraday wrote up these lectures, lovingly bound them and presented them to Davy as a gift. Davy was impressed and some months later, after suffering an eye injury in an explosion, turned to Faraday to act as his assistant.

Not long after, Davy decided to retire and take a two-year leisurely tour of Europe, meeting many of the continent's top scientists along the way. He asked Faraday to join him and his wife, half as assistant, half as valet. The science part of this was a success; the valet part less so. But Faraday dutifully played his roles, emptying his master's chamber pot each morning, while aiding in a number of important scientific discoveries along the way, including a wonderful caper in Florence where Davy and Faraday used Galileo's old lens to burn a diamond, reducing it, for the first time, to Carbon.

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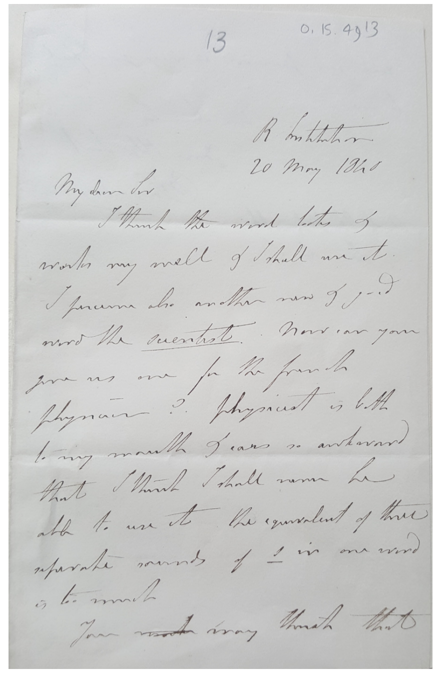
[3] According to the rest of the internet, Faraday complains about three separate sounds of "i". The rest of the internet is wrong and can't read Faraday's writing. The original letter is in the Wren library in Trinity College and is shown on the next page. I'm grateful to Frank James, editor of Faraday's correspondence, for help with this.

Back in England, Faraday started work at the Royal Institution. He would remain there for over 45 years. An early attempt to study electricity and magnetism was abandoned after a priority dispute with his former mentor Davy and it was only after Davy's death in 1829 that Faraday turned his attentions fully to the subject. He made his discovery of induction on 28th October, 1831. The initial experiment involved two, separated coils of wire, both wrapped around the same magnet. Turning on a current in one wire induces a momentary current in the second. Soon after, he found that a current is also induced by passing a loop of wire over a magnet. The discovery of induction underlies the electrical dynamo and motor, which convert mechanical energy into electrical energy and vice-versa.



13 0.15.49[13]

R Institution
20 May 1840

My dear Sir

I think the word [illegible] of works very well & I shall use it. I perceive also another new & good word the scientist. Now can you give us one for the french physicien? Physicist is both to my mouth & ears so awkward that I think I shall never be able to use it the equivalent of three separate sounds of i in one word is too much

Your very thankful [illegible]

Figure 40:

Faraday was not a great theorist and the mathematical expression that we have called Faraday's law is due to Maxwell. Yet Faraday's intuition led him to make one of the most important contributions of all time to theoretical physics: he was the first to propose the idea of the field.

As Faraday's research into electromagnetism increased, he found himself lacking the vocabulary needed to describe the phenomena he was seeing. Since he didn't exactly receive a classical education, he turned to William Whewell, then Master of Trinity, for some advice. Between them, they cooked up the words 'anode', 'cathode', 'ion', 'dielectric', 'diamagnetic' and 'paramagnetic'. They also suggested the electric charge be renamed 'Franklinic' in honour of Benjamin Franklin. That one didn't stick.

The last years of Faraday's life were spent in the same way as Einstein: seeking a unified theory of gravity and electromagnetism. The following quote describes what is, perhaps, the first genuine attempt at unification:

> Gravity: Surely his force must be capable of an experimental relation to Electricity, Magnetism and the other forces, so as to bind it up with them in reciprocal action and equivalent effect. Consider for a moment how to set about touching this matter by facts and trial ...
>
> *Faraday, 19th March, 1849.*

As this quote makes clear, Faraday's approach to this problem includes something that Einstein's did not: experiment. Ultimately, neither of them found a connection between electromagnetism and gravity. But it could be argued that Faraday made the more important contribution: while a null theory is useless, a null experiment tells you something about Nature.

4.2 One Last Thing: The Displacement Current

We've now worked our way through most of the Maxwell equations. We've looked at Gauss' law (which is really equivalent to Coulomb's law)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{4.5}$$

and the law that says there are no magnetic monopoles

$$\nabla \cdot \mathbf{B} = 0 \tag{4.6}$$

and Ampère's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{4.7}$$

and now also Faraday's law

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{4.8}$$

In fact, there's only one term left to discuss. When fields change with time, there is an extra term that appears in Ampère's law, which reads in full:

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{4.9}$$

This extra term is called the *displacement current*. It's not a great name because it's not a current. Nonetheless, as you can see, it sits in the equation in the same place as the current which is where the name comes from.

