Part III quantum mechanics preparatory workshop notes

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In this workshop we will revise the major lessons of QM before you get started with the Part III theoretical physics courses. We will do this by recapping the main ideas, and then working through a series of examples. This is designed to refresh knowledge that may have been lost or gone rusty. You will not find anything here beyond standard QM material, it is however presented in a way that is meant to maximise the applicability to Part III (mainly QFT) and may differ from the way you learnt it in your previous QM courses.

The topics we will touch are roughly:

- Hilbert Space, Observables, etc.
- Dirac (Bra-Ket) Notation, Schrödinger & Heisenberg Picture
- Probabilities & Measurement
- Harmonic Oscillator, Spin Algebra, Ladder Operators
- Perturbation Techniques

Furthermore, you are encouraged to bring any questions or specific problems about QM to the live session so we can talk about them in detail.

Please get in touch with comments and suggestions, or if you find errors in this script, at *sxes2@cam.ac.uk*.

Units & Notation

Since most formulas and expressions in QM and QFT have many factors of c and $\hbar$, it is very convenient to work in a units such that $c = \hbar = 1$. In other words, we measure all physical quantities in relation to the fundamental constants c, $\hbar$, etc., and appropriate combinations thereof. This is called *natural units*. At the level of our formulas, you can imagine we rescaled all coordinates, parameters, functions, etc. by a combination of the natural constants (in the fashion of a coordinate change) to obtain new, unit-free coordinates, parameters, functions. This is sometimes called 'nondimensionalization'.

We sometimes use $\vec{\nabla}$ (read *nabla*) to mean the the following. For a function $(\vec{\nabla} f)_i = \partial_{x^i} f$, the components of the gradient $\mathrm{d}f$. For a vector field $\vec{v}(x)$ we write its divergence as $\nabla \cdot v$ and its curl as $\vec{\nabla} \times \vec{v}$.

Recommended Material

- PQM lecture notes by David Skinner, *www.damtp.cam.ac.uk/user/dbs26* (!)
- Past PQM example sheets, *www.damtp.cam.ac.uk/user/examples* (!)
- Griffiths, *Introduction to Quantum Mechanics*
- Dirac, *The Principles of Quantum Mechanics*
- Weinberg, *Lectures on Quantum Mechanics*
- For QFT: Nair, *QFT – A Modern Perspective*

1 States, observables and measurements

The central equation in quantum mechanics is, of course, the Schrödinger equation

$$i\partial_t \Psi(t,x) = \left(\frac{(-i\partial_x)^2}{2m} + V(x) \right) \Psi(t,x) \tag{1}$$

describing the time evolution of the function $\Psi(t,x)$ in terms of a partial differential equation. It involves the potential energy $V(x)\Psi(t,x)$ and the kinetic energy $-\frac{1}{2m}\partial_x^2 \Psi(t,x)$. The sum of potential and kinetic energy is called the Hamiltonian, and is already in classical mechanics responsible for the time evolution of a system. Recall that $H(x,p) = \frac{p^2}{2m} + V(x)$ and the equation of motion for any quantity $f(x,p)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \{f, H\}_{\mathrm{PB}} = \frac{\partial f}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial x} \tag{2}$$

using the Poisson bracket $\{\cdot,\cdot\}_{\text{PB}}$ between the two functions H and f. In words we can say that the Hamiltonian function H generates time evolution by *acting on* other functions via the Poisson bracket $\{\cdot, H\}_{\text{PB}}$. So in particular we get

$$\dot{x} = \frac{p}{m} \qquad \dot{p} = -\partial_x V(x) \tag{3}$$

describing a classical trajectory $\gamma(t) = (x(t), p(t))$ in phase space. Note that, given two solutions $\gamma_1(t)$ and $\gamma_2(t)$, the sum $\gamma_1 + \gamma_2$ is not necessarily also a solution of the equations of motion. In fact, this can generically only be the case if the system eq. (3) is *linear*, i.e. the potential is at most quadratic.

Of course we know from Schrödinger's (thought) experiments with his cat that if two states of a system are physical, i.e. they solve the fundamental equation of motion of nature, then so is any linear combination of the two. Say the fundamental equation of motion of nature is given by some differential operator L acting on a function f describing the current state of a system: $L(f) = 0$. The previous comments imply that L must be a linear operator and hence its solution space be a vector space! By now we are of course already talking about the Schrödinger equation (1) with the operator given by

$$L = -i\partial_t + \frac{(-i\partial_x)^2}{2m} + V(x)\,. \tag{4}$$

We also know that the functions $\Psi(t,x)$ on which this operator acts have a very special interpretation, their mod-square $\rho(t,x) = |\Psi(t,x)|^2$ is a probability distribution on configuration space. It gives the probability of finding a particle at given point (or rather in a given subset) of configuration space. For this last sentence to make sense, the total probability of finding the particle at any point of configuration space needs to be unity, or at the very least finite, so that we can normalise Ψ if need be[1]

$$\int_{\mathbb{R}} \mathrm{d}x\, \rho(t,x) = \int_{\mathbb{R}} \mathrm{d}x\, |\Psi(t,x)|^2 < \infty\,. \tag{5}$$

It will turn out to be very rewarding to notice that the solutions of eq. (1) have the analytic structure of a Hilbert space. Let's recall the definition. A Hilbert space $\mathcal{H}$ is a complex vector space with an inner product $\langle\psi|\xi\rangle$ between vectors, which satisfies

1. $\langle\psi|\xi\rangle = \langle\xi|\psi\rangle^*$
2. $\langle\psi|\alpha_1\xi_1 + \alpha_2\xi_2\rangle = \alpha_1\langle\psi|\xi_1\rangle + \alpha_2\langle\psi|\xi_2\rangle$, with $\alpha_1, \alpha_2 \in \mathbb{C}$
3. $\|\psi\|^2 = \langle\psi|\psi\rangle \geq 0$ and $\langle\psi|\psi\rangle = 0$ iff $|\psi\rangle = 0$

[1] We loosen the condition of unit total probability so that the solution space remains a vector space. This has to be taken into account later.

Clearly these conditions are very sensible and may seem nearly too obvious to even mention. You will however soon encounter very important physical theories which violate some of these conditions, and dealing with this issue will take a good deal of work.

