# 6 Turbulence

When the speed of fluid flows increases beyond some critical value, things have a tendency to go a bit squirly. The calm, serene laminar flows that we've seen in earlier chapters become unstable and are replaced by something messy and dirty, with the fluid moving in seemingly random directions, eddies forming and stretching, before disintegrating into smaller eddies. This is turbulent flow.

There is every reason to believe that turbulent flow is correctly described by the Navier-Stokes equation, not least computer simulations which, in this context, go by the name of 'DNS, standing for "direct numerical simulation". But understanding the full details of turbulent flow remains, to put it mildly, a formidable problem. Turbulence kicks in when the Reynolds number is greater than some critical value $Re > Re_{\rm crit}$. The exact number depends on the kind of flow we're looking at, but a ballpark figure is

$$Re_{\rm crit} \sim 10^3$$

At these speeds, the advective term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ in the Navier-Stokes equation is important. This is the only non-linear term in the equation and it drives the system to a chaotic state, with the motion wildly dependent on the initial conditions. The challenge is to understand this motion.

This challenge is, it turns out, hard. Despite many decades of study, turbulence remains poorly understood. It is clear that it is not feasible to find explicit solutions exhibiting turbulence. Instead, we will retreat and look at averaged properties of flows. This might seem like a strange thing to do. After all, the Navier-Stokes equation is, at the end of a day, just a differential equation and, as such, its behaviour is entirely deterministic. Nonetheless, turbulent motion appears to be random. This can be traced to the sensitive dependence on initial conditions that characterises chaotic systems. To proceed, we will embrace this randomness and work in a statistical sense. Rather than trying to analyse any specific solution, we will instead try to extract properties of appropriately averaged solutions.

Our goals in this section will be limited. We won't look at any specific turbulent flows, such as boundary layers or wakes. Instead, we will just try to understand some very general properties that are shared by all turbulent flows, at least in some regime. Nor will we study the interesting behaviour that happens for flows around the critical Reynolds number $Re_{\rm crit}$, where instabilities develop. Instead we focus on what happens with $Re \gg Re_{\rm crit}$, a regime known as *fully developed turbulence.*

## 6.1 Mean Flow

As we mentioned above, to understand turbulence it's necessary to think on a more probabilistic level about the Navier-Stokes equation. But given that the Navier-Stokes equation is purely deterministic, it's not obvious what this means. If we're going to think about averaged properties, the first question we should ask is: what are we actually averaging over?

There are different answers that we could give to this. One way to proceed is to average over different initial conditions to the Navier-Stokes equation. We could pick some collection of initial conditions, all of which look similar. Because of the chaotic nature of the equation, each will give rise to very different solutions. We could then try to figure out average properties of these solutions. This is known as the *ensemble average* and is similar to the philosophy underlying Kinetic Theory and Statistical Mechanics.

Alternatively, we could do something that feels more physical. A turbulent velocity field $\mathbf{u}(\mathbf{x}, t)$ varies rapidly in both space and time and we could choose to average over either of these. There is a general expectation (although no proof) that, for a typical flow, it doesn't matter which average we choose: all should give the same answer. This goes by the name of the *ergodic hypothesis*.

Here we will average over time (because it turns out to be the simplest). We decompose the complicated turbulent flow $\mathbf{u}(\mathbf{x}, t)$ into an averaged, mean flow $\mathbf{U}(\mathbf{x}, t)$ together with some fluctuations $\delta\mathbf{u}(\mathbf{x}, t)$,

$$\mathbf{u} = \mathbf{U} + \delta\mathbf{u} \tag{6.1}$$

where to define the mean flow we average over some time scale $T$,

$$\mathbf{U}(\mathbf{x}, t) = \langle \mathbf{u}(\mathbf{x}, t) \rangle := \frac{1}{T} \int_t^{t+T} dt' \; \mathbf{u}(\mathbf{x}, t')$$

This is called *Reynolds averaging*. There are two options for how to think about the time scale $T$,

- We could simply take $T \to \infty$. In this case, we have a steady mean flow $\mathbf{U}(\mathbf{x})$.
- Alternatively, we may have a situation in which there are two different time scales in the flow. The turbulent fluctuations occur over some short time scale $\tau_{\text{short}}$, which is superimposed on some averaged flow which takes place over some much longer time scale $\tau_{\text{long}}$. In this case we could take $\tau_{\text{short}} \ll T \ll \tau_{\text{long}}$ to get a mean velocity field $\mathbf{U}(\mathbf{x}, t)$ which varies only over the long time scale.

In what follows, we'll adopt the second of these. This isn't for any particularly well-motivated physical reason, but simply because it's not much more effort to do this and it obviously includes the $T \to \infty$ situation as a special case in which $\mathbf{U}(\mathbf{x})$ is stationary.

Since our mean flow is $\mathbf{U} = \langle \mathbf{u} \rangle$, the complicated velocity fluctuations are $\delta\mathbf{u} = \mathbf{u} - \langle \mathbf{u} \rangle$. By construction, this means that the average of the fluctuations vanishes:

$$\langle \delta\mathbf{u} \rangle = 0 \tag{6.2}$$

Importantly the Reynolds averaging commutes with spatial differentiation so if our fluid is incompressible then both the mean flow and the fluctuations must be separately incompressible,

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \langle \nabla \cdot \mathbf{u} \rangle = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{U} = 0 \quad \Rightarrow \quad \nabla \cdot \delta\mathbf{u} = 0$$

We do a similar averaging for other fields, including the pressure which we write as

$$P = \langle P \rangle + \delta P \tag{6.3}$$

with $\langle P \rangle$ defined by a time-average in the same way as (6.1). Again, we have designed things so that the average fluctuation necessarily vanishes: $\langle \delta P \rangle = 0$.

We will actually explore the averaged properties of the Navier-Stokes equation twice in these lectures. Our focus in this section will on deriving an equation for the mean flow $\mathbf{U}$ after integrating out the fluctuations. This won't take us particularly far, not least because it feels like we're throwing out the baby and keeping the bath water since, for many situations, the fluctuations are much more interesting than the mean flow! Nonetheless, we include this approach because it gives some intuition for the difficulties involved. Moreover, this is a popular approach when modelling turbulence in situations where there is clearly some overarching mean flow, with turbulence bubbling away underneath (it turns out that this takes often place in a regime of Reynolds numbers $10^3 \lesssim Re \lesssim 10^5$) and will allow us to define some commonly used concepts such as "Reynolds stress" and "eddy viscosity". Then, in Section 6.3, we will retrace the same steps, this time focussing on the fluctuations themselves. It's only in this second approach that we'll start to make some real progress.

