5. Systems of Particles

So far, we've only considered the motion of a single particle. If our goal is to understand everything in the Universe, this is a little limiting. In this section, we take a small step forwards: we will describe the dynamics of N, interacting particles.

The first thing that we do is put a label $i = 1, \ldots, N$ on everything. The $i^{\rm th}$ particle has mass m_i, position $\mathbf{x}_i$ and momentum $\mathbf{p}_i = m_i \dot{\mathbf{x}}_i$. (A word of warning: do not confuse the label i on the vectors with index notation for vectors!) Newton's second law should now be written for each particle,

$$\dot{\mathbf{p}}_i = \mathbf{F}_i$$

where $\mathbf{F}_i$ is the force acting on the $i^{\rm th}$ particle. The novelty is that the force $\mathbf{F}_i$ can be split into two parts: an external force $\mathbf{F}_i^{\rm ext}$ (for example, if the whole system sits in a gravitational field) and a force due to the presence of the other particles. We write

$$\mathbf{F}_i = \mathbf{F}_i^{\rm ext} + \sum_{j \neq i} \mathbf{F}_{ij}$$

where $\mathbf{F}_{ij}$ is the force on particle i due to particle j. At this stage, we get to provide a more precise definition of Newton's third law. Recall the slogan: every action has an equal and opposite reaction. In equations this means,

- **N3 Revisited:** $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$

In particular, this form of the third law holds for both gravitational and Coulomb forces. However, we will soon find a need to present an even stronger version of Newton's third law.

5.1 Centre of Mass Motion

The total mass of the system is

$$M = \sum_{i=1}^{N} m_i$$

We define the *centre of mass* to be

$$\mathbf{R} = \frac{1}{M} \sum_{i=1}^{N} m_i \mathbf{x}_i$$

The total momentum of the system, $\mathbf{P}$, can then be written entirely in terms of the centre of mass motion,

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i = M\dot{\mathbf{R}}$$

We can now look at how the centre of mass moves. We have

$$\dot{\mathbf{P}} = \sum_i \dot{\mathbf{p}}_i = \sum_i \left(\mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{ij} \right) = \sum_i \mathbf{F}_i^{\text{ext}} + \sum_{i<j} (\mathbf{F}_{ij} + \mathbf{F}_{ji})$$

But Newton's third law tells us that $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ and the last term vanishes, leaving

$$\dot{\mathbf{P}} = \sum_i \mathbf{F}_i^{\text{ext}} \tag{5.1}$$

This is an important formula. It tells us if you just want to know the motion of the centre of mass of a system of particles, then only the external forces count. If you throw a wriggling, squealing cat then its internal forces $\mathbf{F}_{ij}$ can change its orientation, but they can do nothing to change the path of its centre of mass. That is dictated by gravity alone. (Actually, this statement is only true for conservative forces. The shape of the cat could change friction coefficients which would, in turn, change the external forces).

It's hard to overstate the importance of (5.1). Without it, the whole Newtonian framework for mechanics would come crashing down. After all, nothing that we really describe is truly a point particle. Certainly not a planet or a cat, but even something as simple as an electron has an internal spin. Yet none of these details matter because everything, regardless of the details, any object acts as a point particle if we just focus on the position of its centre of mass.

5.1.1 Conservation of Momentum

There is a trivial consequence to (5.1). If there is no net external force on the system, so $\sum_i \mathbf{F}_i^{\text{ext}} = 0$, then the total momentum of the system is conserved: $\dot{\mathbf{P}} = 0$.

5.1.2 Angular Momentum

The total angular momentum of the system about the origin is defined as

$$\mathbf{L} = \sum_i \mathbf{x}_i \times \mathbf{p}_i$$

Recall that when we take the time derivative of angular momentum, we get $d/dt(\mathbf{x}_i \times \mathbf{p}_i) = \dot{\mathbf{x}}_i \times \mathbf{p}_i + \mathbf{x}_i \times \dot{\mathbf{p}}_i = \mathbf{x}_i \times \dot{\mathbf{p}}_i$ because $\mathbf{p}_i$ is parallel to $\dot{\mathbf{x}}_i$. Using this, the change in the total angular momentum is

$$\frac{d\mathbf{L}}{dt} = \sum_i \mathbf{x}_i \times \dot{\mathbf{p}}_i = \sum_i \mathbf{x}_i \times \left(\mathbf{F}_i^{\text{ext}} + \sum_{j\neq i} \mathbf{F}_{ij} \right) = \boldsymbol{\tau} + \sum_i \sum_{j\neq i} \mathbf{x}_i \times \mathbf{F}_{ij}$$

where $\boldsymbol{\tau} \equiv \sum_i \mathbf{x}_i \times \mathbf{F}_i^{\text{ext}}$ is the *total external torque*. The second term above still involves the internal forces. What are we going to do about it? Since $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, we can write it as

$$\sum_i \sum_{i\neq j} \mathbf{x}_i \times \mathbf{F}_{ij} = \sum_{i<j} (\mathbf{x}_i - \mathbf{x}_j) \times \mathbf{F}_{ij}$$

This would vanish if the force between the i^{th} and j^{th} particle is parallel to the line $(\mathbf{x}_i - \mathbf{x}_j)$ joining the two particles. This is indeed true for both gravitational and Coulomb forces and this requirement is sometimes elevated to a strong form of Newton's third law:

- **N3 Revisited Again:** $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ and is parallel to $(\mathbf{x}_i - \mathbf{x}_j)$.

In situations where this strong form of Newton's third law holds, the change in total angular momentum is again due only to external forces,

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} \tag{5.2}$$

5.1.3 Energy

The total kinetic energy of the system of particles is

$$T = \frac{1}{2} \sum_i m_i \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i$$

We can decompose the position of each particle as

$$\mathbf{x}_i = \mathbf{R} + \mathbf{y}_i$$

where $\mathbf{y}_i$ is the position of the particle i relative to the centre of mass. In particular, since $\sum_i m_i \mathbf{x}_i = M\mathbf{R}$, the $\mathbf{y}_i$ must obey the constraint $\sum_i m_i \mathbf{y}_i = 0$. The kinetic energy can then be written as

$$\begin{aligned} T &= \frac{1}{2} \sum_i m_i \left(\dot{\mathbf{R}} + \dot{\mathbf{y}}_i \right)^2 \\ &= \frac{1}{2} \sum_i m_i \dot{\mathbf{R}}^2 + \dot{\mathbf{R}} \cdot \sum_i m_i \dot{\mathbf{y}}_i + \frac{1}{2} \sum_i m_i \dot{\mathbf{y}}_i{}^2 \\ &= \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \sum_i m_i \dot{\mathbf{y}}_i{}^2 \end{aligned} \tag{5.3}$$

This tells us that the kinetic energy splits up into the kinetic energy of the centre of mass, together with the kinetic energy of the particles moving around the centre of mass.

