Part III General Relativity Preparatory Workshop

João Melo, David Tong

This workshop will gradually build up the machinery necessary to describe some simple examples of the physics behind General Relativity. The objective is getting some physical intuition to ground the mathematical discussion in the main lecture course. We shall start by reviewing Euler-Lagrange equations and index notation in the case of the dynamics of non-relativistic particles; then we move to the relativistic case, sticking to special relativity; finally we discuss the equivalence principle which sets the stage for GR; we end with a discussion of some simple predictions from General Relativity (that have been experimentally tested!).

This will mostly consist of exercises intercut with just the necessary information to be able to proceed to the next exercise. For more details consult the notes provided on the course webpage. Starred exercises will be covered in class, non-starred exercises will only be covered if time allows.

1. Non-Relativistic Particles

Our tool of choice throughout these lectures is the action. The advantage of the action is that it makes various symmetries manifest. And, as we shall see, there are some deep symmetries in the theory of general relativity that must be maintained. This greatly limits the kinds of equations which we can consider and, ultimately, will lead us inexorably to the Einstein equations.

Therefore, we start by reviewing this principle in the simple case of a non-relativistic particle. We describe the position of a particle by coordinates x^i where, for now, we take $i = 1, 2, 3$ for a particle moving in 3-dim space. Importantly, there is no need to identify the coordinates x^i with the (x, y, z) axes of Euclidean space; they could be any coordinate system of your choice.

We shall also use the *Einstein summation convention*, which amounts to saying that if an index is repeated twice (we usually call that a *dummy index*) we sum over all values that index can take; if it only appears once, it is implied that that expression is valid, independently, for all possible values of the index (these are *free indices*); and, if an index appears more than twice, you have made a mistake.

We want a way to describe how the particle moves between fixed initial and final positions,

$$x^i(t_1) = x^i_{\text{initial}}, \qquad \text{and}, \qquad x^i(t_2) = x^i_{\text{final}} \tag{1.1}$$

To do this, we consider all possible paths $x^i(t)$ subject to the boundary conditions above. To each of these paths, we assign a number called the *action* S. This is defined as

$$S[x^i(t)] = \int_{t_1}^{t_2} \mathrm{d}t\, L(x^i(t), \dot{x}^i(t)) \tag{1.2}$$

where the function $L(x^i, \dot{x}^i)$ is the *Lagrangian* which specifies the dynamics of the system. The action is a functional; this means that you hand it an entire function worth of information, $x^i(t)$, and it spits back only a single number.

The *principle of least action* is the statement that the true path taken by the particle is an extremum of S. Although this is a statement about the path as a whole, it is entirely equivalent to a set of differential equations which govern the dynamics. These are known as the *Euler-Lagrange equations*.

Problem 1.1.* Take the action evaluated on a given path $x^i(t)$, vary that path slightly $x^i(t) + \delta x^i(t)$, and, by imposing that $x^i(t)$ is an extremum of the action, derive the Euler-Lagrange equations,

$$\frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = 0 \tag{1.3}$$

We shall first consider the non-relativistic motion of a particle of mass m in flat Euclidean space $\mathbb{R}^3$. For once, the coordinates $x^i = (x, y, z)$ actually are the usual Cartesian coordinates. The Lagrangian that describes the motion is simply the kinetic energy,

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \tag{1.4}$$

The Euler-Lagrange equations (1.3) applied to this Lagrangian simply tell us that $\ddot{x}^i = 0$, which is the statement that free particles move at constant velocity in straight lines.

Imagine we want to use some funky coordinate system. In that case, the infinitesimal distance between any two points, x^i and $x^i + \mathrm{d}x^i$, called the *line element*, will no longer be given by Pythagoras's theorem $ds^2 = \mathrm{d}x^i\,\mathrm{d}x^i$, instead, it will, in general, take the form

$$ds^2 = g_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{1.5}$$

Where the 3×3 matrix g_{ij} is called the *metric*. The metric is symmetric: $g_{ij} = g_{ji}$ since the anti-symmetric part drops out of the distance when contracted with $\mathrm{d}x^i\,\mathrm{d}x^j$. Further it is positive definite and non-degenerate, so its inverse exists.

Now imagine, we're simply given some coordinate system and a line element as in (1.5). Is this necessarily just flat space on some funky coordinate system, or can it be that some metrics are in essence distinct from the flat space metric? It turns out the answer is no, as in, not all metrics correspond just flat space in a different coordinate system. Those metrics are what we use to describe *curved space*. How to make these notions more rigorous, and tell whether a given metric describes flat or curved space will be covered in more detail in the course of the main lectures.