### 6.1.1 The Reynolds Averaged Navier-Stokes Equation

If we substitute the decomposition (6.1) and (6.3) into the Navier-Stokes equation, we have

$$\rho \left( \frac{\partial (\mathbf{U} + \delta\mathbf{u})}{\partial t} + (\mathbf{U} + \delta\mathbf{u}) \cdot \nabla (\mathbf{U} + \delta\mathbf{u}) \right) = -\nabla(\langle P \rangle + \delta P) + \mu \nabla^2 (\mathbf{U} + \delta\mathbf{u}) \tag{6.4}$$

We've neglected any further forces acting on the fluid, such as the forcing term needed to drive turbulence, but they can be added as needed. Now we average this equation and use the fact that $\langle \delta\mathbf{u}\rangle = \langle \tilde{P}\rangle = 0$. We need to be a little careful with the time derivative term: we have

$$\left\langle \frac{\partial(\delta\mathbf{u})}{\partial t}\right\rangle = \frac{1}{T}\int_t^{t+T} dt'\ \frac{\partial(\delta\mathbf{u})}{\partial t'} = \frac{1}{T}\left[\delta\mathbf{u}(\mathbf{x},t+T) - \delta\mathbf{u}(\mathbf{x},t)\right]$$
$$= \frac{1}{T}\frac{\partial}{\partial t}\int_t^{t+T} dt'\ \delta\mathbf{u}(\mathbf{x},t') = \frac{\partial\langle\delta\mathbf{u}\rangle}{\partial t} = 0$$

where, again, we've used the fact that $\langle\delta\mathbf{u}\rangle = 0$. We're then left with

$$\rho\left(\frac{\partial\mathbf{U}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{U} + \langle\delta\mathbf{u}\cdot\nabla\delta\mathbf{u}\rangle\right) = -\nabla\langle P\rangle + \mu\nabla^2\mathbf{U}$$

This is *almost* the Navier-Stokes equation for the averaged velocity $\mathbf{U}$. The only difference is the term $\delta\mathbf{u}\cdot\nabla\delta\mathbf{u}$, quadratic in the fluctuations, that wasn't killed by averaging. We take this over to the right-hand side and treat it as part of the stress tensor, writing the Navier-Stokes equation for the averaged flow in the form (3.7)

$$\rho\left(\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x^j}\right) = \frac{\partial\sigma_{ij}}{\partial x^j} \tag{6.5}$$

This is the *Reynolds' averaged Navier-Stokes* equation. The stress tensor on the right-hand side is

$$\sigma_{ij} = -\langle P\rangle\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x^j} + \frac{\partial U_j}{\partial x^i}\right) - \rho\langle\delta u_i\delta u_j\rangle \tag{6.6}$$

where we've used the fact that $\nabla\cdot\delta\mathbf{u} = 0$ in writing it in this form. We see that, in this approximation, the role of the fluctuations is to guide the mean flow through the additional term

$$R_{ij} = \rho\langle\delta u_i\delta u_j\rangle$$

in the stress tensor. This is known as the *Reynolds stress* or, sometimes, the *turbulent stress*. (Actually, more often the Reynolds stress is defined without the factor of $\rho$, even though that isn't, strictly, a stress.) So if we want to understand how the mean flow flows, we need to understand something about the variance of the fluctuations $\langle\delta u_i\delta u_j\rangle$.

### Finding Closure

Our next task is to get an expression for this extra contribution to the stress tensor $R_{ij}$. To this end, if we subtract the averaged Navier-Stokes equation (6.5) from our starting point (6.4), we have

$$\rho\left(\frac{\partial(\delta\mathbf{u})}{\partial t}+(\mathbf{U}\cdot\nabla)\delta\mathbf{u}+(\delta\mathbf{u}\cdot\nabla)\mathbf{U}+(\delta\mathbf{u}\cdot\nabla)\delta\mathbf{u}-\langle\delta\mathbf{u}\cdot\nabla\delta\mathbf{u}\rangle\right)=-\nabla\delta P+\mu\nabla^2\delta\mathbf{u}$$

If we multiply this by $\delta\mathbf{u}$, we get the following expression for the tensor $\delta u_i\delta u_j$

$$\rho\left(\frac{\partial(\delta u_i\delta u_j)}{\partial t}+U_l\frac{\partial(\delta u_i\delta u_j)}{\partial x^l}+\delta u_i\delta u_l\frac{\partial U_j}{\partial x^l}+\delta u_j\delta u_l\frac{\partial U_i}{\partial x^l}+\frac{\partial}{\partial x^l}(\delta u_l\delta u_i\delta u_j)\right.$$
$$\left.+\,\delta u_i\frac{\partial R_{lj}}{\partial x^l}+\delta u_j\frac{\partial R_{li}}{\partial x^l}\right)=-\delta u_i\frac{\partial(\delta P)}{\partial x^j}-\delta u_j\frac{\partial(\delta P)}{\partial x^i}+\mu^2\left(\delta u_i\nabla^2\delta u_j+\delta u_j\nabla^2\delta u_i\right)$$

We now take the average to get an equation for the Reynolds' stress tensor that, schematically, takes the form

$$\frac{\partial R_{ij}}{\partial t}+(\mathbf{U}\cdot\nabla)R_{ij}=-\rho\frac{\partial}{\partial x^l}\langle\delta u_l\delta u_i\delta u_j\rangle+\text{other stuff}$$

where the other stuff includes other averages such as $\langle\delta P\,\delta\mathbf{u}\rangle$. The key point is that we can get ourselves an equation for $R_{ij}$, but it involves a 3-point average $\langle\delta\mathbf{u}^3\rangle$. And if we try to get an equation for $\langle\delta\mathbf{u}^3\rangle$ then you probably won't be surprised to hear that it involves $\langle\delta\mathbf{u}^4\rangle$, and so on. We find that we have an infinite hierarchy of equations. This is not unusual in physics when doing this kind of analysis. (An analogous situation arises in Kinetic Theory when deriving the Boltzmann equation where it is called the BBGKY hierarchy.) Within the context of turbulence, this is known as the *closure problem*: the set of equations don't close and keep forcing you to look at the next order in fluctuations.

What to do about it? Well, there is no mathematically well-defined way to truncate this infinite series of equations. Nor is there a physical reason to expect some simplification to occur. Turbulence is a strongly coupled problem and to do things properly, you really need to worry about this infinite series of equations. Of course, that's not particularly practical. So to proceed, the usual strategy is just to make something up. This made-up thing is unlikely to have any real justification behind it for the simple reason that no such justification exists. But these made-up approaches to have a name: they are collectively called "closure models". There are many.

Here's the simplest example of a made-up thing, due to Boussinesq. Suppose that, for some reason, the three-point averages $\langle \delta u^3 \rangle$ and higher are actually unimportant. Then we can look for an expression for the Reynolds' stress $R_{ij}$ that depends only on the mean flow $\mathbf{U}$. One, particularly simple option is to postulate that it takes the form

$$R_{ij} = -\mu_T \left( \frac{\partial U_i}{\partial x^j} + \frac{\partial U_j}{\partial x^i} \right) + \frac{2}{3} K \delta_{ij} \tag{6.7}$$

which depends on two, unknown constant $\mu_T$ and $K$. The latter has a nice physical interpretation: it is the kinetic energy in the fluctuations $K = \frac{1}{2}\rho \langle \delta \mathbf{u} \cdot \delta \mathbf{u} \rangle$. The former has a nice name: it is called the *turbulent viscosity*, or sometimes the *eddy viscosity*. This guess for the Reynolds' stress has the nice effect of simply renormalising the stress tensor $\sigma_{ij}$ on the right-hand side of the averaged Navier-Stokes equation which, from (6.6), becomes

$$\sigma_{ij} = -\left( \langle P \rangle + \frac{2}{3} K \right) \delta_{ij} + (\mu + \mu_T) \left( \frac{\partial U_i}{\partial x^j} + \frac{\partial U_j}{\partial x^i} \right)$$

This takes the same form as the usual stress tensor, but with an effective pressure $P_{\text{eff}} = \langle P \rangle + \frac{2}{3} K$ and an effective viscosity $\mu_{\text{eff}} = \mu + \mu_T$. The end result is that this guess has led us back to the original Navier-Stokes equation, but with an extra contribution to the pressure and a shifted value of the viscosity.

There are many other, more sophisticated closure models, in which one tries to incorporate $\langle \delta \mathbf{u}^3 \rangle$ corrections and so on, and then gives up at some higher order. They may be more sophisticated, but it's not obvious that they are more right and we won't discuss them here. Instead, we will reset and go in a different direction.