We can repeat the analysis that lead to the construction of the potential energy. When the i^{th} particle moves along a trajectory $\mathcal{C}_i$, the difference in kinetic energies is given by

$$T(t_2) - T(t_1) = \sum_i \int_{\mathcal{C}_i} \mathbf{F}_i^{\text{ext}} \cdot d\mathbf{x}_i + \sum_i \sum_{j \neq i} \int_{\mathcal{C}_i} \mathbf{F}_{ij} \cdot d\mathbf{x}_i$$

If we want to define a potential energy, we require that both external and internal forces are conservative. We usually do this by asking that

- Conservative External Forces: $\mathbf{F}_i^{\text{ext}} = -\nabla_i V_i(\mathbf{x}_i)$
- Conservative Internal Forces: $\mathbf{F}_{ij} = -\nabla_i V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|)$

Note that, for once, we are not using the summation convention here. We are also working with the definition $\nabla_i \equiv \partial/\partial \mathbf{x}_i$. In particular, internal forces of this kind obey the stronger version of Newton's third law if we take the potentials to further obey $V_{ij} = V_{ji}$. With these assumptions, we can define a conserved energy given by

$$E = T + \sum_i V_i(\mathbf{x}_i) + \sum_{i<j} V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|)$$

5.1.4 In Praise of Conservation Laws

> *Semper Eadem*, the motto of Trinity College, celebrating conservation laws since 1546

Above we have introduced three quantities that, under the right circumstances, are conserved: momentum, angular momentum and energy. There is a beautiful theorem, due to Emmy Noether, which relates these conserved quantities to symmetries of space and time. You will prove this theorem in a later Classical Dynamics course, but here we give just a taster[4] of this result, together with some motivation.

- Conservation of momentum follows from the translational invariance of space. In our formulation, we saw that momentum is conserved if the total external force vanishes. But without an external force pushing the particles one way or another, any point in space is just as good as any other. This is the deep reason for momentum conservation.

[4] A proof of Noether's theorem first needs the basics of the Lagrangian formulation of classical mechanics. An introduction can be found at http://www.damtp.cam.ac.uk/user/tong/dynamics.html

- Conservation of angular momentum follows from the rotational invariance of space. Again, there are hints of this already in what we have seen since a vanishing external torque can be guaranteed if the background force is central, and therefore rotational symmetric.

- Conservation of energy follows from invariance under time translations. This means that it doesn't matter when you do an experiment, the laws of physics remain unchanged. We can see one aspect of this in our discussion of potential energy in Section 2 where it was important that there was no explicit time dependence. (This is not to say that the potential energy doesn't change with time. But it only changes because the position of the particle changes, not because the potential function itself is changing).

5.1.5 Why the Two Body Problem is Really a One Body Problem

Solving the dynamics of N mutually interacting particles is hard. Here "hard" means that no one knows how to do it unless the forces between the particles are of a very special type (e.g. harmonic oscillators).

However, when there are no external forces present, the case of two particles actually reduces to the kind of one particle problem that we met in the last section. Here we see why.

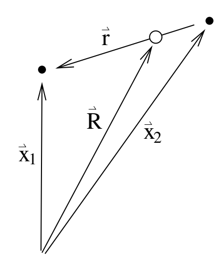

Figure 22: The particles are the black dots; the centre of mass is the white dot.

We have already defined the centre of mass,

$$M\mathbf{R} = m_1\mathbf{x}_1 + m_2\mathbf{x}_2$$

We'll also define the relative separation,

$$\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$$

Then we can write

$$\mathbf{x}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r} \quad \text{and} \quad \mathbf{x}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}$$

We assume that there are no external forces at work on the system, so $\mathbf{F}_i^{\text{ext}} = 0$ which ensures that the centre of mass travels with constant velocity: $\ddot{\mathbf{R}} = 0$. Meanwhile, the relative motion is governed by

$$\ddot{\mathbf{r}} = \ddot{\mathbf{x}}_1 - \ddot{\mathbf{x}}_2 = \frac{1}{m_1}\mathbf{F}_{12} - \frac{1}{m_2}\mathbf{F}_{21} = \frac{m_1 + m_2}{m_1 m_2}\,\mathbf{F}_{12}$$

where, in the last step, we've used Newton's third law $\mathbf{F}_{12} = -\mathbf{F}_{21}$. The equation of motion for the relative position can then be written as

$$\mu\ddot{\mathbf{r}} = \mathbf{F}_{12}$$

where μ is the *reduced mass*

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

But this is really nice. It means that we've already solved the problem of two mutually interacting particles because their centre of mass motion is trivial, while their relative separation reduces to the kind of problem that we've already seen. In particular, if they interact through a central force of the kind $\mathbf{F}_{12} = -\nabla V(\mathbf{r})$ — which is true for both gravitational and electrostatic forces — then we simply need to adopt the methods of Section 4, with m in (4.1) replaced by μ.

In the limit when one of the particles involved is very heavy, say $m_2 \gg m_1$, then $\mu \approx m_1$ and the heavy object remains essentially fixed, with the lighter object orbiting around it. For example, the centre of mass of the Earth and Sun is very close to the centre of the Sun. Even for the Earth and moon, the centre of mass is 1000 miles below the surface of the Earth.

5.2 Collisions

You met collisions in last term's mechanics course. This subject is strictly speaking off-syllabus but, nonetheless, there's a couple of interesting things to say. Of particular interest are *elastic* collisions, in which both kinetic energy and momentum are conserved. As we have seen, such collisions will result from any conservative inter-particle force between the two particles.

Consider the situation of a particle travelling with velocity $\mathbf{v}$, colliding with a second, stationary particle. After the collision, the two particles have velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. Even without knowing anything else about the interaction, there is a pleasing, simple result that we can derive. Conservation of energy tells us

$$\frac{1}{2}m\mathbf{v}^2 = \frac{1}{2}m\mathbf{v}_1^2 + \frac{1}{2}m\mathbf{v}_2^2$$

while the conservation of momentum reads

$$m\mathbf{v} = m\mathbf{v}_1 + m\mathbf{v}_2 \tag{5.4}$$

Squaring this second equation, and comparing to the first, we learn that the cross-term on the right-hand side must vanish. This tells us that

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0 \tag{5.5}$$

In other words, either one of the particles is stationary, or the two particles scatter at right-angles.