In QM, the two relevant Hilbert spaces are the familiar $\mathbb{C}^n$, and the space of complex valued functions on $\mathbb{R}^d$ with the inner product

$$\langle\psi|\xi\rangle = \int \mathrm{d}^d x\, \psi^*(x)\xi(x)\,. \tag{6}$$

The latter example is infinite dimensional, and we think of every function $\psi(x)$ as a vector in this infinite dimensional space. The passage from a finite to infinite dimensional vector space can roughly be thought of as making the index i of a vector ψ_i continuous, so the component form together a function $\psi(x)$. However, since there are infinitely many coordinates, we have to include a new requirement: for a vector to be sensible, it better have a finite length, i.e. $\langle\psi|\psi\rangle < \infty$, which we recognize as eq. (5). We denote the Hilbert space of functions satisfying this constraint by $L^2(\mathbb{R}^d)$.

From your linear algebra course you should be familiar with the concept of a basis and a change of basis. In short, a vector is independent of a basis, but the components which we use to describe the vector are not. A basis is specified by a set of basis vectors, i.e. a linearly independent spanning set of vectors, and the components of any vector in that basis are simply the projection of the vector on the basis elements. The components of a vector in two different bases are related by a matrix, whose entries are the inner products of basis elements. In the case of complex functions, you should be familiar with the two most common bases. The position space and momentum space basis. The components of a vector $|\psi\rangle$ are given by

$$\psi(x) = \langle x|\psi\rangle \qquad \text{and} \qquad \tilde{\psi}(p) = \langle p|\psi\rangle \tag{7}$$

in the respective bases. We put the tilde over the Fourier transform of ψ to emphasise that it is a different function, though related, to ψ. Since we claimed that the set of basis elements for position space $|x\rangle$ and momentum space $|p\rangle$ form a basis, they had better satisfy the completeness relation:

$$\int \mathrm{d}^d x\, |x\rangle\langle x| = \mathbb{1} \qquad \text{and} \qquad \int \mathrm{d}^d p\, |p\rangle\langle p| = \mathbb{1}\,. \tag{8}$$

Finally, if $\psi(x)$ and $\tilde{\psi}(p)$ are the components of the same vector, just in expressed in different bases, then there should be a linear transformation relating the two. Indeed, using eq. (8), we find

$$\tilde{\psi}(p) = \langle p|\psi\rangle = \langle p|\mathbb{1}|\psi\rangle = \int \mathrm{d}^d x\, \langle p|x\rangle\langle x|\psi\rangle = \int \mathrm{d}^d x\, \langle p|x\rangle\psi(x) \tag{9}$$

which we recognise as the Fourier transform of $\psi(x)$ given that the matrix elements of the change of basis are $\langle p|x\rangle \propto e^{-ip\cdot x}$. Hence taking the Fourier transform of a function corresponds simply to changing basis in this Hilbert space.

Q 1.1 *Use the canonical commutation relation $[\hat{x}, \hat{p}] = i$ to show that*

$$e^{ia\hat{p}}\hat{x}e^{-ia\hat{p}} = \hat{x} + a \tag{10}$$

and hence, without loss of generality,

$$e^{-ia\hat{p}}|x\rangle = |x + a\rangle\,. \tag{11}$$

From this deduce that $\langle p|x\rangle \propto e^{-ixp}$ and finally fix the normalization by unitarity.

Observables

While the state of a physical system is represented by a vector in a Hilbert space, the possible observables, i.e. measurable quantities, are represented by operators (think infinite dimensional matrices). When the system is in a given state ψ, the expected value of an observable $\mathcal{O}$, represented by the operator $\hat{\mathcal{O}}$, is defined by

$$\langle\hat{\mathcal{O}}\rangle_\psi = \frac{\langle\psi|\hat{\mathcal{O}}|\psi\rangle}{\langle\psi|\psi\rangle}\,. \tag{12}$$

If this quantity is supposed to be measurable in the real world, it had better be a real number. Setting the imaginary part of $\langle\hat{\mathcal{O}}\rangle_\psi$ to zero leads us to the requirement $\hat{\mathcal{O}} = \hat{\mathcal{O}}^\dagger$, i.e. that the operator representing an observable be Hermitian, or self-adjoint.

Hermitian operators acting on a Hilbert space have a few properties that are very handy for calculations, and also crucial for their interpretation as observables. As soon as we know that an operator is Hermitian, we know that

- its eigenvectors $|\psi_i\rangle$ form a complete basis, $\mathbb{1} = \sum_i |\psi_i\rangle\langle\psi_i|$,
- its eigenvectors (eigenspaces) are mutually orthogonal $\langle\psi_i|\psi_j\rangle = \delta_{ij}$,
- its eigenvalues λ_i are real.

Q 1.2 *Prove the second and third property of Hermitian operators above.*

The most fundamental observable of a system is of course its total energy, represented by the Hermitian operator $\hat{H}$, the Hamiltonian. Further observables are position, momentum, spin, etc, and all these quantities are

represented by Hermitian operators. There are also many important non-Hermitian operators, a particularly significant class are the Unitary operators, which preserve the norm on $\mathcal{H}$.

Often it is convenient to define an operator in terms of is spectral decomposition, that is, its eigenvectors and eigenvalues, e.g. the position operator $\hat{x}$ is defined in the position basis as

$$\hat{\vec{x}} = \int \mathrm{d}^d x \, \vec{x} |x\rangle\langle x| \,. \tag{13}$$

These spectral decompositions are very useful in explicit computations. Finding the matrix elements of an operator in a given basis works just as in usual linear algebra, the only difference being that the indices on a matrix have now become continuous parameters. Hence matrix elements are functions (or more generally distributions) in two variables, e.g. $\mathcal{O}(x,y) = \langle x|\hat{\mathcal{O}}|y\rangle$.

Q 1.3 *For $|\psi\rangle \in L^2(\mathbb{R})$, write down the vectors $|\psi\rangle$, $\hat{x}|\psi\rangle$, $\hat{p}|\psi\rangle$, as well as the operators $\hat{x}$, $\hat{p}$, in both position and momentum basis.*

Note that in a given basis, we can understand the 'matrix elements' of an operator as an integral kernel $(\hat{\mathcal{O}}\psi)(x) = \int \mathrm{d}^d y \, \mathcal{O}(x,y)\psi(y)$.

