Quantum Field Theory: Example Sheet 2

Dr David Tong, October 2007

1. A string has classical Hamiltonian given by

\[ H = \sum_{n=1}^{\infty} \left( \frac{1}{2} p_n^2 + \frac{1}{2} \omega_n^2 q_n^2 \right) \]

where \( \omega_n \) is the frequency of the \( n \)th mode. (Compare this Hamiltonian to the Lagrangian (3) in Example Sheet 1. We have set the mass per unit length in that question to \( \sigma = 1 \) to simplify some of the formulae a little). After quantization, \( q_n \) and \( p_n \) become operators satisfying

\[ [q_n, q_m] = [p_n, p_m] = 0 \quad \text{and} \quad [q_n, p_m] = i \delta_{nm} \]

Introduce creation and annihilation operators \( a_n \) and \( a_n^\dagger \),

\[ a_n = \sqrt{\frac{\omega_n}{2}} q_n + i \sqrt{\frac{\omega_n}{2}} p_n \quad \text{and} \quad a_n^\dagger = \sqrt{\frac{\omega_n}{2}} q_n - i \sqrt{\frac{\omega_n}{2}} p_n \]

Show that they satisfy the commutation relations

\[ [a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0 \quad \text{and} \quad [a_n, a_m^\dagger] = \delta_{nm} \]

Show that the Hamiltonian of the system can be written in the form

\[ H = \sum_{n=1}^{\infty} \frac{1}{2} \omega_n (a_n a_n^\dagger + a_n^\dagger a_n) \]

Given the existence of a ground state \( |0\rangle \) such that \( a_n |0\rangle = 0 \), explain how, after removing the vacuum energy, the Hamiltonian can be expressed as

\[ H = \sum_{n=1}^{\infty} \omega_n a_n^\dagger a_n \]

Show further that \( [H, a_n^\dagger] = \omega_n a_n^\dagger \) and hence calculate the energy of the state

\[ |l_1, l_2, \ldots, l_N\rangle = (a_1^\dagger)^{l_1} (a_2^\dagger)^{l_2} \cdots (a_N^\dagger)^{l_N} |0\rangle \]
2. The Fourier decomposition of a real scalar field and its conjugate momentum in the Schrödinger picture is given by

\[ \phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[ a_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p} \cdot \vec{x}} \right] \]

\[ \pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \frac{E_p}{2} \left[ a_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} - a_{\vec{p}}^\dagger e^{-i\vec{p} \cdot \vec{x}} \right] \]

Show that the commutation relations

\[ [\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0 \quad \text{and} \quad [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \]

imply that

\[ [a_{\vec{p}}, a_{\vec{q}}] = [a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger] = 0 \quad \text{and} \quad [a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \]

3. Consider a real scalar field with the Lagrangian

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \quad (1) \]

Show that, after normal ordering, the conserved four-momentum \( P^\mu = \int d^3x T^{0\mu} \) takes the operator form

\[ P^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu a_{\vec{p}}^\dagger a_{\vec{p}} \]

where \( p^0 = E_\vec{p} \) in this expression. From this expression for \( P^\mu \) verify that if \( \phi(x) \) is now in the Heisenberg picture, then

\[ [P^\mu, \phi(x)] = -i \partial^\mu \phi(x) \]

4. Show that in the Heisenberg picture,

\[ \dot{\phi}(x) = i[H, \phi(x)] = \pi(x) \quad \text{and} \quad \dot{\pi}(x) = i[H, \pi(x)] = \nabla^2 \phi(x) - m^2 \phi(x) \]

Hence show that the operator \( \phi(x) \) satisfies the Klein-Gordon equation.

5. Let \( \phi(x) \) be a real scalar field in the Heisenberg picture. Show that the relativistically normalized one-particle states \( |p\rangle = \sqrt{2E_\vec{p}} a_{\vec{p}}^\dagger |0\rangle \) satisfy

\[ \langle 0 | \phi(x) | p \rangle = e^{-i p \cdot x} \]
6. In Example Sheet 1, you showed that the classical angular momentum of field is given by

\[ Q_i = \frac{1}{2} \epsilon_{ijk} \int d^3x \ (x^j T^{0k} - x^k T^{0j}) \]

Write down the explicit form of the angular momentum for a free real scalar field with Lagrangian (1). Show that, after normal ordering, the quantum operator \( Q_i \) can be written as

\[ Q_i = \frac{i}{2} \epsilon_{ijk} \int \frac{d^3p}{(2\pi)^3} a_p^\dagger \left( p^j \frac{\partial}{\partial p_k} - p^k \frac{\partial}{\partial p_j} \right) a_p \]

Hence confirm that the quanta of the scalar field have spin zero (i.e. a stationary one-particle state \(|\vec{p} = 0\rangle\) has zero angular momentum).

7. The purpose of this question is to introduce you to non-relativistic quantum field theory. This is the only place you will encounter such a thing in this course. Consider the Lagrangian for a complex scalar field \( \psi \) given by

\[ \mathcal{L} = +i\bar{\psi} \partial_0 \psi - \frac{1}{2m} \nabla \bar{\psi} \cdot \nabla \psi \]

Determine the equation of motion, the energy-momentum tensor and the conserved current arising from the symmetry \( \psi \rightarrow e^{i\alpha} \psi \). Show that the momentum conjugate to \( \psi \) is \( i\psi^* \) and compute the classical Hamiltonian.

We now wish to quantize this theory. We will work in the Schrödinger picture. Explain why the correct commutation relations are

\[ [\psi(\vec{x}), \psi(\vec{y})] = [\psi^\dagger(\vec{x}), \psi^\dagger(\vec{y})] = 0 \quad \text{and} \quad [\psi(\vec{x}), \psi^\dagger(\vec{y})] = \delta^{(3)}(\vec{x} - \vec{y}) \]

Expand the fields in a Fourier decomposition as

\[ \psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} \]
\[ \psi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}}^\dagger e^{-i\vec{p} \cdot \vec{x}} \]

Determine the commutation relations obeyed by \( a_{\vec{p}} \) and \( a_{\vec{p}}^\dagger \). Why do we have only a single set of creation and annihilation operators \( a_{\vec{p}}, a_{\vec{p}}^\dagger \) even though \( \psi \) is complex? What is the physical significance of this fact? Show that one particle states have the energy appropriate to a free non-relativistic particle of mass \( m \).
8. Show that the time ordered product $T (\phi(x_1)\phi(x_2))$ and the normal ordered product $\phi(x_1)\phi(x_2)$ are both symmetric under the interchange of $x_1$ and $x_2$. Deduce that the Feynman propagator $\Delta_F(x_1 - x_2)$ has the same symmetry property.

9. Verify Wick’s theorem for the case of three scalar fields:

$$T (\phi(x_1)\phi(x_2)\phi(x_3)) = :\phi(x_1)\phi(x_2)\phi(x_3): + \phi(x_1)\Delta_F(x_2 - x_3) + \phi(x_2)\Delta_F(x_3 - x_1) + \phi(x_3)\Delta_F(x_1 - x_2)$$

10. Consider the scalar Yukawa theory given by the Lagrangian

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - M^2 \psi^* \psi - \frac{1}{2} m^2 \phi^2 - g \psi^* \psi \phi$$

Compute the amplitude for

- “Nucleon-Anti-Nucleon” annihilation $\psi + \bar{\psi} \to \phi$ at order $g$
- “Nucleon-Meson” scattering $\phi + \psi \to \phi + \psi$ at order $g^2$