Although the conservation of energy and momentum gives us some information about the collision, it is not enough to uniquely determine the final outcome. It's easy to see why: we have six unknowns in the two velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, but just four equations in (5.4) and (5.5).

Acting on Impulse

When particles are subjected to short, sharp shocks – such as the type that arise in collisions – one often talks about *impulse* instead of force. If a force $\mathbf{F}$ acts for just a short time Δt, then the impulse $\mathbf{I}$ experienced by the particle is defined to be

$$\mathbf{I} = \int_t^{t+\Delta t} \mathbf{F}\, dt = \Delta \mathbf{p}$$

The second equality above follows from Newton's second law and tells us that the impulse is the same as the change of momentum.

5.2.1 Bouncing Balls

For particles constrained to move along a line (i.e. in one dimension), the same counting that we did above tells us that the conservation of energy and momentum is enough to tell us everything. Here we look at a couple of examples. First, place a small ball of mass m on top of a large ball of mass M and drop both so that they hit the floor with speed u. How fast does the smaller ball fly back up?

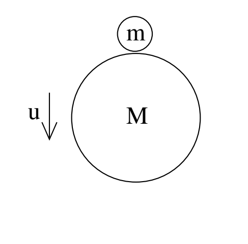

Figure 23:

It's best to think of the small ball as very slightly separated from the larger one. Assuming all collisions are elastic, the big ball then hits the ground first and bounces back up with the same speed u, whereupon it immediately collides with the small ball. After this collision, we'll call the speed of the small ball v and the speed of the large ball V. Conservation of energy and momentum then tell us

$$mu^2 + Mu^2 = mv^2 + MV^2 \quad \text{and} \quad Mu - mu = mv + MV$$

Note that we've measured velocity upwards: hence the initial momentum of the small ball is the only one to come with a minus sign.

Just jumping in and solving these as simultaneous equations will lead to a quadratic and some messy algebra. There's a slightly slicker way. We write the two equations as

$$M(V-u)(V+u) = m(u-v)(u+v) \quad \text{and} \quad M(u-V) = m(v+u)$$

Dividing one by the other gives $V+u = v-u$. We can now use this and the momentum conservation equation to eliminate V. We find

$$v = \frac{3M-m}{M+m}u$$

You can try this at home with a tennis ball and basketball. But trust the maths. It's telling you that the speed will be almost three times greater. This means that the kinetic energy (and therefore the height reached by the tennis ball) will be almost nine times greater. You have been warned!

5.2.2 More Bouncing Balls and the Digits of π

Here's another example. The question seems a little arbitrary, but the answer is quite extraordinary. Consider two balls shown in the figure. The rightmost ball has mass m. The leftmost ball is much heavier: it has the rather strange mass $M = 16 \times 100^N \times m$ where N is an integer.

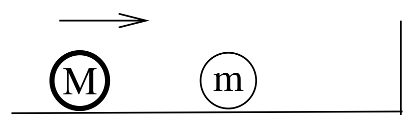

Figure 24:

We give the heavy ball a small kick so it rolls to the right. It collides elastically with the light ball which then flies off towards the wall. The collision with the wall is also elastic and the light ball bounces off with the same speed it arrived at, heading back towards the heavy ball. The process keeps repeating: the light ball bounces off the heavy one, bounces off the wall, and returns to collide yet again with the heavy ball. Note that the total energy is conserved in all processes but the total momentum is not conserved in the collision with the wall.

A priori, there are two possible outcomes of this. It may be that the heavy ball moves all the way to the right where it too bounces off the wall (and, of course, the light ball which is trapped between it and the wall). Or, it may be that the light ball eventually collides enough times that the heavy ball turns around and starts moving towards the left.

Which of these two possibilities occurs will be decided by the dynamics. Below, we'll see that it's actually the latter scenario that takes place: the heavy ball does not reach the wall. The question that we want to ask is: how many times, $p(N)$, does the heavy ball hit the lighter one before it turns around and starts heading in the opposite direction?

The answer to this question is one of the most ridiculous things I've ever seen in physics. It is[5]

$$p(N) - 1 = \text{The first } N+1 \text{ digits of } \pi$$

In other words, $p(0) - 1 = 3$, $p(1) - 1 = 31$, $p(2) - 1 = 314$, $p(3) - 1 = 3141$ and so on.

In case it's not obvious, let me explain why you should also find this result ridiculous. The number π is, of course, ubiquitous in physics. But this is very different from the decimal expansion of the number. As the name suggests, the *digits* of π in a decimal expansion have as much to do with biology as mathematics. But we subtly inserted the relevant biological fact in the original question by insisting that the mass of the big ball is $M = 16 \times 10^{2N} \times m$. This seemingly innocuous factor of 10 will prove to be the reason that the expansion of π comes out in base 10.

Let's now try to prove this unlikely result. Let u_n be the velocity of the heavy ball and v_n be the velocity of the light ball after the n^{th} collision between them. Conservation of energy and momentum tell us that

$$\begin{aligned} Mu_{n+1}^2 + mv_{n+1}^2 &= Mu_n^2 + mv_n^2 \\ Mu_{n+1} + mv_{n+1} &= Mu_n - mv_n \end{aligned}$$

Rearranging these reveals some nice algebraic simplifications. Despite the quadratic nature of the energy conservation equation, the relationship between the velocities before and after is actually linear,

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = A \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

where the matrix A depends only on the ratio of masses which we denote as $x = m/M$ and is given by

$$A = \frac{1}{1+x} \begin{pmatrix} 1-x & -2x \\ 2 & 1-x \end{pmatrix}$$

Since we start with the only the heavy ball moving, $(u_0, v_0) = (u_0, 0)$. The velocities after the n^{th} collision between the balls are

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = A^n \begin{pmatrix} u_0 \\ 0 \end{pmatrix} \tag{5.6}$$

[5]This is a variation of a problem first stated in the 2003 in the paper "*Playing Pool with π*" by Gregory Galperin. The proof in this paper uses purely geometric techniques. I'm grateful to Joe Minahan for help constructing this example, together with the proof below.