Q* 1.4 *What are $\rho(x) = \langle\psi|\delta^{(3)}(\hat{x}-x)|\psi\rangle$ and $\vec{j}(x) = \frac{1}{2m}\langle\psi|\{\hat{\vec{p}}, \delta^{(3)}(\hat{x}-x)\}|\psi\rangle$ in terms of the position space wave function $\psi(x)$? Can you write $\nabla \cdot j(x)$ in a similar way? Here $\{A,B\} = AB + BA$ is the anti-commutator.*

Another way of defining an operator is to declare how it acts on a set of basis vectors. Again for the position operator, we can define it in terms of its action on the position space basis, $\hat{\vec{x}}|x\rangle = \vec{x}|x\rangle$. Naturally, the definition in terms of its spectrum is a special case of this.

Q 1.5 *Working in position basis show that $\hat{p}$ is Hermitian, i.e. that $\langle\psi|\hat{p}|\xi\rangle = (\langle\xi|\hat{p}|\psi\rangle)^*$ for arbitrary $|\xi\rangle, |\psi\rangle \in L^2(\mathbb{R})$.*

Incidentally, we can now use the basis independent notation in our Hilbert space, which is also known as *Dirac notation*, to write the Schrödinger equation (1) compactly as

$$i\partial_t|\Psi\rangle = \hat{H}|\Psi\rangle \,, \tag{14}$$

where $\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{x})$.

Q* 1.6 *Recall that in 3d a particle with charge e moving in a time independent magnetic field has Hamiltonian $H(x,p) = \frac{1}{2m}(\vec{p} - e\vec{A}(x))^2$, where the electromagnetic potential $\vec{A}$ determines the magnetic field via $\vec{B} = \vec{\nabla} \times \vec{A}$.*

(The electric field is given by $\vec{E} = -\partial_t \vec{A}$ and hence vanishes since $\vec{A}$ is time independent.)

a) Derive the classical equation of motion $\vec{F} = m\ddot{\vec{x}} = e(\vec{E} + \dot{\vec{x}} \times \vec{B})$.

b) What is the momentum $\vec{p}_A = m\dot{\vec{x}}$ in terms of $\vec{x}$ and $\vec{p}$. Calculate the Poisson bracket $\{(p_A)_i, (p_A)_j\}_{PB}$.

c) What is the quantum Hamiltonian in terms of $\hat{\vec{p}}$ and $\vec{A}(\hat{x})$? Is there a normal ordering ambiguity in promoting the functions to operators? (Hint: remember the Coulomb gauge fixing condition $\nabla \cdot A = 0$.)

d) After canonically quantizing ($[\hat{x}^i, \hat{p}_j] = i\delta^i_j$), what is $\hat{\vec{p}}_A$ in position representation? Show that the commutator becomes $[(\hat{p}_A)_i, (\hat{p}_A)_j] = ie\epsilon_{ijk}B_k(\hat{x})$.

This is a simple example of a system with a gauge symmetry, which will become very important in QFT and GR and you will learn a lot about it over the next terms. The non-vanishing of the commutator $[(p_A)_i, (p_A)_j]$ indicates that the particle moves in a *curved space*. In this example the curvature does not reside in (position-)space, but rather in an 'internal' space. See e.g. Nair or Nakahara for more.

Probabilities

Given that we can calculate the *expectation value* of an observable in a state ψ using eq. (12), it is natural to ask what the outcome of a single measurement will be. The probabilistic nature of QM suggests that this question is ill-posed. We can only sensibly ask for the *probability* of measuring some outcome. Recall that the expectation is in general a sum over all possible outcomes, weighted by their probability. Inserting the spectral decomposition of the operator $\hat{A}$ in terms of its eigenvalues a_i and (orthonormal) eigenvectors $|i\rangle$ yields

$$\sum_i \lambda_i p(\lambda_i) = \langle \hat{A} \rangle_\psi = \sum_i a_i \frac{|\langle i|\psi\rangle|^2}{\|\psi\|^2} \,. \tag{15}$$

We recognise that the probability of measuring a given eigenvalue a_i is given by

$$p(a_i) = \frac{|\langle i|\psi\rangle|^2}{\|\psi\|^2} = \langle |i\rangle\langle i| \rangle_\psi = \langle P_A(i) \rangle_\psi \tag{16}$$

where we defined the *projection operator* on the i^{th} eigenstate of $\hat{A}$.

Q 1.7 *Check that $P_A(i)$ is indeed a projection operator (i.e. $P^2 = P$) and observable (i.e. $P^\dagger = P$). Construct the projection operator projecting on two states $P_A(\{i, j\})$, then on an arbitrary subset of the spectrum of $\hat{A}$.*

If we want to compute the probability to find an eigenvalue within a certain range of the spectrum, we likewise take the expectation value of the projection operator onto that part of the spectrum.

Q 1.8 *a) Show that $\langle\psi|\psi\rangle$ is constant in time using eq. (14) (with general $\hat{H}$) and the product rule for differentiation. What property of $\hat{H}$ is crucial?*

b) Using the Dirac notation form for $\nabla \cdot j$, evaluate $\partial_t \rho + \nabla \cdot j$ using the Schrödinger equation (14) with a Hamiltonian $\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{x})$. What is the interpretation of ρ, $\vec{j}$, and the differential equation they satisfy?

c) What is the relation between a) and b)?

d) What happens if the Hamiltonian is not Hermitian? How can we interpret this?

Now that we know the probability for each possible outcome of a measurement, we can go ahead and actually perform it! A crucial part of (the Copenhagen interpretation of) QM is that measurements cause the wave function to collapse. Suppose we measure A, and observe a value a_i (corresponding to the i^{th} eigenvalue of $\hat{A}$). The measurement will cause the wave function to collapse

$$|\psi\rangle \longrightarrow P_A(i)|\psi\rangle = |i\rangle\langle i|\psi\rangle\,. \tag{17}$$

This is an instant projection. It represents a non-continuous, non-unitary evolution. (There exist other interpretations of QM which avoid wavefunction collapse, see e.g. decoherence, however there exist no generally accepted resolutions of the so called measurement problem.) Note that before and after the measurement the time evolution is given by the Schrödinger equation.

Uncertainty

An immediate consequence of the representation of observables by operators (again, think matrices) is that observables no longer necessarily commute, e.g. the commutator between the position and momentum operator is

$$[\hat{x}, \hat{p}] = i \tag{18}$$

and is important enough to carry the name canonical commutation relation. (It is called canonical because is follows from replacing the Poisson bracket in classical mechanics by the commutator

$$\{x, p\}_{\text{PB}} = 1 \qquad \rightarrow \qquad -i[\hat{x}, \hat{p}] = 1 \tag{19}$$

which is an ad hoc, yet canonical, procedure.)