## 6.2 Some Dimensional Analysis

"Big whorls have little whorls
Which feed on their velocity,
And little whorls have lesser whorls
And so on to viscosity."

*Lewis Fry Richardson*

Turbulence is one of the great problems in physics. To make progress it's clear that we're going to have to break out some pretty powerful machinery. And things don't get more powerful than dimensional analysis. In this section, we will use dimensional analysis to get a handle on one very specific property of turbulence: what happens to the energy?

The set of ideas described here is due to Richardson, Taylor and Kolmogorov. These ideas culminated in a series of papers by Kolmogorov in 1941 and fluid dynamicists often refer to this argument, rather elliptically, as K41. It is, I think, one of the greatest applications of dimensional analysis in all of physics.

To get started, we need a few facts about turbulent flows. First is the observation that turbulence is very much a dissipative phenomenon: if you leave a turbulent fluid alone, it will quickly relax back to equilibrium with the turbulent properties dying away. This means that something must be feeding the turbulence to keep it alive. In other words, there has to be some injection of energy into the system. This could be due to some external pressure difference, some shear effect due to gravity, or some teaspoon stirring the fluid. The details won't concern us and we'll model this energy injection by some external force density $\mathbf{f}(\mathbf{x}, \mathbf{t})$, as in our original Navier-Stokes equation (3.2). The work done by this force is

$$\text{Work Done} = \int d^3x \; \mathbf{f} \cdot \mathbf{u}$$

Although we're doing work on the system, the turbulent flow doesn't speed up over time. Or, said more precisely, we're not interested in situations where the fluid speeds up over time. Instead, this energy drains away through dissipation. We've already seen that dissipation occurs due to viscosity. The kinetic energy of the fluid is K.E. $= \frac{1}{2}\rho \int d^3x \, \mathbf{u}^2$ and, from (3.8), the energy lost is

$$\frac{\partial(\text{K.E.})}{\partial t} = -\rho\nu \int d^3x \; \left|\frac{\partial u_i}{\partial x^j}\right|^2$$

In a steady state, we simply equate the work done with the lost energy

$$\text{Work Done} = \rho\nu \int d^3x \; \left|\frac{\partial u_i}{\partial x^j}\right|^2 \tag{6.8}$$

The key to understanding the physics is to appreciate that the processes on either side of this equation take place at very different scales. On the left-hand side, the driving force is a macroscopic phenomenon, typically comparable to the size of the entire system. Meanwhile, on the right-hand side dissipation is a phenomenon that occurs at a much smaller level. This shows up in the equation above because dissipation is greatest when there are large gradients in velocity. So energy goes in at the largest scale, and out at the smallest. We would like to put some equations to these words to make them more concrete.

We start by quantifying the work done. Suppose that the driving force takes place over some large length scale $L$. This is sometimes called the *outer scale*. Over this scale, the mean velocity field will vary with some magnitude $\Delta U$. The Reynolds number (3.17) for the flow is roughly

$$Re \sim \frac{\Delta U\, L}{\nu}$$

and, by assumption, we have $Re \gg 1$.

The turbulent flow is not laminar, but swirling in many directions. It's useful to think of this in terms of vorticity, with the different swirling referred to as eddies. In this somewhat cartoon picture, the flow on the large, outer scale consists of eddies of size $L$.

Now our first stab at some dimensional analysis. We will focus our attention on $\epsilon$, the work done per unit mass, defined by

$$\epsilon = \frac{\text{Work Done}}{\rho V} \tag{6.9}$$

where $V$ is the volume of the system. This has dimension $[\epsilon] = L^2T^{-3}$. Since this energy is injected on the large outer scale $L$, we expect that it manifests itself in terms of the macroscopic quantities $\Delta U$ and $L$ that we introduced above. Dimensional analysis means that there is a unique possibility, namely

$$\epsilon \sim \frac{(\Delta U)^3}{L} \tag{6.10}$$

What about the energy dissipated? The relation (6.8) equates the work done with the energy lost, so

$$\frac{(\Delta U)^3}{L} \sim \frac{\nu}{V}\int d^3x\ |\nabla \mathbf{u}|^2 \tag{6.11}$$

Suppose that the dissipation also comes from these same, macroscopically large velocity gradients. Then we would have $|\nabla \mathbf{u}| \sim \Delta U/L$, which would give a dissipation rate

$$|\nabla \mathbf{u}| \sim \frac{\Delta U}{L} \quad\Rightarrow\quad \frac{\nu}{V}\int d^3x\ |\nabla \mathbf{u}|^2 \sim \frac{\nu(\Delta U)^2}{L^2} \sim \frac{1}{Re}\frac{(\Delta U)^3}{L} \tag{6.12}$$

But that's nowhere near enough! It's less dissipation than we need by a factor of $Re$ and, as we have stressed, turbulent flow takes place at values of the Reynolds number $Re \gg 1$. It must be the case that dissipation takes place with a larger $|\nabla \mathbf{u}|$ than that caused by the driving force. Which means that there must larger gradients of $\mathbf{u}$ and so physics taking place on some smaller scale. That, of course, agrees with experimental observations of turbulence, where there are features on many different scales. We would like to construct a simple model of this.

### 6.2.1 Scale Invariance

Energy is injected at some length scale $L$ where it causes eddies of size $L$. But, as we have seen, these eddies don't dissipate enough energy and structures on smaller scales must form. It's useful to think of these new length scales emerging as the original, large eddies break up into smaller ones.

There's nothing clean and simple going on here where, for example, the initial eddy neatly splits in two. Instead, as we have stressed, turbulence is a messy and complicated phenomenon and it consists of eddies of all possible sizes, at least within a range, bounded above by the outer scale $L$ and, as we will see shortly, bounded below by a much smaller length scale $l_0$.

As the larger eddies break up, they lose energy which is fed into the smaller eddies below them. The eddies of size $l$ have some velocity difference $\Delta u_l$. These eddies are being fed some energy by the bigger boys above them but, at the same time, they're losing energy as they themselves decay into the smaller eddies below. The key assumption here is that, at least in some regime of scales, this process takes place with no dissipation at all. Indeed, we've seen that the dissipation due to the very largest eddies (6.12) is suppressed by $1/Re$, and we make the approximation that this can be ignored completely. This is essentially the statement that viscosity is irrelevant for this aspect of turbulence. This assumption means that the eddies at size $l$ receive some energy (per time per unit mass) $\epsilon$ and promptly pass it down to smaller scales. This process is known as the *energy cascade* and was first proposed by Richardson.

The kinetic energy of eddies at scale $l$ and below is $\sim (\Delta u_l)^2$. (Note: rather unusually, we're talking about the energy in vortices of size $\lesssim l$ rather than the more usual formulation of vortices between size, say $l$ and $l + dl$.) Suppose that these eddies hold on to the energy for some "cascade time" $\tau_l$ which, as the notation suggests, also depends on $l$. The energy passing through is equal to (6.10) and, at a given scale $l$, is

$$\frac{(\Delta u_l)^2}{\tau_l} \sim \frac{(\Delta U)^3}{L} \sim \epsilon \tag{6.13}$$

The next question is: what is the cascade time $\tau_l$? On dimensional grounds, there's only one possibility: this it must be

$$\tau_l \sim \frac{l}{\Delta u_l} \tag{6.14}$$

Note that it's important that we don't allow the viscosity $\nu$ to sneak into this formula since that carries dimensions $[\nu] = L^2 T^{-1}$ and messes up the dimensional analysis. This is a reiteration of a point that we made above: the viscosity is irrelevant for the energy cascade since no dissipation is taking place. If we now substitute (6.14) into (6.13), we find that there is a scale invariance in the energy cascade, with the velocities of eddies of size $l$ obeying the same formula independent of $l$,

$$\frac{(\Delta u_l)^3}{l} \sim \frac{(\Delta U)^3}{L} \sim \epsilon$$

Rearranging this, we find that the velocities of eddies of size $l$ scale as

$$\Delta u_l \sim (\epsilon l)^{1/3} \tag{6.15}$$

This is known as the *Kolmogorov-Obhukov* law. (Later, in 6.3, we will give a more precise formulation of this law and see that it is better viewed as originally written as $(\Delta u)^3 \sim \epsilon l$.)