The smart way to compute the matrix A^n is to first diagonalise A .The eigenvalues of A are easily computed to be $e^{\pm i\theta}$ where

$$\cos\theta = \frac{1-x}{1+x}$$

Using this, we can write

$$A^n = S\begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta}\end{pmatrix} S^{-1} \qquad \text{with } \; S = \frac{1}{1+x}\begin{pmatrix} i\sqrt{x} & -i\sqrt{x} \\ 1 & 1\end{pmatrix} \tag{5.7}$$

and the velocities after the n^{th} collision are given by

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \frac{u_0}{\sqrt{x}}\begin{pmatrix} \sqrt{x}\cos n\theta \\ \sin n\theta \end{pmatrix}$$

We want to know how many collisions, p, it takes before the heavy ball starts moving in the opposite direction. This occurs when $\cos n\theta < 0$, which means that p must obey

$$(p-1)\theta < \frac{\pi}{2} \quad \text{while} \quad p\theta > \frac{\pi}{2}$$

To get a feel for this, we'll make an approximation. Since $x = m/M$, we can expand $\cos\theta \approx 1 - \frac{1}{2}\theta^2 \approx 1 - 2x$, which gives us $\theta \approx 2\sqrt{x}$. Using our rather strange choice of mass, $x = 10^{-2N}/16$, so $\theta \approx 10^{-N}/2$. If the corrections to this approximation are unimportant, the number of collisions p is the largest integer such that $(p-1)\times 10^{-N} < \pi$ while $p \times 10^{-N} > \pi$. The answer is

$$p(N) - 1 = [10^N \pi]$$

which means the integer part of $10^N\pi$. This is the same thing as the first $N+1$ digits of π.

Finally, we should check whether the approximations that we made above are valid. Is there some way the higher order terms that we neglected can change the answer? Although we should check this, we won't. Because it turns out to be quite tricky. If you're interested, some relevant details can be found in the original paper cited above.

5.3 Variable Mass Problems

Recall that the correct version of Newton's second law is

$$\dot{\mathbf{p}} = \mathbf{F} \tag{5.8}$$

where $\mathbf{p} = m\dot{\mathbf{x}}$ is the momentum. This coincides with the more familiar $m\ddot{\mathbf{x}} = \mathbf{F}$ only when the mass of the object is unchanging. Here we will look at a few situations where the mass actually does change. There are two canonical examples: things falling apart and things gathering other stuff. We'll treat them each in turn.

5.3.1 Rockets: Things Fall Apart

A rocket moves in a straight line with velocity $v(t)$. The mass of the rocket, $m(t)$, changes with time because it propels itself forward by spitting out fuel behind. Suppose that the fuel is ejected at a speed u *relative* to the rocket. Our goal is to figure out how the speed of the rocket changes over time.

You might think that we should just plug this into Newton's second law (5.8) to get "$d(mv)/dt = F$". But this isn't quite right. The equation (5.8) refers to the momentum of the entire system, which in this case includes the rocket and the ejected fuel. And we need to take both into account.

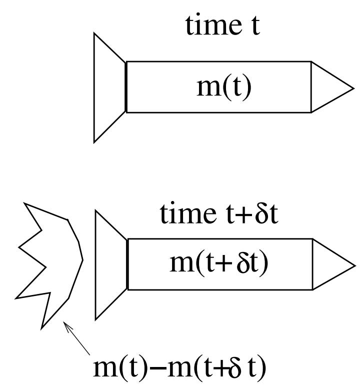

Figure 25:

To proceed, it's best to go back to first principles and work infinitesimally. At time t, the momentum of the rocket is

$$p(t) = m(t)v(t)$$

After a short interval δt, this momentum is split between the momentum of the rocket and the momentum of the recently ejected fuel,

$$p(t + \delta t) = p_{\text{rocket}}(t + \delta t) + p_{\text{fuel}}(t + \delta t)$$

The momentum of the rocket at this later time is given by

$$\begin{aligned} p_{\text{rocket}}(t + \delta t) &= m(t + \delta t)v(t + \delta t) \\ &\approx \left(m(t) + \frac{dm}{dt}\delta t\right)\left(v(t) + \frac{dv}{dt}\delta t\right) \\ &\approx m(t)v(t) + \left(v\frac{dm}{dt} + m\frac{dv}{dt}\right)\delta t + \mathcal{O}(\delta t^2) \end{aligned}$$

where we've Taylor expanded the mass and velocity and kept terms up to order δt. Similarly, the momentum of the fuel ejected between time t and $t + \delta t$ is

$$\begin{aligned} p_{\text{fuel}}(t + \delta t) &= [m(t) - m(t + \delta t)]\,[v(t) - u] \\ &\approx -\frac{dm}{dt}\delta t\ [v(t) - u] + \mathcal{O}(\delta t^2) \end{aligned}$$

Notice that the speed of the fuel is $v - u$; this is because the fuel has speed u relative to the rocket. In fact, there's a small subtlety here. Does the fuel travel at velocity $v(t) - u$ or $v(t + \delta t) - u$ or some average of the two? In fact, it doesn't matter. The difference only shows up at order δt^2 and doesn't affect our final answer. Adding together these two momenta, we have the result

$$p(t+\delta t) = p(t) + \left(m(t)\frac{dv}{dt} + u\frac{dm}{dt}\right)\delta t + \mathcal{O}(\delta t^2) \tag{5.9}$$

At this stage, we can use Newton's second law in the form (5.8) which, using the definition of the derivative, is given by

$$\frac{p(t+\delta t) - p(t)}{\delta t} = F$$

Comparing this to (5.9), we arrive at the *Tsiolkovsky rocket equation*

$$m(t)\frac{dv}{dt} + u\frac{dm}{dt} = F \tag{5.10}$$

Apparently, this equation was first derived only in 1903.