Q 1.9 *Using the position basis expressions for $\hat{x}$ and $\hat{p}$ find the position space expression for the commutator $[\hat{p}, f(\hat{x})]$, for a function $f \in C^\infty(\mathbb{R})$, by letting it act on an arbitrary vector. Compare this with what you get without choosing a basis, by writing f as a Taylor series (assuming that it is analytic).*

The effect of this is that the uncertainty in different observables may become correlated, even if they were classically uncorrelated. Recall that the uncertainty (variance) for the value of an observable A is given by

$$\sigma_A^2(\psi) = \langle (\hat{A} - \langle \hat{A} \rangle_\psi)^2 \rangle_\psi = \langle \hat{A}^2 \rangle_\psi - \langle \hat{A} \rangle_\psi^2 = \langle \hat{A}_0^2 \rangle_\psi \tag{20}$$

where for convenience we introduced the shorthand $\hat{A}_0 = \hat{A} - \mathbb{1}\langle \hat{A} \rangle_\psi$. The interpretation of this uncertainty is that some properties of the system simply do not have an exact value, but are smeared out over an interval σ_A around the expected value, depending on the state.[2]

Q 1.10 *Show that the uncertainty of the observable A in the state ψ satisfies $\sigma_A^2(\psi) \geq 0$, with $\sigma_A^2(\psi) = 0$ iff ψ is an eigenvector of $\hat{A}$.*

Now, it follows from the positivity of the norm ($\|\cdot\| \geq 0$) that the product of the uncertainties of two observables obeys

$$\sigma_A^2(\psi)\sigma_B^2(\psi) \geq \left(\frac{1}{2} \langle \{\hat{A}_0, \hat{B}_0\} \rangle_\psi \right)^2 + \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle_\psi \right)^2 , \tag{21}$$

the Schrödinger uncertainty relation. Note that the symmetric part $\{\hat{A}_0, \hat{B}_0\}$, is in direct analogy with the classical version of this relation. The quantum contribution comes from the commutator $[\hat{A}, \hat{B}]$, and has no classical analogue. This becomes particularly interesting when $\hat{A}$ and $\hat{B}$ are conjugate, so that the commutator is i. Then we can write a weaker, but more practical bound

$$\sigma_A(\psi)\sigma_B(\psi) \geq 1/2 \tag{22}$$

where the right hand side is now independent of the state ψ.

Q* 1.11 *Show eq. (21) by considering the function $f(\lambda) = \langle |\hat{A}_0 + \lambda \hat{B}_0|^2 \rangle_\psi$ and its minimum, for $\lambda \in \mathbb{C}$.*

[2] In particular, it doesn't mean that we are not able, technologically or otherwise, to measure the precise value – the system just does not have a fixed value for A. The right picture is that of a wave, it simply does not make sense to ask for its 'location' to higher accuracy than its wavelength.

2 Unitarity, the Heisenberg picture and perturbation theory

Unitary transformations

Many computations become much easier if we choose a good basis for the Hilbert space. The basis independence of the result is guaranteed if we only allow basis changes that preserve the norm $\|\cdot\|$. This amounts to the requirement of unitarity of the operator U implementing the basis change

$$U^\dagger U = \mathbb{1}\,. \tag{23}$$

Unitary operators on our Hilbert space are the analogue of orthogonal matrices on $\mathbb{R}^n$.

Now, the Schrödinger equation (14) says that the evolution of a physical system is given by the operator

$$\hat{U}(t) = \exp(-it\hat{H}) \tag{24}$$

which is sensibly called the *time evolution operator*.

Q 2.1 *a) Check that $|\Psi(t)\rangle = \hat{U}(t)|\psi\rangle$ solves the Schrödinger equation (14).*

b) Check that the evolution operator is unitary if $\hat{H}$ is Hermitian.

c) Check that the evolution operator for evolving by a time $t_1 + t_2$ can be obtained by first evolving by t_1 and then by t_2.

In other words, the action of $\hat{U}(t)$ on states generates time translations.

Q 2.2 *a) What is the probability to find a system in an eigenstate $|j\rangle$ of an observable A at time $T > 0$ if it was in the eigenstate $|i\rangle$ at time 0? (Hint: what is $\psi(t)$ for $T > t > 0$?)*

b) What if $|i\rangle$ is an energy eigenstate?

Q* 2.3 *Use the result from the last question to compute the probability density for a free particle, $\hat{H} = \frac{\hat{p}^2}{2m}$, initially in a position eigenstate at $x = 0$, to be measured at position x at time t. (You may have to go to imaginary time $t \to -it_E$ to compute an integral and go back to real time $t_E \to it$.)*

There is in general a close relationship between unitary operators and transformations. Note that for every Hermitian operator $\hat{A}$ there is an associated family of unitary operators $\hat{U}_A(\theta) = \exp(-i\theta\hat{A})$. Physically, $\hat{U}_A(\theta)$ generates a shift by θ in the observable conjugate to A, e.g. the momentum operator $\hat{p}$ generates translations of $\hat{x}$.

Q 2.4 *Show that the operator* $\hat{U}_p(a) = \exp(-ia\hat{p})$ *indeed generates a shift of the observable* $\hat{x}$ *by the amount* a *in the sense that* $\hat{U}_p(a)|x\rangle = |x+a\rangle$ *as well as*

$$e^{-ia[\hat{p},\cdot]} f(\hat{x}) = \hat{U}_p(a) f(\hat{x}) \hat{U}_p(a)^\dagger = f(\hat{x}-a)\,. \tag{25}$$

(Hint: you can show that the operator relations hold when acting on a complete set of basis vectors, and hence, by linearity, on all of the Hilbert space.)

The exponential of the commutator action is defined as

$$e^{-ia[\hat{p},\cdot]} f(\hat{x}) = f(\hat{x}) - ia[\hat{p}, f(\hat{x})] + \frac{(ia)^2}{2}[\hat{p},[\hat{p}, f(\hat{x})]] - \frac{(ia)^3}{3!}[\hat{p},[\hat{p},[\hat{p}, f(\hat{x})]]] + \cdots\,. \tag{26}$$

As observed earlier, the operator $\hat{U}_A$ is unitary, i.e. preserves total probability, if $\hat{A}$ is Hermitian, i.e. is an observable.