### Viscosity Brings the Cascade to a Halt

The energy cascade does not involve dissipation, merely a transfer of energy from large scales to small. But at some point this energy cascade should come to a halt and the energy $\epsilon$ should be dissipated into heat. To understand when this happens, we return to the statement (6.11) which says that the energy in is equal to the energy out. We saw that this certainly wasn't satisfied by the dissipation from large eddies. But now we can ask: for what scale $l_0$ does this energy balance hold?

The eddies of size $l_0$ have velocity differentials $\Delta u_0$ and (6.11) holds if

$$\frac{\nu(\Delta u_0)^2}{l_0^2} \sim \epsilon$$

But we also know from (6.15) how $\Delta u_0$ and $l_0$ are related. This gives us the *Kolmogorov scale*, also known as the *inner scale*,

$$l_0 \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \sim \left(\frac{\nu^3 L}{(\Delta U)^3}\right)^{1/4} \sim \frac{L}{(Re)^{3/4}}$$

Clearly $l_0 \ll L$ since $Re \gg 1$. This is where energy is finally dissipated to heat. Note that the Kolmogorov scale can also be determined by dimensional analysis: it is the unique length scale that can be formed from the energy dissipation rate $\epsilon$ and the viscosity $\nu$.

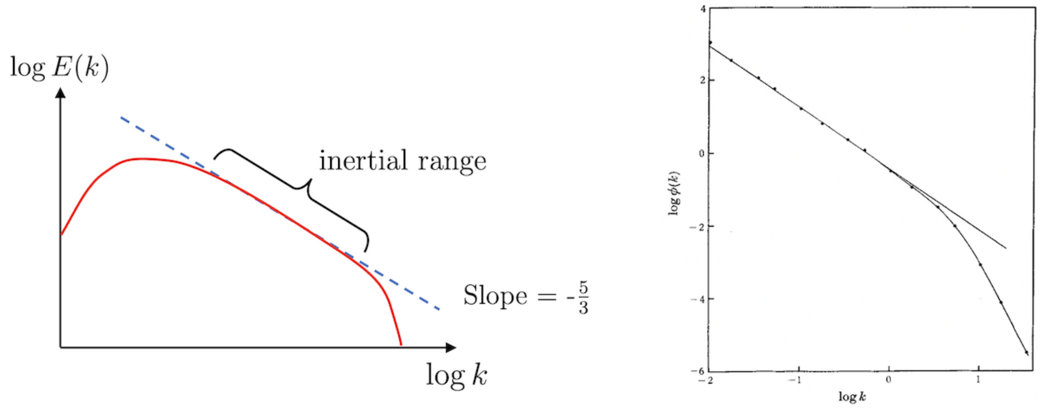


**Figure 35**. On the left, a sketch of the expected behaviour of $E(k)$ based on dimensional analysis. The energy is injected at small $k$ and dissipated at large $k$, with the characteristic $E(k) \sim k^{-5/3}$ in the inertial range. On the right, data.

This finishes our crude, dimensional analysis approach to turbulence. The energy is injected at some scale $L$ and dissipated at the much smaller scale $l_0 = L/(Re)^{3/4}$. The scales in-between, $l_0 \ll l \ll L$ are called the *inertial range* and exhibit a scale invariant energy cascade. Note that we're not really used the Navier-Stokes equation at any point in the analysis. Everything follows from the hypothesis that, in the inertial range, big eddies cascade down into smaller eddies in a way that does not involve any dissipation.

### Wavenumbers

We can also phrase the energy cascade in terms of wavenumbers $k \sim 1/l$. Let $E(k)\,dk$ be the kinetic energy per unit mass storied in eddies with wavenumber between $k$ and $k + dk$. Then $E(k)$ has dimension $[E] = L^3T^{-2}$. On dimensional grounds, we must have

$$E(k) \sim \epsilon^{2/3} k^{-5/3} \tag{6.16}$$

The expected behaviour is sketched on the left of figure 35. This $k^{-5/3}$ behaviour matches well with experiment. The first test was done with a probe attached to a ship which sailed back and forth in a tidal channel just off Vancouver Island. The data[15] is shown in the right-hand side of Figure 35, with the straight line having slope $-5/3$.

[15]This is taken from Grant, Stewart and Moilliet, "Turbulence Spectra from a Tidal Channel". The function $\phi(k)$ shown on the vertical axis is closely related to $E(k)$ described above: it is $2E = k^2\partial^2\phi/\partial k^2 - k\partial\phi/\partial k$.

We can reconcile the result $E(k) \sim k^{-5/3}$ with our previous analysis. If we integrate over all wavenumbers larger than $k$, we have

$$\int_k^\infty dk'\; E(k') \sim \epsilon^{2/3} k^{-2/3} \sim (\epsilon l)^{2/3} \sim (\Delta u_l)^2$$

This is the kinetic energy $(\Delta u_l)^2$ which, from (6.15), we see should indeed scale as $(\Delta u_l)^2 \sim (\epsilon l)^{2/3}$, in agreement with (6.15).

#### Briefly, Intermittency

There is a slight problem with the dimensional analysis that we described above. It's not really correct. The subtlety comes because we assume that the energy cascade retains no memory of the outer scale $L$ on which we initially inject energy. If this scale could sneak into the energy cascade, then it would infect our dimensional analysis and the result (6.16) could be corrected to

$$E(k) \sim \epsilon^{2/3} k^{-5/3} (kL)^\zeta$$

for some $\zeta$ known as the *intermittency exponent*, the name arising because the experimental manifestation is that turbulent flows have periods in which the velocity fluctuations are weak, interspersed with intermittent bursts in which the fluctuations are much larger. In many situations, this exponent seems to be small. But it is not known how to calculate it.

There are close similarities between this story and what's seen in so-called critical points in phase transitions. There too one sees scale invariance and a naive dimensional analysis argument (known in that context as "mean field theory") suggests a particular value for certain exponents. But that's not the value that is seen experimentally. The flaw in that context is that the short distance UV cut-off (i.e. the atomic scale) unexpectedly sneaks in to the dimensional analysis and contributes what's known as an "anomalous critical exponent". It is entirely analogous to the intermittency exponent in turbulence, except that now this involve the long distance IR cut-off. You can read more about critical points and how to compute anomalous exponents in the lectures on Statistical Field Theory.

### 6.3 Velocity Correlations

In this section, we put a little more meat on the dimensional analysis argument above. In particular, we will derive a more rigorous version of the Kolmogorov-Obhukov law (6.15).

We will do this by returning to the averaging procedure that we introduced in Section 6.1, but now with a focus on the fluctuations $\delta\mathbf{u}$ rather than the mean flow $\mathbf{U}$. To do this, it's simplest if we assume that there are only fluctuations with a vanishing mean flow $\mathbf{U} = 0$. This means, of course, that $\delta\mathbf{u} = \mathbf{u}$ and we can drop the $\delta$'s. With no background flow governing the fluctuations, all points in space and time are, statistically at least, the same. This is known as homogeneous and isotropic turbulence and it offers the simplest setting where we may hope to understand a little of what's going on.