An Example: A Free Rocket in Space

Let's solve the rocket equation when there is no external force, $F = 0$. We can write it as

$$\frac{dv}{dt} = -\frac{u}{m}\frac{dm}{dt}$$

which can be trivially integrated to give

$$v(t) = v_0 + u\log\left(\frac{m_0}{m(t)}\right)$$

Here we have chosen the rocket to have speed v_0 when its mass is m_0. We see that burning rocket fuel will only increase your speed logarithmically. If we further assume that the rocket burns fuel at a constant rate,

$$\frac{dm}{dt} = -\alpha$$

then we have $m(t) = m_0 - \alpha t$. (Note that $\alpha > 0$ means that $dm/dt < 0$ as it should be). In this case, the velocity of the rocket is

$$v(t) = v_0 - u\log\left(1 - \frac{\alpha t}{m_0}\right)$$

Notice that this solution only makes sense for times $t < m_0/\alpha$. This is because at time $t = m_0\alpha$, all of the fuel runs out which, in our somewhat silly model, means that the rocket has disappeared entirely. For these times $t < m_0/\alpha$, we can integrate once more to get the position

$$x = v_0 t + \frac{u m_0}{\alpha}\left[\left(1 - \frac{\alpha t}{m_0}\right)\log\left(1 - \frac{\alpha t}{m_0}\right) + \frac{\alpha t}{m_0}\right]$$

Another Example: A Rocket with Linear Drag

Here's a slightly more involved example. The initial mass of the rocket is m_0 and we will still burn fuel at a constant rate, so $\dot{m} = -\alpha$. But now the rocket is subject to linear drag, $F = -\gamma v$, presumably because it has encountered some sticky alien intergalactic golden syrup or something. If the rocket starts from rest, how fast is it going after it has burned one half of its mass as fuel?

With linear drag, the rocket equation (5.10) becomes

$$m\dot{v} + u\dot{m} = -\gamma v \tag{5.11}$$

We can already get a feel for what's going on by looking at this equation. Since $\dot{m} = -\alpha$, rearranging we get

$$m\dot{v} = \alpha u - \gamma v$$

This means that we will continue to accelerate through the sticky alien goo if we're travelling slowly and burning fuel fast enough so that $\alpha u > \gamma v$. But as our speed approaches $v = \alpha u/\gamma$, the acceleration slows down and we expect this to be the limiting velocity. However, if we were travelling too fast to begin with, so $\gamma v > \alpha u$, then we will slow down until we again hit the limiting speed $v = \alpha u/\gamma$.

Let's now look in more detail at the solution. We could solve the rocket equation (5.11) to get $v(t)$, but since the question doesn't ask about velocity as a function of time, we'll be much better off thinking of velocity as a function of mass: $v = v(m)$. Then

$$\dot{v} = \frac{dv}{dm}\dot{m} = -\alpha\frac{dv}{dm}$$

Using this, the rocket equation becomes

$$-\alpha m\frac{dv}{dm} - \alpha u = -\gamma v$$

This can be happily integrated using a few basic steps,

$$\frac{dv}{dm} = \frac{\gamma v - \alpha u}{\alpha m} \quad\quad \Rightarrow \quad\quad \int \frac{dv}{\gamma v - \alpha u} = \int \frac{dm}{\alpha m}$$

Before integrating, we need to decide whether the denominator on the left-hand side is positive or negative. (Because integrating will give us a log and the argument of log has to be positive). Because we stated above that the rocket starts from rest, we have $\gamma v < \alpha u$ meaning that the left-hand side is negative. Integrating then gives

$$\frac{1}{\gamma}\log\left(\frac{\alpha u - \gamma v}{\alpha u}\right) = \frac{1}{\alpha}\log\left(\frac{m}{m_0}\right)$$

Here the denominators that we introduced in the argument of both logs are there on dimensional grounds. (Remember that the argument of log has to be dimensionless). The factor of m_0 is an integration constant; the factor of αu tells us that the velocity vanishes when $m = m_0$. Rearranging, we get the final answer

$$v = \frac{\alpha u}{\gamma}\left(1 - \left(\frac{m}{m_0}\right)^{\gamma/\alpha}\right)$$

We see that the behaviour is in agreement with our discussion after (5.11); as m decreases, v increases to towards the limiting velocity $v = \alpha u/\gamma$. But it never reaches this velocity until all the mass of the rocket is burnt as fuel. In particular, we can answer the question posed at the beginning simply by setting $m = m_0/2$.

5.3.2 Avalanches: Stuff Gathering Other Stuff

It's somewhat more natural to come up with examples where things fall apart and the mass decreases. But, for completeness, let's discuss a situation where the mass increases: avalanches. I should confess up front that avalanches are very poorly understood and the model below holds no claim to realism.

We'll denote the mass of snow moving in the avalanche as $m(t)$. We'll further assume that all the snow moves down the hill at the same speed $v(t)$, picking up extra snow as it goes. We can use the rocket equation (5.10), with $u = v$ since the the snow lying on the ground which is picked up has speed v relative to the avalanche. Ignoring friction, but including the force due to gravity, the rocket equation becomes

$$m\frac{dv}{dt} + v\frac{dm}{dt} = mg\sin\theta$$

where θ is the angle that the slope makes with the ground. Because the snow lying on the ground had no momentum, we do get the naive equation that comes from simply plugging the momentum of the avalanche into (5.8)

$$\frac{d}{dt}(mv) = mg\sin\theta$$

Suppose that the snow has density ρ, height h and all of it is picked up as the avalanche passes over. Then after the avalanche has moved a distance x down the slope, it has picked up a mass $m(t) = \rho h x(t)$. The equation of motion is

$$\frac{d}{dt}(\rho h x v) = \rho h x g \sin\theta$$

At this point, it is best to think of velocity as a function of position: $v = v(x)$. Then we can write $d/dt = v\, d/dx$ so

$$v\frac{d}{dx}(xv) = xg\sin\theta$$

This is again easily integrated in a few standard manoeuvres. If we first multiply both sides by x, we have

$$xv\frac{d}{dx}(xv) = x^2 g\sin\theta \quad \Rightarrow \quad \frac{1}{2}(xv)^2 = \frac{1}{3}x^3 g\sin\theta$$

where we've set the integration constant to zero so that $v = 0$ when we start at $x = 0$. Rearranging now gives the speed as a function of position,

$$v = \sqrt{\frac{2}{3}xg\sin\theta}$$

If we integrate this once more, we get

$$x = \frac{1}{6}gt^2\sin\theta$$

where we again set the integration constant to zero by assuming that $x = 0$ when $t = 0$. It's worth mentioning that this is a factor of 1/3 smaller than the result we get for an object that doesn't gather mass as it goes which, taken at face value, suggests that you should be able to outrun an avalanche, at least if you didn't have to worry about friction. Personally, I wouldn't bet on it.