So what does this extra term do? Well, something quite remarkable. But before we get to this, there's a story to tell you.

The first four equations above (4.5), (4.6), (4.7) and (4.8) — which include Ampère's law in unmodified form — were arrived at through many decades of painstaking experimental work to try to understand the phenomena of electricity and magnetism. Of course, it took theoretical physicists and mathematicians to express these laws in the elegant language of vector calculus. But all the hard work to uncover the laws came from experiment.

The displacement current term is different. This was arrived at by pure thought alone. This is one of Maxwell's contributions to the subject and, in part, why his name now lords over all four equations. He realised that the laws of electromagnetism captured by (4.5) to (4.8) are not internally consistent: the displacement current term *has* to be there. Moreover, once you add it, there are astonishing consequences.

4.2.1 Why Ampère's Law is Not Enough

We'll look at the consequences in the next section. But for now, let's just see why the unmodified Ampère law (4.7) is inconsistent. We simply need to take the divergence to find

$$\mu_0 \nabla \cdot \mathbf{J} = \nabla \cdot (\nabla \times \mathbf{B}) = 0$$

This means that any current that flows into a given volume has to also flow out. But we know that's not always the case. To give a simple example, we can imagine putting lots of charge in a small region and watching it disperse. Since the charge is leaving the central region, the current does not obey $\nabla \cdot \mathbf{J} = 0$, seemingly in violation of Ampère's law.

There is a standard thought experiment involving circuits which is usually invoked to demonstrate the need to amend Ampère's law. This is shown in the figure. The idea is to cook up a situation where currents are changing over time. To do this, we hook it up to a capacitor — which can be thought of as two conducting plates with a gap between them — to a circuit of resistance R. The circuit includes a switch. When the switch is closed, the current will flow out of the capacitor and through the circuit, ultimately heating up the resistor.

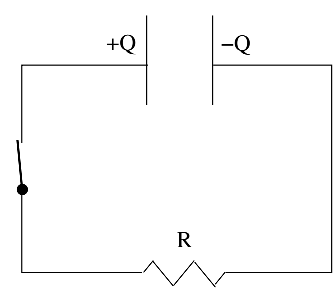


Figure 41:

So what's the problem here? Let's try to compute the magnetic field created by the current at some point along the circuit using Ampère's law. We can take a curve C that surrounds the wire and surface S with boundary C. If we chose S to be the obvious choice, cutting through the wire, then the calculation is the same as we saw in Section 3.1. We have

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I \tag{4.10}$$

where I is the current through the wire which, in this case, is changing with time.

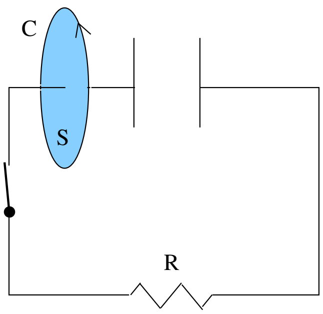


Figure 42: This choice of surface suggests there is a magnetic field

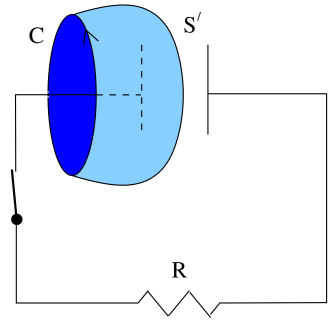


Figure 43: This choice of surface suggests there is none.

Suppose, however, that we instead decided to bound the curve C with the surface S', which now sneaks through the gap between the capacitor plates. Now there is no current passing through S', so if we were to use Ampère's law, we would conclude that there is no magnetic field

$$\int_C \mathbf{B} \cdot d\mathbf{r} = 0 \tag{4.11}$$

This is in contradiction to our first calculation (4.10). So what's going on here? Well, Ampère's law only holds for steady currents that are not changing with time. And we've deliberately put together a situation where I is time dependent to see the limitations of the law.

Adding the Displacement Current

Let's now see how adding the displacement current (4.9) fixes the situation. We'll first look at the abstract issue that Ampère's law requires $\nabla \cdot \mathbf{J} = 0$. If we add the displacement current, then taking the divergence of (4.9) gives

$$\mu_0 \left(\nabla \cdot \mathbf{J} + \epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} \right) = \nabla \cdot (\nabla \times \mathbf{B}) = 0$$

But, using Gauss's law, we can write $\epsilon_0 \nabla \cdot \mathbf{E} = \rho$, so the equation above becomes

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

which is the continuity equation that tells us that electric charge is locally conserved. It's only with the addition of the displacement current that Maxwell's equations become consistent with the conservation of charge.