This is again in direct analogy with classical mechanics, where every observable (function on phase space), generates a *canonical transformation* on the phase space via the Poisson bracket

$$\begin{aligned} &\mathcal{O} \to e^{a\{f,\cdot\}_{\text{PB}}} \mathcal{O} \\ &\quad = \mathcal{O} + a\{f,\mathcal{O}\}_{\text{PB}} + \frac{a^2}{2}\{f,\{f,\mathcal{O}\}_{\text{PB}}\}_{\text{PB}} + \frac{a^3}{3!}\{f,\{f,\{f,\mathcal{O}\}_{\text{PB}}\}_{\text{PB}}\}_{\text{PB}} + \cdots \end{aligned} \tag{27}$$

for any function $\mathcal{O}(x,p)$. Canonical transformations are coordinate transformations on phase space which preserve the symplectic (or Poisson bracket) structure. This fits in place nicely with the statement that unitary operators effect basis changes on Hilbert space.

If you want to learn more about this perspective on quantum mechanics you should look up *geometric quantization*, e.g. in chapter 20 of Nair's 'modern perspective on QFT', or in 'Symplectic geometry and geometric quantization' by Blau.

Heisenberg Picture

Extending the line of argument from the previous section, we could ask for the time evolution of observables directly. Using the Schrödinger equation (14) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathcal{O}\rangle_\Psi = -i\langle[\mathcal{O},\hat{H}]\rangle_\Psi \tag{28}$$

where the time dependence sits purely with the state $|\Psi(t)\rangle$ and $\mathcal{O}$ can be any operator. It is not hard to see that you get the same answer if instead of using the Schrödinger equation, you used the Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{O}(t) = -i[\mathcal{O}(t),\hat{H}] \tag{29}$$

for the time dependence of an *operator*, with the states frozen at a fixed point in time.

Q 2.5 *Check that both a) (14) with* $\frac{d}{dt}\mathcal{O} = 0$ *and b) eq. (29) with* $\frac{d}{dt}|\Psi\rangle = 0$ *lead to the right equation of motion for expectation values eq. (28).*

It is not difficult to see that the solution of the Heisenberg equation is given by

$$\mathcal{O}(t) = \hat{U}(t - t_0)^\dagger \mathcal{O}(t_0)\hat{U}(t - t_0) \tag{30}$$

which could of course be obtained directly by inspection of the matrix element $\langle\Psi|\mathcal{O}|\Psi\rangle$, and stripping the time evolution operator off the state, onto the operator.

In summary, we can either view the states as evolving in time and the operators as being constant

$$|\Psi(t)\rangle = \hat{U}(t)|\psi\rangle\,, \qquad \mathcal{O}(t) = \mathcal{O}(0)\,, \tag{31}$$

or vice versa

$$|\Psi(t)\rangle = |\psi\rangle\,, \qquad \mathcal{O}(t) = \hat{U}(t)^\dagger \mathcal{O}(0)\hat{U}(t)\,. \tag{32}$$

Since these two formulations describe the same dynamics, the difference between them is completely unobservable. They are merely different pictures of the same underlying physics. They carry the names Schrödinger and Heisenberg picture respectively.

A common trick in QFT is to work in an intermediate framework, called the interaction picture. In this picture the evolution due to the simpler part of the Hamiltonian (usually the quadratic bit) is put into the operators, and the evolution due to the more complicated parts (the interactions) are put into the states.

Q 2.6 *Using* $\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{x})$*, determine the commutation relation* $[\hat{x}(t), \hat{p}(t)]$ *from the requirement that the operators* $\hat{x}(t)$ *and* $\hat{p}(t)$ *satisfy an equation analogous to the classical equation of motion eq. (3). Note the close similarity to the classical computation involving the Poisson bracket.*

Perturbation Techniques

There are not many quantum systems that admit an exact analytic solution (finding the eigenstates and their energies), so in general a more widely applicable method for extracting the physics from a Hamiltonian is needed. One way is to split the Hamiltonian $\hat{H}$ in question into a part that we can solve, $\hat{H}_0$, and treat the remainder as a perturbation.

$$\hat{H} = \hat{H}_0 + \hat{H}_I\,. \tag{33}$$

Of course this split is arbitrary, but typically $\hat{H}_0$ will describe a particle with either no, or quadratic potential. For this to be useful, there needs to be some sense in which the contribution from $\hat{H}_I$ is much smaller than that from $\hat{H}_0$, so we expand the quantities of interest in a series.

In QFT you will be mainly concerned with calculating *transition amplitudes*. Recall from Q 2.2 that the probability to measure a particle in state $|i\rangle$ at time 0 and then in state $|j\rangle$ at time t is

$$P_{i\to j}(t) = |\langle j| \exp(-it\hat{H})|i\rangle|^2 \tag{34}$$

where $|i\rangle$ are the normalized eigenvectors of some observable, e.g. position eigenstates, momentum eigenstates or energy eigenstates.

To make use of the split into a known Hamiltonian and a perturbation we need to use the operator identity

$$\exp(A+\epsilon B) = e^A + \epsilon \int_0^1 \mathrm{d}s\, e^{(1-s)A} B e^{sA} + \mathcal{O}(\epsilon^2) \tag{35}$$

or equivalently

$$\exp(-it(\hat{H}_0+\hat{H}_I)) = e^{-it\hat{H}_0} + \int_0^t \mathrm{d}t'\, e^{-i(t-t')\hat{H}_0}\hat{H}_I e^{-it'\hat{H}_0} + \mathcal{O}(\hat{H}_I^2)\,. \tag{36}$$

A mnemonic for this is that the perturbation $\hat{H}_I$ is inserted at every possible point along the interval of evolution $[0,t]$, dressed by evolution under the free H_0 before and after.

Q* 2.7 *Show the above relation e.g. by expanding both sides in their Taylor expansions and counting powers of A and B. (Hint: you may need the binomial coefficient.)*

This expansion is particularly useful if the states of interest are eigenstates of the *unperturbed* system $\hat{H}_0$.