5.4 Rigid Bodies

So far, we've only discussed "particles", objects with no extended size. But what happens to more complicated objects that can twist and turn as they move? The simplest example is a *rigid body*. This is a collection of N particles, constrained so that the relative distance between any two points, i and j, is fixed:

$$|\mathbf{x}_i - \mathbf{x}_j| = \text{fixed}$$

A rigid body can undergo only two types of motion: its centre of mass can move; and it can rotate. We'll start by considering just the rotations. In Section 5.4.5, we'll combine the rotations with the centre of mass motion.

5.4.1 Angular Velocity

We fix some point in the rigid body and consider rotation about this point. To describe these rotations, we need the concept of angular velocity. We'll begin by considering a single particle which is rotating around the z-axis, as shown in the figure. The position and velocity of the particle are given by

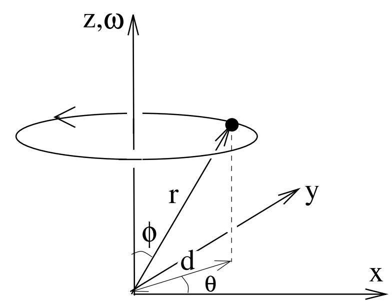

Figure 26:

$$\mathbf{x} = (d\cos\theta, d\sin\theta, z) \quad \Rightarrow \quad \dot{\mathbf{x}} = (-\dot{\theta}d\sin\theta, \dot{\theta}d\cos\theta, 0)$$

We can write this by introducing a new vector $\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{z}}$,

$$\dot{\mathbf{x}} = \boldsymbol{\omega} \times \mathbf{x}$$

The vector $\boldsymbol{\omega}$ is called the *angular velocity*. In general we can write $\boldsymbol{\omega} = \omega\hat{\mathbf{n}}$. Here the magnitude, $\omega = |\dot{\theta}|$ is the angular speed of rotation, while the unit vector $\hat{\mathbf{n}}$ points along the axis of rotation, defined in a right-handed sense. (Curl the fingers of your right hand in the direction of rotation: your thumb points in the direction of $\boldsymbol{\omega}$).

The speed of the particle is then given by

$$v = |\dot{\mathbf{x}}| = r\omega\sin\phi = d\omega$$

where

$$d = |\hat{\mathbf{n}} \times \mathbf{x}| = r\sin\phi$$

is the perpendicular distance to the axis of rotation as shown in the figure. Finally, we will also need an expression for the kinetic energy of this particle as it rotates about the axis $\hat{\mathbf{n}}$ through the origin; it is

$$T = \frac{1}{2}m\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} = \frac{1}{2}m(\boldsymbol{\omega}\times\mathbf{x})\cdot(\boldsymbol{\omega}\times\mathbf{x}) = \frac{1}{2}md^2\omega^2 \qquad (5.12)$$

5.4.2 The Moment of Inertia

Now let's return to our main theme and look at a collection of N particles which make up a rigid body. The fact that the object is rigid means that all particles rotate with the same angular velocity,

$$\dot{\mathbf{x}}_i = \boldsymbol{\omega} \times \mathbf{x}_i$$

This ensures that the relative distance between points remains fixed as it should:

$$\begin{aligned}\frac{d}{dt}|\mathbf{x}_i - \mathbf{x}_j|^2 &= 2(\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j)\cdot(\mathbf{x}_i - \mathbf{x}_j) \\ &= 2[\boldsymbol{\omega}\times(\mathbf{x}_i - \mathbf{x}_j)]\cdot(\mathbf{x}_i - \mathbf{x}_j) = 0\end{aligned}$$

We can write the kinetic energy for a rigid body as

$$T = \frac{1}{2}\sum_i m_i \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i = \frac{1}{2} I\omega^2$$

where

$$I \equiv \sum_{i=1}^{N} m_i d_i^2$$

is the *moment of inertia*. Notice the similarity between the rotational kinetic energy $\frac{1}{2}I\omega^2$ and the translational kinetic energy $\frac{1}{2}Mv^2$. The moment of inertia is to rotations what the mass is to translations. The bigger I, the more energy you need to supply to the body to make it spin.

The moment of inertia also plays a role in the angular momentum of the rigid body. We have

$$\mathbf{L} = \sum_i m_i \mathbf{x}_i \times \dot{\mathbf{x}}_i = \sum_i m_i \mathbf{x}_i \times (\boldsymbol{\omega}\times\mathbf{x}_i)$$

If we write $\boldsymbol{\omega} = \omega\hat{\mathbf{n}}$, for a unit vector $\hat{\mathbf{n}}$, then the magnitude of the angular momentum in the direction of $\boldsymbol{\omega}$ is

$$\begin{aligned}\mathbf{L}\cdot\hat{\mathbf{n}} &= \omega\sum_i m_i\Big(\mathbf{x}_i\times(\hat{\mathbf{n}}\times\mathbf{x}_i)\Big)\cdot\hat{\mathbf{n}} \\ &= \omega\sum_i m_i(\mathbf{x}_i\times\hat{\mathbf{n}})\cdot(\mathbf{x}_i\times\hat{\mathbf{n}}) \\ &= I\omega\end{aligned}$$

We saw earlier in (5.2) that acting with a torque τ changes the angular momentum: $\dot{\mathbf{L}} = \tau$. For a rigid body, we learn that if the torque is in the same direction as the angular velocity, so $\tau = \tau\hat{\mathbf{n}}$, then the change in the angular velocity is simply

$$I\dot{\omega} = \tau$$

Calculating the Moment of Inertia

It's often useful to treat rigid bodies as continuous objects. This means that we replace the discrete particle masses m_i with a continuous density distribution $\rho(\mathbf{x})$. In this course, we will nearly always be interested in uniform objects for which the density ρ is constant. (Although a spatially dependent ρ doesn't add any more conceptual difficulties). The total mass of the body is then given by a volume integral

$$M = \int \rho(\mathbf{x})\, dV$$

and the moment of inertia is

$$I = \int \rho(\mathbf{x})\, x_\perp^2\, dV = \int \rho(\mathbf{x})\, (x \sin\phi)^2\, dV$$

where $x_\perp = x\sin\phi$ is the perpendicular distance from the point $\mathbf{x}$ to the axis of rotation. Let's look at some simple examples.