Now let's return to our puzzle of the circuit and capacitor. Without the displacement current we found that $\mathbf{B} = 0$ when we chose the surface S' which passes between the capacitor plates. But the displacement current tells us that we missed something, because the build up of charge on the capacitor plates leads to a time-dependent electric field between the plates. For static situations, we computed this in (2.10): it is

$$E = \frac{Q}{\epsilon_0 A}$$

where A is the area of each plate and Q is the charge that sits on each plate, and we are ignoring the edge effects which is acceptable as long as the size of the plates is much bigger than the gap between them. Since Q is increasing over time, the electric field is also increasing

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I(t)$$

So now if we repeat the calculation of $\mathbf{B}$ using the surface S', we find an extra term from (4.9) which gives

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \int_{S'} \mu_0 \epsilon_0 \frac{\partial E}{\partial t} = \mu_0 I$$

This is the same answer (4.10) that we found using Ampère's law applied to the surface S.

Great. So we see why the Maxwell equations need the extra term known as the displacement current. Now the important thing is: what do we do with it? As we'll now see, the addition of the displacement current leads to one of the most wonderful discoveries in physics: the explanation for light.

4.3 And There Was Light

The emergence of light comes from looking for solutions of Maxwell's equations in which the electric and magnetic fields change with time, even in the absence of any external charges or currents. This means that we're dealing with the Maxwell equations in vacuum:

$$\nabla \cdot \mathbf{E} = 0 \quad \text{and} \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

The essence of the physics lies in the two Maxwell equations on the right: if the electric field shakes, it causes the magnetic field to shake which, in turn, causes the electric field to shake, and so on. To derive the equations governing these oscillations, we start by computing the second time derivative of the electric field,

$$\mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{\partial}{\partial t}\left(\nabla\times\mathbf{B}\right) = \nabla\times\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\left(\nabla\times\mathbf{E}\right) \tag{4.12}$$

To complete the derivation, we need the identity

$$\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E}$$

But, the first of Maxwell equations tells us that $\nabla\cdot\mathbf{E} = 0$ in vacuum, so the first term above vanishes. We find that each component of the electric field satisfies,

$$\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} - \nabla^2\mathbf{E} = 0 \tag{4.13}$$

This is the wave equation. The speed of the waves, c, is given by

$$c = \sqrt{\frac{1}{\mu_0\epsilon_0}}$$

Identical manipulations hold for the magnetic field. We have

$$\frac{\partial^2\mathbf{B}}{\partial t^2} = -\frac{\partial}{\partial t}\left(\nabla\times\mathbf{E}\right) = -\nabla\times\frac{\partial\mathbf{E}}{\partial t} = -\frac{1}{\mu_0\epsilon_0}\nabla\times\left(\nabla\times\mathbf{B}\right) = \frac{1}{\mu_0\epsilon_0}\nabla^2\mathbf{B}$$

where, in the last equality, we have made use of the vector identity (4.12), now applied to the magnetic field $\mathbf{B}$, together with the Maxwell equation $\nabla\cdot\mathbf{B} = 0$. We again find that each component of the magnetic field satisfies the wave equation,

$$\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} - \nabla^2\mathbf{B} = 0 \tag{4.14}$$

The waves of the magnetic field travel at the same speed c as those of the electric field. What is this speed? At the very beginning of these lectures we provided the numerical values of the electric constant

$$\epsilon_0 = 8.854187817\times 10^{-12}\ m^{-3}\,Kg^{-1}\,s^2\,C^2$$

and the magnetic constant,

$$\mu_0 = 4\pi\times 10^{-7}\ m\,Kg\,C^{-2}$$

Plugging in these numbers gives the speed of electric and magnetic waves to be

$$c = 299792458\ ms^{-1}$$

But this is something that we've seen before. It's the speed of light! This, of course, is because these electromagnetic waves *are* light. In the words of the man himself

"The velocity of transverse undulations in our hypothetical medium, calculated from the electro-magnetic experiments of MM. Kohlrausch and Weber, agrees so exactly with the velocity of light calculated from the optical experiments of M. Fizeau, that we can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena"

James Clerk Maxwell

The simple calculation that we have just seen represents one of the most important moments in physics. Not only are electric and magnetic phenomena unified in the Maxwell equations, but now optics – one of the oldest fields in science – is seen to be captured by these equations as well.

4.3.1 Solving the Wave Equation

We've derived two wave equations, one for $\mathbf{E}$ and one for $\mathbf{B}$. We can solve these independently, but it's important to keep in our mind that the solutions must also obey the original Maxwell equations. This will then give rise to a relationship between $\mathbf{E}$ and $\mathbf{B}$. Let's see how this works.

We'll start by looking for a special class of solutions in which waves propagate in the x-direction and do not depend on y and z. These are called *plane-waves* because, by construction, the fields $\mathbf{E}$ and $\mathbf{B}$ will be constant in the (y, z) plane for fixed x and t.