(An aside: if the restriction to vanishing $\mathbf{U}$ seems to restrictive then there is another way to think about things. We could, alternatively, zoom into some patch where $\mathbf{U}$ is approximately constant and then boost to a frame in which it vanishes. In this way, our analysis should hold locally even for general mean flows $\mathbf{U}(\mathbf{x})$.)

The averages that we encountered in the previous section involved fluctuations at the same point in space. For example, the Reynolds' stress tensor is

$$R_{ij} = \rho\langle\delta u_i(\mathbf{x})\delta u_j(\mathbf{x})\rangle$$

with the two velocities evaluated at the same point. Because we no longer have a background flow $\mathbf{U}(\mathbf{x})$, the system has translational invariance. This means, among other things, that $R_{ij}$ doesn't depend on the point $\mathbf{x}$ at which it's evaluated.

There is now an obvious generalisation in which the correlation between velocity fields is computed at different points,

$$C_{ij}(\mathbf{x}_1, \mathbf{x}_2) = \langle u_i(\mathbf{x}_1)u_j(\mathbf{x}_2)\rangle$$

A number of constraints on this correlation function follow simply from the symmetries of our problem which are enhanced because we're assuming $\mathbf{U} = 0$. First, translational invariance means that it is only a function of the difference $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ and we write $C_{ij}(\mathbf{x}_1, \mathbf{x}_2) = C_{ij}(\mathbf{r})$. Second, isotropy, together with parity invariance, means that the six components of a general symmetric tensor are reduced to just two,

$$C_{ij}(\mathbf{r}) = C_{TT}(r)\,(\delta_{ij} - \hat{r}_i\hat{r}_j) + C_{LL}(r)\hat{r}_i\hat{r}_j \tag{6.17}$$

Here $C_{TT}(r)$ is the transverse correlation function and $C_{LL}(r)$ the longitudinal correlation function. Note, in particular, that $C_{ij}(\mathbf{r}) = C_{ij}(-\mathbf{r})$, which follows by parity invariance.

When $r=0$, so the two points in the correlation function coincide, we have a handle on the correlation function: it must take the form

$$C_{ij}(0) = \frac{2}{3}\mathcal{E}\delta_{ij} \tag{6.18}$$

where $\mathcal{E} = \frac{1}{2}\langle \mathbf{u}\cdot\mathbf{u}\rangle$ is the kinetic energy (divided by the density $\rho$). This coincides with the expression (6.7) when $\mathbf{U}=0$ (with $\mathcal{E} = K/\rho$). But while (6.7) was pulled out of thin air, here (6.18) follows because, in the absence of a background mean flow, the only symmetric two-tensor that we have at our disposal is $\delta_{ij}$.

There's one last constraint that comes from the fact that the fluid is incompressible, $\nabla\cdot\mathbf{u}=0$, which means that

$$\frac{\partial C_{ij}}{\partial r^i} = 0$$

We can use this to relate $C_{TT}(r)$ and $C_{LL}(r)$. To do this, we write $\hat{r}_k = r_k/r$ and make use of the identities

$$\frac{\partial \hat{r}_k}{\partial r^i} = \frac{\delta_{ik}}{r} - \frac{r_i r_k}{r^3} \quad \Rightarrow \quad \frac{\partial(\hat{r}_i\hat{r}_j)}{\partial r^i} = 2\frac{\hat{r}_j}{r}$$

Then

$$\frac{\partial C_{ij}}{\partial r^i} = \frac{dC_{TT}}{dr}\hat{r}_i\left(\delta_{ij} - \hat{r}_i\hat{r}_j\right) + \frac{dC_{LL}}{dr}\hat{r}_i\hat{r}_i\hat{r}_j + 2(C_{LL}-C_{TT})\frac{\hat{r}_j}{r}$$

The first term vanishes and we're left with the simple expression

$$C_{TT}(r) = C_{LL} + \frac{r}{2}\frac{dC_{LL}}{dr} \tag{6.19}$$

This is known as the *von Kármán relation.*

In what follows, we'll also have need for the closely related *structure function.* This looks at the correlation between the difference in the velocity fluctuations between two points,

$$S_{ij}(\mathbf{r}) = \langle (u_i(\mathbf{x}_1) - u_i(\mathbf{x}_2))(u_j(\mathbf{x}_1) - u_j(\mathbf{x}_2))\rangle \tag{6.20}$$

Expanding out the four terms, the structure function can be trivially expressed in terms of the correlation function as

$$S_{ij}(\mathbf{r}) = 2C_{ij}(0) - 2C_{ij}(\mathbf{r}) = \frac{4}{3}\mathcal{E}\delta_{ij} - 2C_{ij}(\mathbf{r})$$

Note, in particular, that $S_{ij}(0) = 0$. As with the correlation function, we can decompose the structure function into transverse and longitudinal pieces

$$S_{ij}(\mathbf{r}) = S_{TT}(r)\left(\delta_{ij} - \hat{r}_i\hat{r}_j\right) + S_{LL}(r)\hat{r}_i\hat{r}_j$$

Comparing to the components of the correlation function, we have

$$S_{LL}(r) = \frac{4}{3}\mathcal{E} - 2C_{LL}(r) \tag{6.21}$$

with a similar expression for the transverse component: $S_{TT}(r) = \frac{4}{3}\mathcal{E} - 2C_{TT}(r)$.

One advantage of working with the structure function is that we can make contact with the simple dimensional analysis arguments of Section 6.2. In particular, if we take the Kolmogorov-Obhukov law (6.15) at face value then it should apply to the structure function, telling us to expect

$$S_{ij}(r) \sim r^{2/3} \quad \text{when} \quad l_0 \ll r \ll L \tag{6.22}$$

This should hold only in the inertial range, as shown. Since the von Kármán relation (6.19) also holds for the structure function, it tells us that, in the inertial range, $S_{TT} = \frac{4}{3}S_{LL}$. (Actually, as part of our analysis we'll get a better understanding of the Kolmogorov-Obhukov law and see that the result $S_{ij} \sim r^{2/3}$ is not exact: nonetheless, we may hope that it's not a wildly inaccurate expectation.)

### 6.3.1 Navier-Stokes for Correlation Functions

We'll attempt to compute the correlation functions using the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{1}{\rho}(\nabla P - \mathbf{f}) + \nu\nabla^2\mathbf{u} \tag{6.23}$$

We've included the driving force $\mathbf{f}$ which, as in Section 6.2, will be responsible for the injection of energy. Because averaging commutes with differentiation (both with respect to time and space), we have

$$\begin{aligned}
\frac{\partial C_{ij}(\mathbf{r};t)}{\partial t} &= \langle \partial_t u_i(\mathbf{x}_1)\, u_j(\mathbf{x}_2)\rangle + \langle u_i(\mathbf{x}_1)\,\partial_t u_j(\mathbf{x}_2)\rangle \\
&= -\frac{\partial}{\partial x_1^k}\langle u_k(\mathbf{x}_1)u_i(\mathbf{x}_1)\, u_j(\mathbf{x}_2)\rangle - \frac{\partial}{\partial x_2^k}\langle u_i(\mathbf{x}_1)u_k(\mathbf{x}_2)u_j(\mathbf{x}_2)\rangle \\
&\quad - \frac{1}{\rho}\langle \partial_i P(\mathbf{x}_1)\, u_j(\mathbf{x}_2)\rangle - \frac{1}{\rho}\langle u_i(\mathbf{x}_1)\partial_j P(\mathbf{x}_2)\rangle \\
&\quad + \frac{1}{\rho}\langle f_i(\mathbf{x}_1)u_j(\mathbf{x}_2)\rangle + \frac{1}{\rho}\langle u_i(\mathbf{x}_1)f_j(\mathbf{x}_2)\rangle \\
&\quad + \nu\langle \nabla^2 u_i(\mathbf{x}_1)\, u_j(\mathbf{x}_2)\rangle + \nu\langle u_i(\mathbf{x}_1)\nabla^2 u_j(\mathbf{x}_2)\rangle
\end{aligned} \tag{6.24}$$

In the first line we have used the incompressibility of the fluid, $\nabla \cdot \mathbf{u} = 0$, to take the derivative outside the average. We can see immediately from this that, as in Section 6.1, to get an equation for the two-point correlation function $C_{ij}(\mathbf{r})$, we need to know something about the three-point function $\langle \delta u^3 \rangle$. If we tried to get an equation for $\langle \delta u^3 \rangle$ then we would, as before, find that it pushes us towards the four-point function $\langle \delta u^4 \rangle$ and so on. This is the same closure problem that we met previously.