A Circular Hoop

A uniform hoop has mass M and radius a. Take the axis of rotation to pass through the centre, perpendicular to the plane of the hoop. This is, perhaps the simplest example, because all points of the hoop lie at the same distance, a, from the centre. The moment of inertia is simply

$$I = Ma^2$$

A Rod

A rod has length l, mass M and uniform density $\rho = M/l$ (strictly this is mass per unit length rather than mass per volume). The moment of inertia about an axis perpendicular to the rod, passing through the end point is

$$I = \int_0^l \rho x^2\ dx = \frac{1}{3}\rho l^3 = \frac{1}{3}Ml^2 \tag{5.13}$$

A Disc

A uniform disc has radius a and mass $M = \pi\rho a^2$. Two dimensional objects, such as the disc, are sometimes referred to as *laminas*. This time we'll look at two different axes of rotation.

We start with an axis of rotation through the centre, perpendicular to the plane of the disc. We can compute the moment of inertia using plane polar coordinates. Recall that we need to include a Jacobian factor of r, so that the infinitesimal area is $dA = rdrd\theta$. The moment of inertia is then

$$I = \int_0^a \int_0^{2\pi} \rho r^2 \; rdrd\theta = \frac{1}{4}\rho\left(2\pi a^4\right) = \frac{1}{2}Ma^2$$

We can also look at an axis rotation that passes through the centre of the disc but, this time, lies within the plane of the disc. We'll choose polar coordinates so that $\theta = 0$ lies along the axis of rotation. Then the point with coordinates (r, θ) lies a distance $r \sin\theta$ away from the axis of rotation. The moment of inertia is now

$$I = \int_0^a \int_0^{2\pi} \rho(r\sin\theta)^2 \; rdrd\theta = \frac{1}{4}Ma^2$$

In fact, these two calculations illustrate a general fact about laminas. If we take the z axis to lie perpendicular to the plane of the lamina, then the moments of inertia about the x and y-axes are $I_x = \int \rho y^2 \, dA$ and $I_y = \int \rho x^2 \, dA$. Meanwhile, the distance of any point to the z-axis is $r = \sqrt{x^2 + y^2}$, so the moment of inertia about the z-axis is

$$I_z = \int \rho(x^2 + y^2)\, dA = I_x + I_y$$

This is known as the *perpendicular axis theorem*

A Sphere

A uniform sphere has radius a and mass $M = \frac{4}{3}\pi\rho a^3$. We pick spherical polar coordinates with the axis $\theta = 0$ pointing along the axis of rotation which passes through the centre of the sphere. A point with coordinates (r, θ, ϕ) has distance $r\sin\theta$ from the axis of rotation. We also have the Jacobian factor $r^2 \sin\theta$, so that the volume element is $dV = r^2 \sin\theta drd\theta d\phi$ with $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$. The moment of inertia is

$$I = \int_0^a \int_0^{\pi} \int_0^{2\pi} \rho(r\sin\theta)^2 \, r^2 \sin\theta \; drd\theta d\phi = \frac{8}{15}\pi\rho a^5 = \frac{2}{5}Ma^2$$

5.4.3 Parallel Axis Theorem

A rigid body has mass M and moment of inertia $I_{\rm CoM}$ about an axis which passes through its centre of mass. Let I be the moment of inertia about a parallel axis that lies a distance h away. Then

$$I = I_{\rm CoM} + Mh^2$$

This is the *parallel axis theorem.*

It is a simple matter to prove this. We pick an origin that sits on the second axis (the one that does not pass through the centre of mass) and label the unit vector along this axis as $\hat{\mathbf{n}}$. Measured from here, the position to any particle can be decomposed as

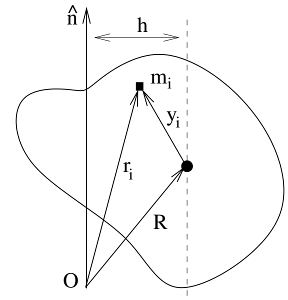

Figure 27:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{y}_i$$

where $\mathbf{R}$ is the centre of mass position and $\mathbf{y_i}$ are constrained to obey $\sum_i m_i \mathbf{y}_i = 0$. We can then write the moment of inertia as

$$\begin{aligned} I &= \sum_i m_i \left(\hat{\mathbf{n}} \times \mathbf{r}_i\right) \cdot \left(\hat{\mathbf{n}} \times \mathbf{r}_i\right) \\ &= \sum_i m_i \left(\hat{\mathbf{n}} \times (\mathbf{R} + \mathbf{y}_i)\right) \cdot \left(\hat{\mathbf{n}} \times (\mathbf{R} + \mathbf{y}_i)\right) \\ &= \sum_i m_i \Big((\hat{\mathbf{n}} \times \mathbf{y}_i) \cdot (\hat{\mathbf{n}} \times \mathbf{y}_i) + 2(\hat{\mathbf{n}} \times \mathbf{y}_i) \cdot (\hat{\mathbf{n}} \times \mathbf{R}) + (\hat{\mathbf{n}} \times \mathbf{R}) \cdot (\hat{\mathbf{n}} \times \mathbf{R}) \Big) \end{aligned}$$

The first term is nothing other than $I_{\rm CoM}$. The middle term vanishes by the constraint $\sum_i m_i \mathbf{y}_i = 0$. (This is because $\mathbf{y}_i$ is only thing that depends on i in the sum. So even though the $\mathbf{y}_i$ are hiding inside some scalar-vector product, you can still move the $\sum m_i$ inside all of this). Finally, the last term contains the factor $(\hat{\mathbf{n}} \times \mathbf{R}) \cdot (\hat{\mathbf{n}} \times \mathbf{R}) = h^2$, where h is the distance between the two axes as shown in the figure. This gives us the result we wanted:

$$I = I_{\rm CoM} + Mh^2$$

Notice that, as a simple corollary, the moment of inertia for an axis which passes through the centre of mass is necessarily lower than that of any parallel axis.

The Disc Again

Let's go back to our disc example, now with an axis that lies perpendicular to the plane of the disc, but passes through a point on the circumference. By the parallel axis theorem, the moment of inertia is

$$I = I_{\rm CoM} + Ma^2 = \frac{3}{2} Ma^2$$

We can also compute this the hard way. If we pick polar coordinates in the plane of the disc, with $\theta = 0$ lying on the vector $\mathbf{a}$ which points from the origin of the disc to the axis of rotation. Then the distance from the axis, d, of a point $\mathbf{r}$ in the disc is given by

$$d^2 = (\mathbf{r} - \mathbf{a})^2 = r^2 + a^2 - 2\mathbf{r}\cdot\mathbf{a} = r^2 + a^2 - 2ar\cos\theta$$

From this we can compute the moment of inertia

$$I = \int_0^a \int_0^{2\pi} \rho(r^2 + a^2 - 2ar\cos\theta)\, rdr\, d\theta = \frac{3}{2}Ma^2$$

in agreement with our result using the parallel axis theorem.