The Maxwell equation $\nabla \cdot \mathbf{E} = 0$ tells us that we must have E_x constant in this case. Any constant electric field can always be added as a solution to the Maxwell equations so, without loss of generality, we'll choose this constant to vanish. We look for solutions of the form

$$\mathbf{E} = (0, E(x,t), 0)$$

where E satisfies the wave equation (4.13) which is now

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \nabla^2 E = 0$$

The most general solution to the wave equation takes the form

$$E(x,t) = f(x - ct) + g(x + ct)$$

Here $f(x-ct)$ describes a wave profile which moves to the right with speed c. (Because, as t increases, x also has to increase to keep f constant). Meanwhile, $g(x+ct)$ describes a wave profile moving to the left with the speed c.

The most important class of solutions of this kind are those which oscillate with a single frequency ω. Such waves are called *monochromatic*. For now, we'll focus on the right-moving waves and take the profile to be the sine function. (We'll look at the option to take cosine waves or other shifts of phase in a moment when we discuss polarisation). We have

$$E = E_0 \sin\left[\omega\left(\frac{x}{c} - t\right)\right]$$

We usually write this as

$$E = E_0 \sin\left(kx - \omega t\right) \tag{4.15}$$

where k is the *wavenumber*. The wave equation (4.13) requires that it is related to the frequency by

$$\omega^2 = c^2 k^2$$

Equations of this kind, expressing frequency in terms of wavenumber, are called *dispersion relations*. Because waves are so important in physics, there's a whole bunch of associated quantities which we can define. They are:

- The quantity ω is more properly called the *angular frequency* and is taken to be positive. The actual frequency $f = \omega/2\pi$ measures how often a wave peak passes you by. But because we will only talk about ω, we will be lazy and just refer to this as frequency.

- The *period* of oscillation is $T = 2\pi/\omega$.

- The *wavelength* of the wave is $\lambda = 2\pi/k$. This is the property of waves that you first learn about in kindergarten. The wavelength of visible light is between $\lambda \sim 3.9 \times 10^{-7}\ m$ and $7 \times 10^{-7}\ m$. At one end of the spectrum, gamma rays have wavelength $\lambda \sim 10^{-12}\ m$ and X-rays around $\lambda \sim 10^{-10}$ to $10^{-8}\ m$. At the other end, radio waves have $\lambda \sim 1\ cm$ to $10\ km$. Of course, the electromagnetic spectrum doesn't stop at these two ends. Solutions exist for all λ.

  Although we grow up thinking about wavelength, moving forward the wavenumber k will turn out to be a more useful description of the wave.

- E_0 is the *amplitude* of the wave.

So far we have only solved for the electric field. To determine the magnetic field, we use $\nabla \cdot \mathbf{B} = 0$ to tell us that B_x is constant and we again set $B_x = 0$. We know that the other components B_y and B_z must obey the wave equation (4.14). But their behaviour is dictated by what the electric field is doing through the Maxwell equation $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$. This tells us that

$$\mathbf{B} = (0, 0, B)$$

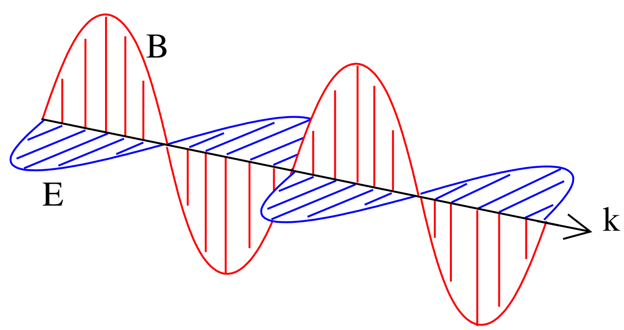
with

$$\frac{\partial B}{\partial t} = -\frac{\partial E}{\partial x} = -kE_0 \cos(kx - \omega t)$$

We find

$$B = \frac{E_0}{c} \sin(kx - \omega t) \tag{4.16}$$

We see that the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields oscillate in phase, but in orthogonal directions. And both oscillate in directions which are orthogonal to the direction in which the wave travels.



Because the Maxwell equations are linear, we're allowed to add any number of solutions of the form (4.15) and (4.16) and we will still have a solution. This sometimes goes by the name of the *principle of superposition*. (We mentioned it earlier when discussing electrostatics). This is a particularly important property in the context of light, because it's what allow light rays travelling in different directions to pass through each other. In other words, it's why we can see anything at all.

The linearity of the Maxwell equations also encourages us to introduce some new notation which, at first sight, looks rather strange. We will often write the solutions (4.15) and (4.16) in complex notation,

$$\mathbf{E} = E_0\, \hat{\mathbf{y}}\, e^{i(kx-\omega t)} \quad , \quad \mathbf{B} = \frac{E_0}{c}\, \hat{\mathbf{z}}\, e^{i(kx-\omega t)} \tag{4.17}$$

This is strange because the physical electric and magnetic fields should certainly be real objects. You should think of them as simply the real parts of the expressions above. But the linearity of the Maxwell equations means both real and imaginary parts of **E** and **B** solve the Maxwell equations. And, more importantly, if we start adding complex **E** and **B** solutions, then the resulting real and imaginary pieces will also solve the Maxwell equations . The advantage of this notation is simply that it's typically easier to manipulate complex numbers than lots of cos and sin formulae.