Before we proceed, a quick comment: it matters in this subject whether the indices i, j are up or down. Once again, the details will be left for the main lectures, but for now, remember that coordinates have superscripts while the metric has two subscripts.

The Lagrangian with an arbitrary metric is the obvious generalisation of (1.4),

$$L = \frac{1}{2}m g_{ij}(x)\dot{x}^i\dot{x}^j \tag{1.6}$$

Problem 1.2.* Prove that the Euler-Lagrange equations arising from (1.6) can be written as

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 \tag{1.7}$$

where

$$\Gamma^i_{jk}(x) = \frac{1}{2} g^{il} (\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{jk}) \tag{1.8}$$

are called the *Christoffel symbols*, where $\partial_i \equiv \frac{\partial}{\partial x^i}$, and g^{ij} is the inverse matrix of g_{ij}.

This equation is called the *geodesic equation* and solutions to this equation are known as *geodesics*.

Problem 1.3. Starting from the standard flat space metric $ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, find the metric in polar coordinates, which are given by,

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

Then compute the Christoffel symbols by directly applying (1.8). Finally, compute the equations of motion arising from the Lagrangian (1.6) (substituting the metric you found) by directly varying the action. Deduce the Chirstoffel symbols from those equations. Which method was quicker?

2. Special Relativity

On our path to full General Relativity we shall first briefly do a pit stop at Special Relativity to set the stage. Therefore we consider a particle moving in Minkowski spacetime $\mathbb{R}^{1,3}$. We will work with Cartesian coordinates $x^\mu = (ct, x, y, z)$, where now $\mu, \nu = 0, 1, 2, 3$. The metric on this spacetime is

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1) \tag{2.1}$$

so that the distance between two neighbouring points labelled by x^μ and $x^\mu + \mathrm{d}x^\mu$ is

$$ds^2 = \eta_{\mu\nu}\, \mathrm{d}x^\mu \, \mathrm{d}x^\nu \tag{2.2}$$

Pairs of points with $ds^2 < 0$ are said to be *timelike separated*; those for which $ds^2 > 0$ are *spacelike separated*; and those for which $ds^2 = 0$ are said to be *lighlike* separated or, more commonly *null*.

Consider the path of a particle through spacetime. In the previous section, we labelled the positions along the path using the time coordinate t for some inertial observer. But, to build a relativistic description of the particle motion, we want time to sit on much the same footing as the spatial coordinates. For this reason, we will introduce a new parameter - let's call it σ - which labels where we are along the worldline of the trajectory. For now it doesn't matter what parametrisation we choose; we will only ask that σ increases monotonically along the trajectory. We'll label the start and end points of the trajectory by σ_1 and σ_2 respectively, with $x^\mu(\sigma_1) = x^\mu_{\text{initial}}$ and $x^\mu(\sigma_2) = x^\mu_{\text{final}}$.

The action for a relativistic particle has a nice geometric interpretation: it extremises the distance between the starting and end points in Minkowski space. A particle with rest mass m follows a timelike trajectory, for which any two points on the curve have $ds^2 < 0$. We therefore take the action to be

$$S = -mc \int_{x_{\text{initial}}}^{x_{\text{final}}} \sqrt{-ds^2} =$$
$$= -mc \int_{\sigma_1}^{\sigma_2} \mathrm{d}\sigma \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}} \tag{2.3}$$

The coefficients in front ensure that the action has dimensions $[S] = \text{Energy} \times \text{Time}$ as it should.

Problem 2.1.* Prove that the action is Lorentz invariant. That is, that it is invariant under $x^\mu \to \Lambda^\mu{}_\rho x^\rho$ where $\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ (i.e. $\Lambda \in O(1,3)$)

Problem 2.2.* Prove that the action is reparametrisation invariant. That is, it is invariant under the change $\sigma \to \tilde{\sigma}(\sigma)$.

These two symmetries are of rather different character. The first one is a true symmetry in the sense that if we find a solution to the equations of motion, then we can act with a Lorentz transformation to generate a new solution. The second one is not really a symmetry, in the sense it does not generate new solutions from old ones. Instead, it is a redundancy in the way we describe the system. It is similar to the gauge "symmetry" of Maxwell and Yang-Mills theory which, despite the name, is also a redundancy rather than a symmetry.