This time, however, there is something that we can say without going down the rabbit hole. First we'll sort out some of the terms in (6.24), and then return to the 3-point function.

**Claim:** The pressure terms vanish: $\langle u_i(\mathbf{x}_1) P(\mathbf{x}_2) \rangle = 0$.

**Proof:** Using homogeneity and isotropy, we must have $\langle P(\mathbf{x}_1) u_i(\mathbf{x}_2) \rangle = f(r)\hat{r}_i$ for some function $f(r)$ where, as before, $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$. But incompressibility tells us that

$$0 = \frac{\partial}{\partial x_1^i} \langle u_i(\mathbf{x}_1) P(\mathbf{x}_2) \rangle = f'(r) + \frac{2}{r} f(r) \quad \Rightarrow \quad f(r) = \frac{\alpha}{r^2}$$

for some constant $\alpha$. But the correlation $\langle u_i(\mathbf{x}_1) P(\mathbf{x}_2) \rangle$ should be finite as $r \to 0$. Which means that we must have $\alpha = 0$ and $\langle u_i(\mathbf{x}_1) P(\mathbf{x}_2) \rangle = 0$. $\square$

Next, we turn our attention to the energy injection terms involving the correlation $\langle u_i(\mathbf{x}_1) f_j(\mathbf{x}_2) \rangle$. To make sense of this, we need to specify the form of the forcing term although, as explained in Section 6.2, the expectation is that the energy will cascade down to smaller scales in a way that is ultimately independent of the forcing term we choose. It turns out that things are particularly simple if we pick a random forcing term that takes the form of Gaussian white noise, meaning that

$$\langle f_i(\mathbf{x}_1, t_1) f_j(\mathbf{x}_2, t_2) \rangle = \delta(t_1 - t_2) \rho^2 \epsilon_{ij}(\mathbf{r}) \tag{6.25}$$

for some choice of function $\epsilon_{ij}(\mathbf{r})$ which we get to decide. We'll take it to be symmetric, so $\epsilon_{ij} = \epsilon_{ji}$ and isotropic so $\epsilon_{ij}(\mathbf{r}) = \epsilon_{ij}(-\mathbf{r})$. We'll shortly see how this tensor $\epsilon_{ij}$ is related to the work done per unit mass $\epsilon$ that played such a key role in our dimensional analysis argument of Section 6.2. One important property of white noise is that the value of the force at any time $t$ is completely uncorrelated with its value at any earlier time $t'$. This will be important shortly.

**Claim:** With the force given by the Gaussian white noise (6.25), the correlation between the force and velocity is give by

$$\frac{1}{\rho}\Big( \langle f_i(\mathbf{x}_1) u_j(\mathbf{x}_2) \rangle + \langle u_i(\mathbf{x}_1) f_j(\mathbf{x}_2) \rangle \Big) = \epsilon_{ij}(\mathbf{r}) \tag{6.26}$$

**Proof:** We integrate up the Navier-Stokes equation (6.23) to get the expression for the velocity field $\mathbf{u}(\mathbf{x}, t)$,

$$\mathbf{u}(\mathbf{x}, t) = \int_0^t dt' \left[ -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}(\nabla P - \mathbf{f}) + \nu \nabla^2 \mathbf{u} \right]$$

with $t \geq t' \geq 0$. Substituting this into the correlation function then gives

$$\langle f_i(\mathbf{x}_1, t) u_j(\mathbf{x}_2, t) \rangle = \int_0^t dt' \, \langle f_i(\mathbf{x}_1, t) \left[ -u_k \frac{\partial u_j}{\partial x^k} - \frac{1}{\rho} \left( \frac{\partial P}{\partial x^j} - f_j \right) + \nu \nabla^2 u_j \right] (\mathbf{x}_2, t') \rangle$$

All the fields $\mathbf{u}(\mathbf{x}, t')$ and $P(\mathbf{x}, t')$ are uncorrelated with the force $\mathbf{f}(\mathbf{x}, t)$ at a later time $t > t'$ because the force is taken to be white noise. The only contribution comes from in this correlation function therefore comes from

$$\langle f_i(\mathbf{x}_1, t) u_j(\mathbf{x}_2, t) \rangle = \frac{1}{\rho} \int_0^t dt' \, \langle f_i(\mathbf{x}_1, t) f_j(\mathbf{x}_2, t') \rangle = \rho \int_0^t dt' \, \delta(t - t') \epsilon_{ij}(\mathbf{r})$$

Now, if we integrate $\int dt' \, \delta(t - t')$ over a range that includes the point $t' = t$ then the integral clearly gives 1. Here, however, the point $t' = t$ sits right at the end of the integral range. This is a bit ambiguous but there's a sensible way to think about it. If we were to extend the integral a little further beyond $t$, the integral clearly gives 1. But this gets a contribution both from our original integral and from the extension. It seems fair to share these. This then gives

$$\int_0^t dt' \, \delta(t - t') = \frac{1}{2} \quad \Rightarrow \quad \langle f_i(\mathbf{x}_1, t) u_j(\mathbf{x}_2, t) \rangle = \frac{\rho}{2} \epsilon_{ij}(\mathbf{r})$$

Adding the second contribution then gives the claimed result (6.26). □

From (6.26), we see that if we take the trace of the tensor $\epsilon_{ij}(\mathbf{r})$ and evaluate it at $\mathbf{r} = 0$ we have the work done (divided by the density),

$$\epsilon_{ii}(0) = \frac{2}{\rho} \langle \mathbf{f} \cdot \mathbf{u} \rangle := 2\epsilon \tag{6.27}$$

On the right-hand side, we have the same $\epsilon$ that we met in Kolmogorov's dimensional analysis argument in Section 6.2. We'll make contact with these ideas later in this section. For now, note that we can make the same tensor decomposition as in (6.17) and write

$$\epsilon_{ij}(\mathbf{r}) = \epsilon_{TT}(r) \left( \delta_{ij} - \hat{r}_i \hat{r}_j \right) + \epsilon_{LL} \hat{r}_i \hat{r}_j$$

This gives $\epsilon_{LL}(r) = \hat{r}_i \hat{r}_j \epsilon_{ij}(\mathbf{r})$. If we evaluate this tensor at $\mathbf{r} = 0$ then we must have $\epsilon_{ij}(0) = \epsilon_{LL}(0)\delta_{ij}$ as there is no other tensor in the game. This then gives

$$\epsilon_{LL}(0) = \frac{2}{3}\epsilon \tag{6.28}$$

Let's pause to take stock. Our equation (6.24) for the correlation function has now become

$$\begin{aligned}\frac{\partial C_{ij}(\mathbf{r};t)}{\partial t} = &-\frac{\partial}{\partial x_1^k}\langle u_k(\mathbf{x}_1)u_i(\mathbf{x}_1)\, u_j(\mathbf{x}_2)\rangle - \frac{\partial}{\partial x_2^k}\langle u_i(\mathbf{x}_1)u_k(\mathbf{x}_2)u_j(\mathbf{x}_2)\rangle \\ &+ \epsilon_{ij}(\mathbf{r}) + 2\nu\nabla^2 C_{ij}(\mathbf{r})\end{aligned} \tag{6.29}$$

which is starting to look a little simpler. Our next task is to better understand the structure of the three-point functions.