However, you should be aware that this notation comes with some danger: whenever you compute something which isn't linear in **E** and **B** — for example, the energy stored in the fields, which is a quadratic quantity — you can't use the complex notation above; you need to take the real part first.

4.3.2 Polarisation

Above we have presented a particular solution to the wave equation. Let's now look at the most general solution with a fixed frequency ω. This means that we look for solutions within the ansatz,

$$\mathbf{E} = \mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \quad \text{and} \quad \mathbf{B} = \mathbf{B}_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{4.18}$$

where, for now, both $\mathbf{E}_0$ and $\mathbf{B}_0$ could be complex-valued vectors. (Again, we only get the physical electric and magnetic fields by taking the real part of these equations). The vector $\mathbf{k}$ is called the *wavevector*. Its magnitude, $|\mathbf{k}| = k$, is the wavenumber and the direction of $\mathbf{k}$ points in the direction of propagation of the wave. The expressions (4.18) already satisfy the wave equations (4.13) and (4.14) if ω and $\mathbf{k}$ obey the dispersion relation $\omega^2 = c^2k^2$.

We get further constraints on $\mathbf{E}_0$, $\mathbf{B}_0$ and $\mathbf{k}$ from the original Maxwell equations. These are

$$\begin{aligned}
\nabla\cdot\mathbf{E} = 0 \quad &\Rightarrow \quad i\mathbf{k}\cdot\mathbf{E}_0 = 0 \\
\nabla\cdot\mathbf{B} = 0 \quad &\Rightarrow \quad i\mathbf{k}\cdot\mathbf{B}_0 = 0 \\
\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad &\Rightarrow \quad i\mathbf{k}\times\mathbf{E}_0 = i\omega\mathbf{B}_0
\end{aligned}$$

Let's now interpret these equations:

Linear Polarisation

Suppose that we take $\mathbf{E}_0$ and $\mathbf{B}_0$ to be real. The first two equations above say that both $\mathbf{E}_0$ and $\mathbf{B}_0$ are orthogonal to the direction of propagation. The last of the equations above says that $\mathbf{E}_0$ and $\mathbf{B}_0$ are also orthogonal to each other. You can check that the fourth Maxwell equation doesn't lead to any further constraints. Using the dispersion relation $\omega = ck$, the last constraint above can be written as

$$\hat{\mathbf{k}} \times (\mathbf{E}_0/c) = \mathbf{B}_0$$

This means that the three vectors $\hat{\mathbf{k}}$, $\mathbf{E}_0/c$ and $\mathbf{B}_0$ form a right-handed orthogonal triad. Waves of this form are said to be *linearly polarised*. The electric and magnetic fields oscillate in fixed directions, both of which are transverse to the direction of propagation.

Circular and Elliptic Polarisation

Suppose that we now take $\mathbf{E}_0$ and $\mathbf{B}_0$ to be complex. The actual electric and magnetic fields are just the real parts of (4.18), but now the polarisation does not point in a fixed direction. To see this, write

$$\mathbf{E}_0 = \boldsymbol{\alpha} - i\boldsymbol{\beta}$$

The real part of the electric field is then

$$\mathbf{E} = \boldsymbol{\alpha}\cos(\mathbf{k}\cdot\mathbf{x} - \omega t) + \boldsymbol{\beta}\sin(\mathbf{k}\cdot\mathbf{x} - \omega t)$$

with Maxwell equations ensuring that $\boldsymbol{\alpha}\cdot\mathbf{k} = \boldsymbol{\beta}\cdot\mathbf{k} = 0$. If we look at the direction of $\mathbf{E}$ at some fixed point in space, say the origin $\mathbf{x} = 0$, we see that it doesn't point in a fixed direction. Instead, it rotates over time within the plane spanned by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (which is the plane perpendicular to $\mathbf{k}$).

A special case arises when the phase of $\mathbf{E}_0$ is $e^{i\pi/4}$, so that $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$, with the further restriction that $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = 0$. Then the direction of $\mathbf{E}$ traces out a circle over time in the plane perpendicular to $\mathbf{k}$. This is called *circular polarisation*. The polarisation is said to be *right-handed* if $\boldsymbol{\beta} = \hat{\mathbf{k}} \times \boldsymbol{\alpha}$ and *left-handed* if $\boldsymbol{\beta} = -\hat{\mathbf{k}} \times \boldsymbol{\alpha}$.

In general, the direction of $\mathbf{E}$ at some point in space will trace out an ellipse in the plane perpendicular to the direction of propagation $\mathbf{k}$. Unsurprisingly, such light is said to have *elliptic polarisation*.

General Wave

A general solution to the wave equation consists of combinations of waves of different wavenumbers and polarisations. It is naturally expressed as a Fourier decomposition by summing over solutions with different wavevectors,

$$\mathbf{E}(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3}\ \mathbf{E}(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

Here, the frequency of each wave depends on the wavevector by the now-familiar dispersion relation $\omega = ck$.