It is hard to overstate the importance of the concept of reparametrisation invariance. A major theme of the lectures is that our theories of physics should not depend on the way we choose to parametrise them. We'll see this again when we come to describe the field equations of general relativity. For now, we'll look at a couple of implications of reparametrisation on the worldline.

First off, because the action is independent of the parametrisation of the worldline, the value of the action evaluated between two points on a given path has an intrinsic meaning. We call this value *proper time*. For a given path $x^\mu(\sigma')$, the proper time between two points, say $\sigma = 0$ and $\sigma' = \sigma$, is

$$\tau(\sigma) = \frac{1}{c} \int_0^\sigma \mathrm{d}\sigma' \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma'} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma'}} \tag{2.4}$$

We recognise this as the time experienced by the particle itself.

Problem 2.3. What does extremising the action mean for the proper time? In Minkowski space, what is the trajectory that maximises proper time? If you have two twins, one stays at rest, and the other goes to a different planet and comes back, what is their relative age when they meet?

Secondly, there's a crucial difference between moving in Euclidean space and moving in Minkowski spacetime. You're not obliged to move in Euclidean space. You can just stop if you want to. In contrast, you can never stop moving in a timelike direction in Minkowski spacetime. You will, sadly, always be dragged inexorably towards the future. Any relativistic formulation of particle mechanics must capture this basic fact.

Before you begin the next exercise just a brief detour into common notation. Even though each object intrinsically has their indices either up or down, it is commonplace to raise and lower indices using the metric. E.g. from some vector in Minkowski spacetime V^μ we define $V_\mu = \eta_{\mu\nu} V^\nu$, and similarly from ω_μ we define $\omega^\mu = \eta^{\mu\nu} \omega_\nu$. This only makes a difference for the $\mu = 0$ component, but the minus sign is extremely important. In some sense, it is the whole content of special relativity. We will see later similar definitions for the general relativistic case of curved spacetime. Further, it is commonplace to denote an index free Minkowski vector without the boldface, just V, to distinguish from Euclidean space. So, for instance, $V^2 = (V \cdot V) = \eta_{\mu\nu} V^\mu V^\nu = V^\mu V_\mu$.

Problem 2.4.* Compute the momentum conjugate to x^μ,

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} \tag{2.5}$$

with $\dot{x}^\mu = \mathrm{d}x^\mu / \mathrm{d}\sigma$, for the action (2.3). How is this equivalent to the usual definition of 4-momentum?

$$p^\mu = m \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{2.6}$$

Compute the square of the 4-momentum, p^2. What does this imply for the number of degrees of freedom of a relativistic particle? How can we reconcile this with the non-relativistic action (1.6)?

Using the results above, compute $(p_0)^2$? What does this mean for our allowed trajectories in Minkowski spacetime?

3. The Equivalence Principle

It is finally time to put gravity into the mix. We won't very rigorous, the proper discussion will be given in the main lectures, this is just to give you a taste of what's to come.

In Newtonian physics, gravity is described by a potential $\Phi(\boldsymbol{x})$ that obeys the Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho \tag{3.1}$$

where ρ is the mass density. Such that the force on a particle is

$$\boldsymbol{F} = -m \boldsymbol{\nabla} \Phi \tag{3.2}$$

This can also be introduced by adding a term $-m\Phi(\boldsymbol{x})$ to the Lagrangian (1.6).

This construction has a well known accident. The quantity that controls the strength of interaction, called the *gravitational mass*, coincides with the usual *inertial mass* appearing in the kinetic term of the action. This is *a priori* not necessary, for instance, the electrostatic interaction obeys very similar laws to these, but the electric charge is completely independent from the inertial mass. However this coincidence in the case of gravity has been experimentally verified to an astounding accuracy (around 10^{-13}) therefore we give it a name, it's the *weak equivalence principle.*

This leads to a famous thought experiment due to Einstein. Imagine you one day wake up to find yourself trapped inside a box that looks like an elevator. This coincidence tells you that you wouldn't be able to figure out by doing some (local) mechanical experiment whether you're on an elevator in Earth, or in some alien spaceship disguised as an elevator which is undergoing constant acceleration with the precise value of the acceleration due to gravity on Earth.

Now there are maybe three possible ways you may be thinking of how to really decide what is your situation. The first one is, well, Newtonian physics only states that the masses are the same, couldn't you do some other kind of experiment, like with electromagnetism, or maybe quantum physics to help you decide? Well, usual everyday objects that were used to verify the weak equivalence principle are made of quantum stuff and are held together by electromagnetic interactions so it would be quite bizarre if these disciplines could tell the difference. This reasoning is also considered to be quite important, so we give it a name *Einstein equivalence principle*, it states that under *no* (local) experiment, mechanical or otherwise, can you tell the difference between constant acceleration and constant gravitational field.