### 6.3.2 The Structure of the Three-Point Function

We write the three-point function as

$$C_{ij,k}(\mathbf{x}_1, \mathbf{x}_2) = \langle u_i(\mathbf{x}_1)u_j(\mathbf{x}_1)u_k(\mathbf{x}_2)\rangle$$

The comma is there to remind us that two of the velocities are evaluated at $\mathbf{x}_1$ and the third at $\mathbf{x}_2$. (The comma doesn't mean differentiation. This isn't general relativity!) By isotropy, we must have

$$C_{ij,k}(\mathbf{x}_1, \mathbf{x}_2) = C_{ij,k}(\mathbf{r}) = C_{ji,k}(\mathbf{r})$$

and by parity invariance,

$$C_{ij,k}(\mathbf{r}) = -C_{ij,k}(-\mathbf{r}) = -\langle u_i(-\mathbf{x}_1)u_j(-\mathbf{x}_1)u_k(-\mathbf{x}_2)\rangle = -\langle u_i(\mathbf{x}_2)u_j(\mathbf{x}_2)u_k(\mathbf{x}_1)\rangle \tag{6.30}$$

with the overall minus sign arising because the correlation function involves an odd number of velocities. In the final equality, we've invoked translational invariance and shifted all arguments by $\mathbf{x}_1 + \mathbf{x}_2$.

In fact, the tensor structure means that we can reduce the correlation function to just three function of $r = |\mathbf{r}|$,

$$C_{ij,k}(\mathbf{r}) = A(r)\delta_{ij}\hat{r}_k + B(r)\left(\delta_{ik}\hat{r}_j + \delta_{jk}\hat{r}_i\right) + D(r)\hat{r}_i\hat{r}_j\hat{r}_k$$

These different functions are further related by the incompressibility of the flow. For the two-point function, this gave us the von Kármán relation (6.19). For the three-point function, we have

**Claim:** Incompressibility gives the relations

$$B = -\frac{1}{2r}\frac{d(r^2 A)}{dr} \tag{6.31}$$

and

$$3A + 2B + D = 0 \tag{6.32}$$

**Proof:** Incompressibility $\nabla \cdot \mathbf{u} = 0$ means that

$$\frac{\partial C_{ij,k}}{\partial x_2^k} = -\frac{\partial C_{ij,k}}{\partial r^k} = 0$$

We use $\partial \hat{r}_k / \partial r^i = \delta_{ik}/r - \hat{r}_i \hat{r}_k / r$ and, after a line or two of algebra, we get

$$\frac{\partial C_{ij,k}}{\partial r^k} = \delta_{ij}\left(A' + \frac{2A}{r} + \frac{2B}{r}\right) + \hat{r}_i \hat{r}_j \left(2B' - \frac{2B}{r} + D' + \frac{2D}{r}\right)$$

Each of these tensor structures must individually vanish. The first gives the relation (6.31). The vanishing of the second term can be written as

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2(D + 2B)\right] = \frac{6B}{r}$$

If we substitute in our expression for $B$ in the recently proved (6.31), this becomes

$$\frac{d}{dr}\left[r^2(D + 2B + 3A)\right] = 0 \quad \Rightarrow \quad 3A + 2B + D = \frac{\text{constant}}{r^2} \tag{6.33}$$

We can fix the constant by looking at $C_{ij,k}(0)$ which must take the value $C_{ij,k}(0) = 0$ for the simple reason that there's no invariant 3-tensor with the right symmetry properties that it can equal. This means that the constant in (6.33) is actually zero and so we get (6.32). $\square$

We can combine the two relations (6.31) and (6.32) to give $D = rA' - A$. The upshot is that the three-point correlation function actually depends on just a single function $A(r)$,

$$C_{ij,k}(\mathbf{r}) = A\delta_{ij}\hat{r}_k - \frac{1}{2}\left(rA' + 2A\right)\left(\delta_{ik}\hat{r}_j + \delta_{jk}\hat{r}_i\right) + (rA' - A)\hat{r}_i\hat{r}_j\hat{r}_k \tag{6.34}$$

Next, it will also be useful to introduce the three-point structure function, which is the obvious generalisation of (6.20),

$$S_{ijk}(\mathbf{r}) = \langle (u_i(\mathbf{x}_1) - u_i(\mathbf{x}_2))(u_j(\mathbf{x}_1) - u_j(\mathbf{x}_2))(u_k(\mathbf{x}_1) - u_k(\mathbf{x}_2))\rangle$$

This is completely symmetric in all three indices. If we expand and cancel terms (remembering that we have translational invariance so $\langle \mathbf{u}(\mathbf{x}_1)^3\rangle = \langle \mathbf{u}(\mathbf{x}_2)^3\rangle$) then we can relate the structure function to the correlation function

$$S_{ijk}(\mathbf{r}) = -2(C_{ij,k} + C_{ik,j} + C_{jk,i})$$

If we substitute in the expression (6.34) we get

$$S_{ijk}(\mathbf{r}) = 2(A + rA')(\delta_{ij}\hat{r}_k + \delta_{ik}\hat{r}_j + \delta_{jk}\hat{r}_i) - 6(rA' - A)\hat{r}_i\hat{r}_j\hat{r}_k \tag{6.35}$$

The fully longitudinal part of the structure function is defined to be

$$S_{LLL}(r) = S_{ijk}(\mathbf{r})\hat{r}_i\hat{r}_j\hat{r}_k$$

From (6.35), we see that this is the same thing as the function $A(r)$, up to an overall constant,

$$S_{LLL}(r) = 12A(r)$$

In what follows, we'll work with $S_{LLL}(r)$ as the object that describes the three-point function.

### 6.3.3 The von Kármán-Howarth Equation

Now we can return to our expression (6.29) for the dynamics of the correlation function. This reads

$$\frac{\partial C_{ij}}{\partial t} = -\frac{\partial C_{ik,j}}{\partial r^k} - \frac{\partial C_{jk,i}}{\partial r^k} + \epsilon_{ij}(\mathbf{r}) + 2\nu\nabla^2 C_{ij}(\mathbf{r})$$

where the indices in the second $\partial C/\partial r$ term have rearranged themselves courtesy of (6.30). We focus on the longitudinal component of the two-point correlator, $C_{LL} = C_{ij}\hat{r}_i\hat{r}_j$ which obeys,

$$\frac{\partial C_{LL}}{\partial t} - \epsilon_{LL} - 2\nu\hat{r}_i\hat{r}_j\nabla^2 C_{ij} = -2\hat{r}_i\hat{r}_j\frac{\partial C_{ik,j}}{\partial r^k} \tag{6.36}$$