4.3.3 An Application: Reflection off a Conductor

There are lots of things to explore with electromagnetic waves and we will see many examples later in the course. For now, we look at a simple application: we will reflect waves off a conductor. We all know from experience that conductors, like metals, look shiny. Here we'll see why.

Suppose that the conductor occupies the half of space $x > 0$. We start by shining the light head-on onto the surface. This means an incident plane wave, travelling in the x-direction,

$$\mathbf{E}_{\text{inc}} = E_0\,\hat{\mathbf{y}}\,e^{i(kx-\omega t)}$$

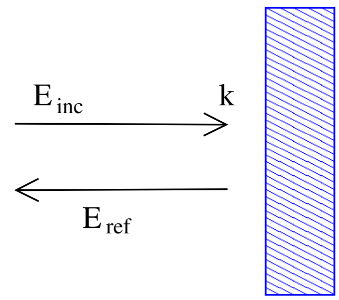


Figure 44:

where, as before, $\omega = ck$. Inside the conductor, we know that we must have $\mathbf{E} = 0$. But the component $\mathbf{E}\cdot\hat{\mathbf{y}}$ lies tangential to the surface and so, by continuity, must also vanish just outside at $x = 0^-$. We achieve this by adding a reflected wave, travelling in the opposite direction

$$\mathbf{E}_{\text{ref}} = -E_0\,\hat{\mathbf{y}}\,e^{i(-kx-\omega t)}$$

So that the combination $\mathbf{E} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{ref}}$ satisfies $E(x = 0) = 0$ as it must. This is illustrated in the figure. (Note, however, that the figure is a little bit misleading: the two waves are shown displaced but, in reality, both fill all of space and should be superposed on top of each other).

We've already seen above that the corresponding magnetic field can be determined by $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$. It is given by $\mathbf{B} = \mathbf{B}_{\text{inc}} + \mathbf{B}_{\text{ref}}$, with

$$\mathbf{B}_{\text{inc}} = \frac{E_0}{c}\,\hat{\mathbf{z}}\,e^{i(kx-\omega t)} \quad \text{and} \quad \mathbf{B}_{\text{ref}} = \frac{E_0}{c}\,\hat{\mathbf{z}}\,e^{i(-kx-\omega t)} \tag{4.19}$$

This obeys $\mathbf{B}\cdot\mathbf{n} = 0$, as it should by continuity. But the tangential component doesn't vanish at the surface. Instead, we have

$$\mathbf{B}\cdot\hat{\mathbf{z}}\big|_{x=0^-} = \frac{2E_0}{c}\,e^{-i\omega t}$$

Since the magnetic field vanishes inside the conductor, we have a discontinuity. But there's no mystery here. We know from our previous discussion (3.6) that this corresponds to a surface current $\mathbf{K}$ induced by the wave

$$\mathbf{K} = \frac{2E_0}{c\mu_0}\,\hat{\mathbf{y}}\,e^{-i\omega t}$$

We see that the surface current oscillates with the frequency of the reflected wave.

Reflection at an Angle

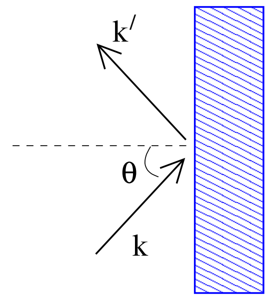


Figure 45:

Let's now try something a little more complicated: we'll send in the original ray at an angle, θ, to the normal as shown in the figure. Our incident electric field is

$$\mathbf{E}_{\rm inc} = E_0\,\hat{\mathbf{y}}\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

where

$$\mathbf{k} = k\cos\theta\,\hat{\mathbf{x}} + k\sin\theta\,\hat{\mathbf{z}}$$

Notice that we've made a specific choice for the polarisation of the electric field: it is out of the page in the figure, tangential to the surface. Now we have two continuity conditions to worry about. We want to add a reflected wave,

$$\mathbf{E}_{\rm ref} = -E_0\,\hat{\boldsymbol{\zeta}}\,e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega' t)}$$

where we've allowed for the possibility that the polarisation $\hat{\boldsymbol{\zeta}}$, the wavevector $\mathbf{k}'$ and frequency ω' are all different from the incident wave. We require two continuity conditions on the electric field

$$(\mathbf{E}_{\rm inc} + \mathbf{E}_{\rm ref})\cdot\hat{\mathbf{n}} = 0 \quad \text{and} \quad (\mathbf{E}_{\rm inc} + \mathbf{E}_{\rm ref})\times\hat{\mathbf{n}} = 0$$

where, for this set-up, the normal vector is $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$. This is achieved by taking $\omega' = \omega$ and $\boldsymbol{\zeta} = \hat{\mathbf{y}}$, so that the reflected wave changes neither frequency nor polarisation. The reflected wavevector is

$$\mathbf{k}' = -k\cos\theta\,\hat{\mathbf{x}} + k\sin\theta\,\hat{\mathbf{z}}$$

We can also check what becomes of the magnetic field. It is $\mathbf{B} = \mathbf{B}_{\rm inc} + \mathbf{B}_{\rm ref}$, with

$$\mathbf{B}_{\rm inc} = \frac{E_0}{c}\,(\hat{\mathbf{k}}\times\hat{\mathbf{y}})\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega' t)} \quad \text{and} \quad \mathbf{B}_{\rm ref} = -\frac{E_0}{c}\,(\hat{\mathbf{k}}'\times\hat{\mathbf{y}})\,e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega' t)}$$