The second way you could conceivably tell the difference is by doing a non-local experiment. Say, instead of dropping one ball you drop two. If you are indeed on Earth there will be tidal forces that pull these balls closer together. This is not ruled out by any experiment (in fact it is confirmed!) and is a legitimate method to distinguish between the two cases. The final method is common sense. Aliens, really?

There is another consequence of these thought experiments and equivalence principles. Any body will take the same trajectory under gravitational interaction regardless of it's composition (also sometimes called the *strong equivalence principle*). We all know this fact here on Earth, all objects fall with the same speed (neglecting friction) be it a hammer or a piece of paper. This suggests that perhaps gravity could be described not by some potential as in electromagnetism but by the geometry of spacetime itself. That is, the whole content of gravity is in allowing the spacetime to have a curved metric,

$$ds^2 = g_{\mu\nu}(x)\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu \tag{3.3}$$

and we generalise the action (2.3) to

$$S = -mc \int_{\sigma_1}^{\sigma_2} \mathrm{d}\sigma \sqrt{-g_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}} \tag{3.4}$$

This is quite a big jump in reasoning. This is because there is no real way to derive that gravity is just the curvature of spacetime, you can only motivate that it could conceivably be the case, and then test it's predictions. And, so far, it has passed every test we throw at it with flying colours.

For example, we can easily recover the equivalence principle in this formulation. It will be entirely equivalent to saying that for any metric we can always find some set of coordinates such that *locally* it will look just like Minkowski space. This turns out to be true, and you'll see it in the main lectures.

Problem 3.1.* Show that if you pick $g_{\mu\nu}$ to be equal to $\eta_{\mu\nu}$ except for the 00-component, which we choose to be,

$$g_{00} = 1 + \frac{2\Phi(x)}{c^2} \tag{3.5}$$

and then you take the non-relativistic limit $c \to \infty$. you recover precisely the correct Newtonian gravity interaction.

Problem 3.2.* Assume the results from problem 3.1. Take the Newtonian potential that arises from a spherical object of mass M,

$$\Phi(r) = -\frac{GM}{r} \tag{3.6}$$

and consider two observers. The first, Alice, is relaxing with a picnic on the ground at radius r_A. The second, Bob, is enjoying a romantic trip for one in a hot air balloon, a distance $r_B = r_A + \Delta r$ higher. If a time interval ΔT_A elapses for Alice, how much time elapses for Bob?

4. General Relativistic Geodesics

The idea of this section is to go a bit beyond what we can by just using the Newtonian limit and the equivalence principle and actually do some GR. It may seem like we're jumping steps but the point is to get some idea of how actual calculations in GR will look like, and derive the original experimental tests that confirmed we were on the right track.

Problem 4.1.* Prove that the equations of motion arising from the action (3.4) can be written as

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = \frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}\sigma}\dot{x}^\mu \tag{4.1}$$

Prove that the RHS of this equation is zero if we choose the parameter $\sigma = a\tau + b$, where τ is the proper time, defined by (2.4) where we substitute $\eta_{\mu\nu} \to g_{\mu\nu}(x)$. These are called *affine parameters*.

Conclude that the equations of motion arising from the action (3.4) are the same as the ones arising from

$$S = \int \mathrm{d}\tau \, g_{\mu\nu}(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \tag{4.2}$$

so long as we supplement them with the constraint

$$g_{\mu\nu}(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = -c^2 \tag{4.3}$$

Although the original action was only defined for massive particles, nowhere in the derivation was the value of m needed. This is in accordance with the equivalence principle. Further (4.2) makes perfect sense for massless particles as well, so long as the constrain imposed is

$$g_{\mu\nu}(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \tag{4.4}$$

Now consider that the metric takes the form,

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{4.5}$$

This is the *Schwarzschild metric* and it describes the spacetime geometry outside a perfectly spherical object such as a star or a black hole. The coordinates θ and ϕ are the usual spherical polar coordinates, with $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$.