We have a little bit of work to do to move those $\hat{r}_i\hat{r}_j$ terms inside the derivatives. On the right-hand side, we have

$$\begin{aligned}\hat{r}_i\hat{r}_j\frac{\partial C_{ik,j}}{\partial r^k} &= \frac{\partial}{\partial r^k}(C_{ik,j}\hat{r}_i\hat{r}_j) - C_{ik,j}\frac{\partial(\hat{r}_i\hat{r}_j)}{\partial r^k} \\ &= \frac{\partial}{\partial r^k}(C_{ik,j}\hat{r}_i\hat{r}_j) - \frac{1}{r}\left(C_{ii,j}\hat{r}_j + C_{ik,k}\hat{r}_i - 2C_{ij,k}\hat{r}_i\hat{r}_j\hat{r}_k\right)\end{aligned} \tag{6.37}$$

where we've again made use of the identity $\partial \hat{r}_i / \partial r^k = \delta_{ik}/r - \hat{r}_i \hat{r}_k / r$. Now, from (6.34), we can compute the various contractions of $C_{ij,k}$ with $\hat{r}$,

$$\begin{aligned}
C_{ik,j}\hat{r}_i\hat{r}_j &= A\hat{r}_k - (rA' + 2A)\,\hat{r}_k + (rA' - A)\hat{r}_k = -2A\hat{r}_k \\
C_{ii,j}\hat{r}_j &= 3A - (rA' + 2A) + (rA' - A) = 0 \\
C_{ik,k}\hat{r}_i &= A - 2\,(rA' + 2A) + (rA' - A) = -rA' - 4A \\
C_{ij,k}\hat{r}_i\hat{r}_j\hat{r}_k &= A - (rA' + 2A) + (rA' - A) = -2A
\end{aligned}$$

So (6.37) becomes

$$\hat{r}_i\hat{r}_j \frac{\partial C_{ik,j}}{\partial r^k} = -2A\frac{\partial}{\partial r^k}(A\hat{r}_k) + A' = -\frac{4A}{r} - A'$$

We have a similar task for the $\nu \nabla^2 C_{ij}$ term in (6.36). For this, it's best to return to the expression (6.17). A slightly tedious exercise in algebra gives

$$\begin{aligned}
\nabla^2 C_{ij} &= \nabla^2 C_{TT}\,(\delta_{ij} - \hat{r}_i\hat{r}_j) + \nabla^2 C_{LL}\hat{r}_i\hat{r}_j + \frac{C_{LL} - C_{TT}}{r^2}\,(2\delta_{ij} - 6\hat{r}_i\hat{r}_j) \\
&\quad + \frac{1}{r}\left(\frac{\partial C_{LL}}{\partial r^k} - \frac{\partial C_{TT}}{\partial r^k}\right)(\delta_{ik}\hat{r}_j + \delta_{jk}\hat{r}_i - 2\hat{r}_i\hat{r}_j\hat{r}_k)
\end{aligned}$$

If we now contract with $\hat{r}_i\hat{r}_j$, the second line disappears and we're left with

$$\hat{r}_i\hat{r}_j \nabla^2 C_{ij} = \nabla^2 C_{LL} - \frac{4}{r^2}(C_{LL} - C_{TT})$$

But now we can use the von Kármán relation (6.19) to write this purely in terms of $C_{LL}$,

$$\begin{aligned}
\hat{r}_i\hat{r}_j \nabla^2 C_{ij} &= \nabla^2 C_{LL} + \frac{2}{r}\frac{\partial C_{LL}}{\partial r} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial C_{LL}}{\partial r}\right) + \frac{2}{r}\frac{\partial C_{LL}}{\partial r} \\
&= \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial C_{LL}}{\partial r}\right)
\end{aligned}$$

Now we can put these pieces back into the expression (6.36), which becomes an equation that relates the longitudinal two-point function $C_{LL}$ with the three-point function $S_{LLL} = 12A$,

$$\frac{\partial C_{LL}}{\partial t} - \epsilon_{LL} - \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial C_{LL}}{\partial r}\right) = \frac{1}{6}\left(S'_{LLL} + \frac{4 S_{LLL}}{r}\right)$$

We can express everything in terms of the structure function using the relation (6.21) which relates $S_{LL} = \frac{4}{3}\mathcal{E} - 2C_{LL}$ with $\mathcal{E} = \frac{1}{2}\langle \mathbf{u}\cdot\mathbf{u}\rangle$ the average kinetic energy (divided by the density). The end result is:

$$\frac{\partial S_{LL}}{\partial t} = \frac{4}{3}\frac{\partial \mathcal{E}}{\partial t} - 2\epsilon_{LL} + \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial S_{LL}}{\partial r}\right) - \frac{1}{3r^4}\frac{\partial}{\partial r}\left(r^4 S_{LLL}\right) \tag{6.38}$$

This is the *von Kármán-Howarth equation*. It tells us how the two-point correlations of the velocity evolve with time. In the limit $\mathbf{r}\to 0$, it reduces to the equation describing energy balance.

### 6.3.4 Kolmogorov's 4/5

it's a short step from the von Kármán-Howarth equation to what we want. We'll focus on the static case where we've reached a kind of equilibrium where, as described in Section 6.2, all the energy fed into the system at large scales is lost to viscosity at small scales. This allows us to drop the time derivatives in (6.38) and we have

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\left(2\nu\frac{\partial S_{LL}}{\partial r} - \frac{S_{LLL}}{3}\right)\right) = 2\epsilon_{LL}(r)$$

We will further assume that the energy is injected on large scales. Following (6.28), we interpret this as the statement

$$\epsilon_{LL}(\mathbf{r}) = \frac{1}{3}\epsilon_{ii}(\mathbf{r}) \approx \frac{2}{3}\epsilon + \ldots$$

where $\epsilon$ is the work done per unit mass that we met in the dimensional analysis argument of Section 6.2. We then integrate our differential equation to get

$$2\nu\frac{\partial S_{LL}}{\partial r} - \frac{1}{3}S_{LLL} = \frac{4}{15}\epsilon r$$

where the constants of integration have been put to zero using the fact that $S_{LL}(0) = S_{LLL}(0) = 0$. Rearranging, we have an expression for the three-point correlations,

$$S_{LLL}(r) = -\frac{4}{5}\epsilon r + 6\nu\frac{\partial S_{LL}}{\partial r} \tag{6.39}$$

This is *Kolmogorov's* $4/5^{\text{th}}$ *'s law*. It's important because the number of exact results about turbulence can be counted on one finger. This is the one. We recognise the first term as a more rigorous version of the Kolmogorov-Obhukov law (6.15) that we derived using dimensional analysis. In fact, this result tells us how to think of the Kolmogorov-Obhukov law: it holds for three-point functions.

The second term in (6.39) is a correction. A naive application of Kolmogorov-Obhukov suggests that the two-point structure function scales as $S_{LL}(r) \sim r^{2/3}$. (We already mentioned this is (6.22).) But that's not the way correlation functions work: just because $\langle u^3 \rangle \sim r$ doesn't mean that $\langle u^p \rangle \sim r^{p/3}$. Nonetheless, if we take this as a ballpark guess for the behaviour of the correlation function then the second term is much smaller than the first if we focus on distance scales that are much larger than the Kolmogorov viscosity scale, $r \gg l_0 \sim (\nu^3/\epsilon)^{1/4}$. This is because if $S_{LL} \sim (\epsilon r)^{2/3}$ then $S'_{LL} \sim \epsilon^{2/3} r^{-1/3}$ and hence $\nu S'_{LL} \sim (l_0/r)^{4/3} \epsilon r \ll \epsilon r$. So, in the inertial range we have the Kolmogorov-Obhukov result

$$S_{LLL}(r) \approx -\frac{4}{5}\epsilon r$$

Except now we know the prefactor. It is $-4/5$.