Note that, in contrast to (4.19), there is now a minus sign in the reflected $\mathbf{B}_{\rm ref}$, but this is simply to absorb a second minus sign coming from the appearance of $\hat{\mathbf{k}}'$ in the polarisation vector. It is simple to check that the normal component $\mathbf{B}\cdot\hat{\mathbf{n}}$ vanishes at the interface, as it must. Meanwhile, the tangential component again gives rise to a surface current.

The main upshot of all of this discussion is relationship between $\mathbf{k}$ and $\mathbf{k}'$ which tells us something that we knew when we were five: the angle of incidence is equal to the angle of reflection. Only now we've derived this from the Maxwell equations. If this is a little underwhelming, we'll derive many more properties of waves later.

4.3.4 James Clerk Maxwell (1831-1879)

Still those papers lay before me,
Problems made express to bore me,
When a silent change came o'er me,
In my hard uneasy chair.
Fire and fog, and candle faded,
Spectral forms the room invaded,
Little creatures, that paraded
On the problems lying there.

James Clerk Maxwell, "A Vision of a Wrangler, of a University, of Pedantry, and of Philosophy"

James Clerk Maxwell was a very smart man. Born in Edinburgh, he was a student, first in his hometown, and later in Cambridge, at Peterhouse and then at Trinity. He held faculty positions at the University of Aberdeen (where they fired him) and Kings College London before returning to Cambridge as the first Cavendish professor of physics.

Perhaps the first very smart thing that Maxwell did was to determine the composition of Saturn's rings. He didn't do this using a telescope. He did it using mathematics! He showed that neither a solid nor a fluid ring could be stable. Such rings could only be made of many small particles. For this he was awarded the Adams Prize. (These days you can win this prize for much much less!)

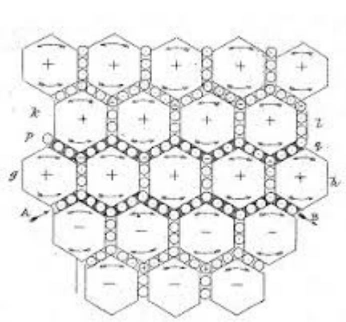


Figure 46: Maxwell's vortices

Maxwell's great work on electromagnetism was accomplished between 1861 and 1862. He started by constructing an elaborate mechanical model of electricity and magnetism in which space is filled by vortices of an incompressible fluid, separated by tiny rotating particles that give rise to electricity. One of his illustrations is shown above. Needless to say, we don't teach this picture of space anymore. From this, he managed to distill everything that was known about electromagnetism into 20 coupled equations in 20 variables. This was the framework in which he discovered the displacement current and its consequences for light.

You might think that the world changed when Maxwell published his work. In fact, no one cared. The equations were too hard for physicists, the physics too hard for mathematicians. Things improved marginally in 1873 when Maxwell reduced his equations to just four, albeit written in quaternion notation. The modern version of Maxwell equations, written in vector calculus notation, is due to Oliver Heaviside in 1881. In all, it took almost 30 years for people to appreciate the significance of Maxwell's acheivement.

Maxwell made a number of other important contributions to science, including the first theory of colour vision and the theory of colour photography. His work on thermodynamics and statistical mechanics deserves at least equal status with his work on electromagnetism. He was the first to understand the distribution of velocities of molecules in a gas, the first to extract an experimental prediction from the theory of atoms and, remarkably, the first (with the help of his wife) to build the experiment and do the measurement, confirming his own theory.

4.4 Transport of Energy: The Poynting Vector

Electromagnetic waves carry energy. This is an important fact: we get most of our energy from the light of the Sun. Here we'd like to understand how to calculate this energy.