Q* 2.8 *Assuming that $|i\rangle, |j\rangle$ are energy eigenstates of $\hat{H}_0$ with different energy eigenvalues, insert eq. (36) into the transition amplitude and find the leading order correction, expressed in terms of the matrix element $\langle i|\hat{H}_I|j\rangle$. (Assume $\langle i|j\rangle = 0$.)*

The result is important enough to carry the name *Fermi's golden rule*. You should find

$$P_{i\to j}(t) = \frac{\sin(\omega t/2)^2}{(\omega/2)^2}|\langle j|\hat{H}_I|i\rangle|^2 \tag{37}$$

with the frequency of oscillation given by $\omega = E_j - E_i$. As an example, consider the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_I = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} + \begin{pmatrix} 0 & \delta \\ \bar{\delta} & 0 \end{pmatrix} \tag{38}$$

where $|\delta| \ll E$.

Q 2.9 *Find the energies and eigenstates for $\delta = 0$. Then use the result of Q 2.8 to find the transition amplitude between the two energy levels to leading order in δ.*

You will have noticed that the computation is tremendously simplified when computing transitions between eigenstates of the unperturbed system. Recall that the Schrödinger and Heisenberg picture are different, yet equivalent, assignments of time evolution between the state and the operators. We can now make use of the above observation by introducing a half-way picture. Since we know the evolution of the state $|\Psi(t)\rangle$ by the unperturbed, exactly solvable Hamiltonian $\hat{H}_0$, we absorb this into the definition of the state, i.e.

$$|\Psi(t)\rangle_I = \exp(it\hat{H}_0)|\Psi(t)\rangle_S \,, \tag{39}$$

where $|\Psi\rangle_I$ and $|\Psi\rangle_S$ are the state in the interaction and Schrödinger pictures respectively. Only the interaction term $\hat{H}_I$ contributes to the time dependence of the state in the interaction picture. Comparison with the Schrödinger equation yields that in this picture the state and the operators have to satisfy the equations

$$i\partial_t|\Psi\rangle_I = \hat{H}_I|\Psi\rangle_I \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{O}_I = i[\hat{H}_0, \mathcal{O}_I] \,. \tag{40}$$

Q 2.10 *Verify this.*

For the next two terms you will be working mainly in this picture.

3 Tensor products, harmonic oscillator and spin

Tensor Products & Identical particles

When combining two physical systems, represented by Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and Hamiltonians $\hat{H}_1$ and $\hat{H}_2$ respectively, we get a system with Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \tag{41}$$

the tensor product of the individual spaces, and a Hamiltonian

$$\hat{H} = \hat{H}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{H}_2 \,. \tag{42}$$

If bases of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are given by $\{|i\rangle\}_{i\in I}$ and $\{|j\rangle\}_{j\in J}$ respectively, then a basis of the tensor product is given by

$$\{|i,j\rangle = |i\rangle \otimes |j\rangle\}_{i\in I, j\in J} \,. \tag{43}$$

A general element of the combined Hilbert space can be expressed as a linear combination of these

$$|\phi\rangle = \sum_{i,j} c_{i,j} |i\rangle \otimes |j\rangle \tag{44}$$

where the $c_{i,j}$ are complex coefficients. Clearly, the number of basis elements and hence the dimension of the combined Hilbert space is $\dim \mathcal{H} = \dim \mathcal{H}_1 \cdot \dim \mathcal{H}_2$. (The generalisation to combining several physical systems follows analogously.)

For the purpose of calculations you can think of the several states and operators living in different worlds; they behave within their Hilbert spaces as you are used to, e.g.

$$\left(\hat{A} \otimes \hat{B}\right) |\psi\rangle \otimes |\xi\rangle = \left(\hat{A}|\psi\rangle\right) \otimes \left(\hat{B}|\xi\rangle\right) \tag{45}$$

and

$$\left(\langle\psi| \otimes \langle\xi|\right) \left(|\psi'\rangle \otimes |\xi'\rangle\right) = \langle\psi|\psi'\rangle\langle\xi|\xi'\rangle \,. \tag{46}$$

In particular the energy of a state $|\psi\rangle \otimes |\xi\rangle$ is given by

$$\left(\hat{H}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{H}_2\right) \left(|\psi\rangle \otimes |\xi\rangle\right) = \left(E_1(\psi) + E_2(\xi)\right) \left(|\psi\rangle \otimes |\xi\rangle\right) \tag{47}$$

where ψ and ξ are eigenstates of $\hat{H}_1$ and $\hat{H}_2$ respectively. Often the trivially acting part is omitted, so we abbreviate $\hat{H}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{H}_2$ as simply $\hat{H}_1 + \hat{H}_2$ and use the index to remind us which subspace the operator acts on.

It is crucial to observe that not every state in the combined system is a product of two individual states, rather, it is in general a superposition of such combined states. This leads to the phenomenon known as entanglement, but we won't have time to dwell on this further in these notes.

Important examples of combined systems occur for constituents describing the same type of particle, e.g. $\mathcal{H}_1 \simeq \mathcal{H}_2 \simeq \mathcal{H}$ and $\hat{H}_1 = \hat{H}_2 = \hat{H}$. Then we find a new requirement that any state has to fulfil. Consider the operator S, called *swap operator*, which we define by its action on the basis elements as

$$\begin{aligned} S : \mathcal{H} \otimes \mathcal{H} &\to \mathcal{H} \otimes \mathcal{H} \\ |i\rangle \otimes |j\rangle &\mapsto |j\rangle \otimes |i\rangle \,. \end{aligned} \tag{48}$$

We can now demand a state in $\mathcal{H} \otimes \mathcal{H}$ be invariant under the action of S,

$$S|\psi\rangle = \alpha|\psi\rangle \tag{49}$$

where α is some phase ($|\alpha| = 1$). Recall that two states $|\psi_1\rangle$ and $|\psi_2\rangle$ are *equivalent*, i.e. describe the same physical state, if they differ only by a

phase, $|\psi_2\rangle = \alpha|\psi_1\rangle$. (This ambiguity may seem innocuous, but has very far reaching and delicate consequences.) Hence we included an arbitrary phase, α, in eq. (49) so as to not lose generality.

Experiments find that in nature only two possibilities are realised, $\alpha = 1$ and $\alpha = -1$. Note that in both cases $S^2|\psi\rangle = |\psi\rangle$.[3] Furthermore, particles with integer spin (bosons) are found to have $\alpha = 1$ while particles with half-integer spin (fermions) have $\alpha = -1$.