For simplicity of notation we usually define the *Schwarszschild radius*

$$R_S = \frac{2GM}{c^2} \tag{4.6}$$

and the function $A(r) = 1 - \frac{2GM}{rc^2}$, such that the metric now looks like,

$$ds^2 = -A(r)\,\mathrm{d}t^2 + A(r)^{-1}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{4.7}$$

Problem 4.2.* Derive the θ equation of motion arising from (4.7). Check that if $\theta = \pi/2$ initially it will remain at that point for all time.

Prove that the angular momentum and the energy of system, defined, respectively by,

$$2l = \frac{\partial L}{\partial \dot{\phi}} \qquad -2E = \frac{\partial L}{\partial \dot{t}} \tag{4.8}$$

are conserved quantities.

Imposing $\dot{\theta} = 0$, the constraint (4.3), and the conservation of energy and angular momentum write the equations of motion in the form

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \frac{1}{2}\frac{E^2}{c^2} \tag{4.9}$$

where the effective potential is given by

$$V_{\text{eff}}(r) = \frac{1}{2}\left(c^2 + \frac{l^2}{r^2}\right)\left(1 - \frac{2GM}{rc^2}\right) \tag{4.10}$$

Prove that, for massless particles, the equation of motion is of the same form, except the effective potential is now

$$V_{\text{null}}(r) = \frac{l^2}{2r^2}\left(1 - \frac{2GM}{rc^2}\right) \tag{4.11}$$

Now let's examine some consequences of these equations. Firstly for massive particles.

Problem 4.3. Take the equation for massive particles (4.9) with (4.10), make the substitution $u = 1/r$, and, assuming $\mathrm{d}u/\mathrm{d}\phi \neq 0$, show that it implies

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u - \frac{GM}{l^2} = \beta\frac{l^2u^2}{GM} \tag{4.12}$$

where

$$\beta = \frac{3G^2M^2}{l^2c^2} \tag{4.13}$$

(Hint: Use the definition of l when going from derivatives wrt to τ to derivatives wrt ϕ)

Now assume $\beta \ll 1$ and expand in powers of β,

$$u = u_0 + \beta u_1 + \dots \tag{4.14}$$

Show that, at 0^{th} order, the equation of motion is

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2} + u_0 - \frac{GM}{l^2} = 0 \tag{4.15}$$

which is solved by

$$u_0(\phi) = \frac{GM}{l^2}(1 + e\cos\phi) \tag{4.16}$$

where e is the eccentricity, a free parameter. This is the Newtonian prediction for the motion of the particle. Note that it is periodic in ϕ.

Now show that, at 1^{st} order, the equation of motion is

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + u_1 = \frac{l^2}{GM} u_0^2 \tag{4.17}$$

which is solved by

$$u_1(\phi) = \frac{GM}{l^2}\left[\left(1 + \frac{e^2}{2}\right) + e\phi \sin\phi - \frac{e^2}{6}\cos(2\phi)\right] \tag{4.18}$$

Interestingly, the GR correction to the Newtonian prediction is no longer periodic in ϕ due to the $\phi \sin\phi$ term. This means that the angular value at which the particle is closest to the origin, called the *perihelion*, will change with time.

These points will be extrema of $u(\phi)$. Check that both at $\phi = 0$ and at $\phi = 2\pi + \delta$ $u(\phi)$ has an extremum, calculating the value of δ and neglecting δ^2 and $\beta\delta$ terms.

This is called the precession of the perihelion and it was famously measured for Mercury, verifying the general relativistic prediction.

And finally, for massless particles,

Problem 4.4. By once again making the substitution $u = 1/r$ show that the equation of motion for a massless particle is

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{3GM}{c^2} u^2 \tag{4.19}$$

Show that, ignoring the RHS, the solution can be written as

$$u = \frac{1}{b}\sin\phi \tag{4.20}$$

for constant b. What is the interpretation of b? (Hint: write the solution in terms of r).

Now work perturbatively in

$$\tilde{\beta} = \frac{GM}{c^2 b} \tag{4.21}$$

by defining

$$u = u_0 + \tilde{\beta} u_1 + \ldots \tag{4.22}$$

By starting with $u_0 = \frac{1}{b}\sin\phi$ at 0^{th} order show that the equation for u_1 at 1^{st} order is

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + u_1 = \frac{3\sin^2\phi}{b} \tag{4.23}$$

which is solved by

$$u_1 = \frac{1}{2b}(3 + 4\cos\phi + \cos(2\phi)) \tag{4.24}$$

Show that the angle at which the particle escapes to $r = \infty$ is now, approximately,

$$\phi \approx -\frac{4GM}{bc^2} \tag{4.25}$$