Our starting point is the expression (4.3) for the energy stored in electric and magnetic fields,

$$U = \int_V d^3x \; \left(\frac{\epsilon_0}{2}\,\mathbf{E}\cdot\mathbf{E} + \frac{1}{2\mu_0}\,\mathbf{B}\cdot\mathbf{B} \right)$$

The expression in brackets is the energy density. Here we have integrated this only over some finite volume V rather than over all of space. This is because we want to understand the way in which energy can leave this volume. We do this by calculating

$$\begin{aligned} \frac{dU}{dt} &= \int_V d^3x \; \left(\epsilon_0 \mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t} \right) \\ &= \int_V d^3x \; \left(\frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B}) - \mathbf{E}\cdot\mathbf{J} - \frac{1}{\mu_0}\mathbf{B}\cdot(\nabla\times\mathbf{E}) \right) \end{aligned}$$

where we've used the two Maxwell equations. Now we use the identity

$$\mathbf{E}\cdot(\nabla\times\mathbf{B})-\mathbf{B}\cdot(\nabla\times\mathbf{E})=-\nabla\cdot(\mathbf{E}\times\mathbf{B})$$

and write

$$\frac{dU}{dt}=-\int_V d^3x\ \mathbf{J}\cdot\mathbf{E}-\frac{1}{\mu_0}\int_S(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{S} \tag{4.20}$$

where we've used the divergence theorem to write the last term. This equation is sometimes called the *Poynting theorem*.

The first term on the right-hand side is related to something that we've already seen in the context of Newtonian mechanics. The work done on a particle of charge q moving with velocity $\mathbf{v}$ for time δt in an electric field is $\delta W = q\mathbf{v}\cdot\mathbf{E}\,\delta t$. The integral $\int_V d^3x\,\mathbf{J}\cdot\mathbf{E}$ above is simply the generalisation of this to currents: it should be thought of as the rate of gain of energy of the particles in the region V. Since it appears with a minus sign in (4.20), it is the rate of loss of energy of the particles.

Now we can interpret (4.20). If we write it as

$$\frac{dU}{dt}+\int_V d^3x\ \mathbf{J}\cdot\mathbf{E}=-\frac{1}{\mu_0}\int_S(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{S}$$

then the left-hand side is the combined change in energy of both fields and particles in region V. Since energy is conserved, the right-hand side must describe the energy that escapes through the surface S of region V. We define the *Poynting vector*

$$\mathbf{S}=\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$$

This is a vector field. It tells us the magnitude and direction of the flow of energy in any point in space. (It is unfortunate that the canonical name for the Poynting vector is $\mathbf{S}$ because it makes it notationally difficult to integrate over a surface which we usually also like to call $\mathbf{S}$. Needless to say, these two things are not the same and hopefully no confusion will arise).

Let's now look at the energy carried in electromagnetic waves. Because the Poynting vector is quadratic in $\mathbf{E}$ and $\mathbf{B}$, we're not allowed to use the complex form of the waves. We need to revert to the real form. For linear polarisation, we write the solutions in the form (4.17), but with arbitrary wavevector $\mathbf{k}$,

$$\mathbf{E}=\mathbf{E}_0\sin(\mathbf{k}\cdot\mathbf{x}-\omega t)\quad\text{and}\quad\mathbf{B}=\frac{1}{c}(\hat{\mathbf{k}}\times\mathbf{E}_0)\sin(\mathbf{k}\cdot\mathbf{x}-\omega t)$$

The Poynting vector is then

$$\mathbf{S} = \frac{E_0^2}{c\mu_0}\,\hat{\mathbf{k}}\,\sin^2(\mathbf{k}\cdot\mathbf{x} - \omega t)$$

Averaging over a period, $T = 2\pi/\omega$, we have

$$\bar{\mathbf{S}} = \frac{E_0^2}{2c\mu_0}\,\hat{\mathbf{k}}$$

We learn that the electromagnetic wave does indeed transport energy in its direction of propagation $\hat{\mathbf{k}}$. It's instructive to compare this to the energy density of the field (4.3). Evaluated on the electromagnetic wave, the energy density is

$$u = \frac{\epsilon_0}{2}\,\mathbf{E}\cdot\mathbf{E} + \frac{1}{2\mu_0}\,\mathbf{B}\cdot\mathbf{B} = \epsilon_0 E_0^2 \sin^2(\mathbf{k}\cdot\mathbf{x} - \omega t)$$

Averaged over a period $T = 2\pi/\omega$, this is

$$\bar{u} = \frac{\epsilon_0 E_0^2}{2}$$

Then, using $c^2 = 1/\epsilon_0\mu_0$,we can write

$$\bar{\mathbf{S}} = c\bar{u}\hat{\mathbf{k}}$$

The interpretation is simply that the energy $\bar{\mathbf{S}}$ is equal to the energy density in the wave $\bar{u}$ times the speed of the wave, c.

4.4.1 The Continuity Equation Revisited

Recall that, way back in Section 1, we introduced the continuity equation for electric charge,

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$$

This equation is not special to electric charge. It must hold for any quantity that is locally conserved.

Now we have encountered another quantity that is locally conserved: energy. In the context of Newtonian mechanics, we are used to thinking of energy as a single number. Now, in field theory, it is better to think of energy density $\mathcal{E}(\mathbf{x}, t)$. This includes the energy in both fields and the energy in particles. Thinking in this way, we notice that (4.20) is simply the integrated version of a continuity equation for energy. We could equally well write it as

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla\cdot\mathbf{S} = 0$$

We see that the Poynting vector $\mathbf{S}$ is to energy what the current $\mathbf{J}$ is to charge.