So if we are combining identical bosons, the product state has to satisfy

$$S|\psi\rangle = |\psi\rangle \tag{50}$$

while for fermions it has to satisfy

$$S|\psi\rangle = -|\psi\rangle\,. \tag{51}$$

In other words, the combined state space is just the *(anti-)symmetric* part of the tensor product. They are spanned by the (anti-)symmetric combinations of basis vectors

$$|i,j\rangle = |i\rangle|j\rangle + |j\rangle|i\rangle \qquad \text{for} \quad i \leq j \tag{52}$$

and

$$|i,j\rangle = |i\rangle|j\rangle - |j\rangle|i\rangle \qquad \text{for} \quad i < j \tag{53}$$

respectively, e.g. a two-fermion system can never be in a state $|\psi\rangle \otimes |\psi\rangle$. This is the famous *Pauli exclusion principle*.

Q 3.1 *What is the dimension of the state space for a spin-$\frac{1}{2}$ particle? What is the dimension of the state space of N distinct spin-$\frac{1}{2}$ particles? What if they are identical fermionic spin-$\frac{1}{2}$ particles? What if they are identical bosonic spin-$\frac{1}{2}$ particles?*

This phase $\alpha = \pm 1$ has important consequences for the thermodynamic properties of the full system and is hence not just a technicality.

Harmonic oscillator

Ladder operators are one of the many avatars of complex methods in physics. Let's start by writing the classical Hamiltonian for the harmonic oscillator not using two real coordinates x, p, but instead one complex coordinate

$$z = x\sqrt{\frac{m\omega}{2}} + p\frac{i}{\sqrt{2m\omega}}\,, \qquad \bar{z} = z^*\,, \tag{54}$$

[3] Using QFT one can in fact prove that this has to be the case, if their interactions are to make sense. Particles for which α is not ± 1, called anyons, are possible in 2d and can be realised in condensed matter lab experiments.

in terms of which the Hamiltonian becomes

$$H(x,p) \to H(z,\bar{z}) = \omega z\bar{z} = \omega|z|^2\,. \tag{55}$$

To quantize we promote these variables to operators ($z \to a, \bar{z} \to a^\dagger$) and write the Hamiltonian as the symmetric product

$$\hat{H} = \frac{\omega}{2}\{a, a^\dagger\} = \frac{\omega}{2}(aa^\dagger + a^\dagger a)\,. \tag{56}$$

Q 3.2 *Show that $\langle\psi|\hat{H}|\psi\rangle \geq 0$ for any state $|\psi\rangle$ by using the property $\|\cdot\|^2 \geq 0$ of Hilbert spaces (and $\omega > 0$).*

Q 3.3 *Write the operators $a, a^\dagger$ in position space and compute their anti-commutator $\{a, a^\dagger\}$. Do you recognise the result? Also compute their commutator $[a, a^\dagger]$ without using position representation, instead using the canonical commutation relations eq. (18).*

Q 3.4 *Show $\langle\psi|\hat{H}|\psi\rangle > 0$ for any state $|\psi\rangle$ by refining your answer to Q 3.2. Find a lower bound for $\langle\hat{H}\rangle_\psi$ and give a condition on any $|\psi\rangle$ saturating this bound.*

Q 3.5 *Show that $a, a^\dagger$ act on eigenstates of $\hat{H}$ by shifting their eigenvalue by ± 1. Since the spectrum of $\hat{H}$ is bounded from below, argue that there must be a state which is annihilated by a.*

In a general system, the state which is annihilated by (all) lowering operators is defined to be the vacuum and given the symbol $|0\rangle$ or $|\Omega\rangle$. Note that this is not the zero element of the Hilbert space, i.e. not $\vec{0}$. It is some non-zero vector, and often a great deal of effort goes just into figuring out what exactly the vacuum of a system is.

Q 3.6 *a) Use the commutation relation $[a, a^\dagger]$ to calculate the norm of the states $|n\rangle = c_n(a^\dagger)^n|0\rangle$. Calculate the overlap $\langle n|m\rangle$. Finally choose c_n such that $\langle n|n\rangle = 1$ and write down $\langle n|m\rangle$ in terms of the normalized states.*

b) Show that you can realize the commutation relation for $a, a^\dagger$ via $a \to \partial_{a^\dagger}$, by showing $a^n f(a^\dagger)|0\rangle = f^{(n)}(a^\dagger)|0\rangle$ for any polynomial f.

c) Repeat a) by realising the commutation relation as in b).

Q* 3.7 *Consider the coherent state $|\lambda\rangle = \exp(\lambda a^\dagger)|0\rangle$ for some $\lambda \in \mathbb{C}$. Show that it is an eigenstate of the annihilation operator a. Calculate the overlap $\langle\lambda|\kappa\rangle$, e.g. by realising the $a, a^\dagger$ commutation relation via $a \to \partial_{a^\dagger}$. Calculate the average energy $\langle\hat{H}\rangle_\lambda = \langle\lambda|H|\lambda\rangle/\langle\lambda|\lambda\rangle$.*

Here are some questions on perturbations of the harmonic oscillator. You may find it useful to recall Fermi's golden rule, eq. (37).

Q 3.8 *Calculate to leading order the transition probability between different levels of the harmonic oscillator, $\hat{H}_0 = \frac{\omega}{2}\{a, a^\dagger\}$, under the anharmonic perturbation $\hat{H}_I = \epsilon\hat{x}^3$. (Hint: express $\hat{x}^3$ in terms of $a, a^\dagger$.)*

Q* 3.9 *Show that the correction to the transmission probability at next-to-leading order is quartic in the perturbation. (You are not required to find this correction.)*

Spin

Spin is an abstract dynamical property of particles and can be thought of by analogy with the familiar angular momentum. Though tempting, it is not quite correct to think of a particle spinning around itself like a top. The analogy is however very deep, which is captured in the statement that the spin and angular momentum operators obey the same *algebra*. Recall that the i^{th} component of the angular momentum operator is defined as

$$\hat{L}_i = (\hat{\vec{x}} \times \hat{\vec{p}})_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k \,, \tag{57}$$

from which we can verify the commutation relations (angular momentum algebra)

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k \,. \tag{58}$$

Note that the definition of $\hat{L}_i$ in terms of position and momentum gives it a natural interpretation as the quantum operator associated to classical angular momentum $\vec{L} = \vec{x} \times \vec{p}$.

Q 3.10 *When promoting the calssical angular momentum to an operator, is there a normal ordering ambiguity in the definition? Use the canonical commutation relation to verify eq.* (58).

As we mentioned before, an observable (Hermitian operator) generates transformations on states. In order to get a feel for what this operator does:

Q* 3.11 *Calculate the action of* $\exp(i\alpha\cdot\hat{L})$ *on a scalar function* $f(x)$. *(Hint 1: go to a basis for* $\mathbb{R}^3$ *in which* $\vec{\alpha}$ *looks simple. Hint 2: you should find that* $f(x) \to f(R(\alpha)x)$, *where* $R(\alpha)$ *is the rotation matrix around the vector* $\vec{\alpha}/\|\alpha\|$ *by an angle* $\|\alpha\|$.*)*

We now create a new set of three operators, $\hat{S}_i$, called spin operators and declare they obey the same commutation relations as the angular momentum operators eq. (58), i.e.

$$[\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk}\hat{S}_k \tag{59}$$

for the components of the spin vector $(\hat{\vec{S}})_i = \hat{S}_i$. So far, this is an abstract algebra. If we want to describe an actual physical system and calculate its properties, we'll have to choose an appropriate *representation* for this algebra.

Q 3.12 *Verify that* $\hat{S}_i = \frac{1}{2}\sigma_i$*, where the three matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{60}$$

are the Pauli matrices, satisfy eq. (59).

Q 3.13 *Calculate the value of the spin* s *for this representation using the relation*

$$\hat{S}^2 = \hat{\vec{S}} \cdot \hat{\vec{S}} = s(s+1)\mathbb{1}\,. \tag{61}$$

What is the Hilbert space that this operators act on? Write down a basis and a generic element of it.

Having a spin-$\frac{1}{2}$ representation and the corresponding space of physical states at hand, we go on to define the Hamiltonian operator. Let's couple the spin to an constant external magnetic field $\vec{B}$ by taking

$$\hat{H} = \hat{\vec{S}} \cdot \vec{B} = \sum_i \hat{S}_i B_i\,, \tag{62}$$

where $\vec{B}$ is some constant vector. For convenience, let's rotate our measuring apparatus such that its z-axis is parallel to the magnetic field, in other words, take $\vec{B} = b\hat{e}_z$.

Q 3.14 *Find the normalized eigenvectors and eigenvalues of the Hamiltonian eq. (62) and denote them by* $|\uparrow\rangle$ *and* $|\downarrow\rangle$ *with energy eigenvalues* $E_\pm$ *respectively. Given a state* $|\psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle$ *at time zero, what is* $|\Psi(t)\rangle$*? i.e. calculate the time evolution operator* $U(t) = \exp(-it\hat{H})$ *for eq. (62) and apply it to* $|\psi\rangle$*.*

Recall that physical states in a quantum theory are described by rays in the Hilbert space. A ray is uniquely determined by a normalized element of the Hilbert space modulo phase. In a spin-1/2 system any normalized element of the Hilbert space can be expressed in the form $|\psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle$ for $c_1 = \sin(\vartheta/2)$ and $c_2 = \cos(\vartheta/2)\, e^{i\varphi}$, up to an overall phase.

Q* 3.15 *By determining the redundancy in the parametrization of physical states by* ϑ *and* φ *determine the physical state space of a spin-1/2 system. Express* $|\Psi(t)\rangle$ *from question Q 3.14 in this form and determine* $\vartheta(t)$ *and* $\varphi(t)$*.*

Q 3.16 *Calculate the average spin* $\vec{S}(t) = \langle\hat{\vec{S}}\rangle_{\Psi(t)}$ *in the state* $\Psi(t)$*. (Hint: look at* S_z *and* $\vec{S}_\perp = (S_x, S_y)$ *separately, and express* $\vec{S}_\perp(t) = R(t)\vec{S}_\perp(0)$ *using a time dependent rotation matrix* R*.)*

Observe the rotation of the spin expectation value around the axis of the magnetic field, called precession. Given the nice behaviour of the expectation value, we can go further and ask how the operator itself evolves (in the Heisenberg picture).

Q* 3.17 *Calculate the time evolution of the operator* $\hat{\vec{S}}$ *in the Heisenberg picture for the Hamiltonian in eq. (62). (Hint 1: you can use the spin algebra to turn the Heisenberg equation into a linear coupled ODE. Hint 2: separate* $\hat{\vec{S}}$ *into* $\hat{S}_z$ *and* $\hat{\vec{S}}_\perp$ *as above.)*

Q* 3.18 *Repeat Q 3.14 without making the simplification* $\vec{B} = b\hat{e}_z$*. If you choose the right basis for the Hilbert space, this exercise requires nearly no additional computation.*

Q* 3.19 *Compute the matrix exponential* $\exp(i\vec{\alpha}\cdot\vec{\sigma})$ *for any vector* $\vec{\alpha}$*. In addition to the algebra eq. (59), you'll need to use* $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$*. A handy way to combine the two is in the equation* $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$*.*

4 The Feynman Propagator

Define the time ordered product of two arbitrary operators in the Heisenberg picture as

$$\mathcal{T}(\mathcal{O}_1(t)\mathcal{O}_2(t')) = \theta(t-t')\mathcal{O}_1(t)\mathcal{O}_2(t') + \theta(t'-t)\mathcal{O}_2(t')\mathcal{O}_1(t) \tag{63}$$

with the Heaviside step function $\theta(x) = (\mathrm{sign}(x)+1)/2$.

Q* 4.1 *Compute the Feynman propagator for the harmonic oscillator, defined as*

$$G_F(t,t') = \langle 0|\mathcal{T}(\hat{x}(t)\hat{x}(t'))|0\rangle\,. \tag{64}$$

Notice how it is a function of $t-t'$, which reflects the time translation symmetry of the system. The Feynman propagator is important because it is a fundamental solution to the equation of motion for the harmonic oscillator.

Q* 4.2 *Show that* G_F *satisfies the equation of motion with delta function source term*

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2\right) G_F(t,t') = -\frac{i}{m}\delta(t-t')\,. \tag{65}$$

The Feynman propagator still knows about the canonical commutation relation of $\hat{x}, \hat{p}$.

Q* 4.3 *Using* $\hat{p}(t) = m\dot{\hat{x}}(t)$ *and taking an appropriate limit of the Feynman propagator, show that*

$$\langle 0|[\hat{x},\hat{p}]|0\rangle = i\,. \tag{66}$$