5.4.4 The Inertia Tensor

The moment of inertia is not inherent to the rigid body itself; it also depends on the axis about which we rotate. There is a more refined quantity which is a property only of the rigid body and contains the necessary information to compute the moment of inertia about any given axis. This is a 3×3 matrix, known as the *inertia tensor* $\mathcal{I}$.

We can already see the inertia tensor sitting in our expression for the kinetic energy of a rotating object, which we write as

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i(\boldsymbol{\omega}\times\mathbf{x}_i)\cdot(\boldsymbol{\omega}\times\mathbf{x}_i) \\ &= \frac{1}{2}\sum_i m_i\Big((\boldsymbol{\omega}\cdot\boldsymbol{\omega})(\mathbf{x}_i\cdot\mathbf{x}_i) - (\mathbf{x}_i\cdot\boldsymbol{\omega})^2\Big) \\ &= \frac{1}{2}\boldsymbol{\omega}^T\mathcal{I}\boldsymbol{\omega} \end{aligned}$$

where the components of the inertia tensor are expressed in terms of the components $(\mathbf{x}_i)_a$, $a = 1, 2, 3$ of the position vectors as

$$\mathcal{I}_{ab} = \sum_i m_i\Big((\mathbf{x}_i\cdot\mathbf{x}_i)\delta_{ab} - (\mathbf{x}_i)_a(\mathbf{x}_i)_b\Big)$$

The moment of inertia about an axis $\hat{\mathbf{n}}$ is encoded in the inertia tensor as

$$I = \hat{\mathbf{n}}^T\mathcal{I}\,\hat{\mathbf{n}}$$

There are many further interesting properties of the inertia tensor. Perhaps the most important is that it relates the angular momentum with the angular velocity. It is not hard to show

$$\mathbf{L} = \mathcal{I}\boldsymbol{\omega}$$

In particular, this means that the angular momentum does not necessarily lie in the same direction as the angular velocity. (This is only true if the object is spinning about an eigenvector of the inertia tensor). This is responsible for many of the weird and wobbly properties of spinning objects. However, a much fuller discussion will have to wait until the next Classical Dynamics course[6].

5.4.5 Motion of Rigid Bodies

So far we have just considered the rotation of a rigid body about some point. Now let's set it free and allow it to move. The most general motion of a rigid body can be described by its centre of mass following some trajectory, $\mathbf{R}(t)$, together with a rotation about the centre of mass. We use our usual notation where the position of any particle in the rigid body is written as

$$\mathbf{r}_i = \mathbf{R} + \mathbf{y}_i \quad \Rightarrow \quad \dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{y}}_i$$

If the body rotates with angular velocity $\boldsymbol{\omega}$ around the centre of mass, we have $\dot{\mathbf{y}}_i = \boldsymbol{\omega} \times \mathbf{y}_i$, which means that we can write

$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{R}) \tag{5.14}$$

The kinetic energy of the rigid body follows from the general calculation (5.3), together with our result (5.12). These give

$$\begin{aligned} T &= \frac{1}{2} M \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + \frac{1}{2} \sum_i m_i \dot{\mathbf{y}}_i \cdot \dot{\mathbf{y}}_i \\ &= \frac{1}{2} M \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + \frac{1}{2} I \omega^2 \end{aligned} \tag{5.15}$$

(Recall, that this calculation needs us to work with the centre of mass $\mathbf{R}$ to ensure that the cross-terms $\dot{\mathbf{R}} \cdot \dot{\mathbf{y}}_i$ drop out in the first line above).

Motion with Rotation about A Different Point

It is certainly most natural to split the motion into the centre of mass trajectory $\mathbf{R}(t)$ together with rotation about the centre of mass. With this choice, Newton's second law (5.1) ensures that $\mathbf{R}(t)$ is dictated only by external forces. Moreover, the kinetic energy splits nicely into translational and rotational energies (5.15). But nothing tells us that we *have* to describe an object in this way. We could, instead, decide that it's better to think of the motion in terms of some other point $\mathbf{Q}$ (say the tip of the nose of dead, rigid cat), together with rotation about $\mathbf{Q}$.

[6]See section 3 of the lecture notes at http://www.damtp.cam.ac.uk/user/tong/dynamics.html

We can derive an expression for such motion using our results above. Let's start by picking $\mathbf{r}_i = \mathbf{Q}$ in (5.14). This tells us

$$\dot{\mathbf{Q}} = \dot{\mathbf{R}} + \boldsymbol{\omega} \times (\mathbf{Q} - \mathbf{R})$$

If we substitute this back into (5.14), we can eliminate $\dot{\mathbf{R}}$ to get an expression for the motion of any point $\mathbf{r}_i$ about $\mathbf{Q}$,

$$\dot{\mathbf{r}}_i = \dot{\mathbf{Q}} + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{Q})$$

There's something a little surprising about this: the angular velocity $\boldsymbol{\omega}$ about any point is the same.

An Example: Roll, Don't Slip

A common example of rigid body motion is an object which rolls along the ground. Let's look at a hoop of radius a as shown in the figure. In this case, the translational speed and the angular speed are related. This comes about if we insist that there is no slipping between the hoop and the ground — a requirement that is usually, quite reasonably, called the *no-slip condition*.

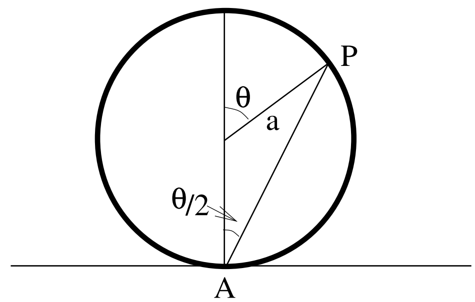

Figure 28:

Consider the point A of the hoop which, at a given instance, is in contact with the ground. The no slip condition is the statement that the point A is instantaneously at rest. In other words, it has no speed relative to the ground. If we denote the angular speed of the hoop as $\dot{\theta}$, the no-slip condition means that the horizontal speed v of the origin is

$$v = a\dot{\theta} \tag{5.16}$$

What, however, is the speed of different point, P on the circumference? Clearly when $\theta = 0$, so P sits at the top of the hoop, the horizontal speed is $a\dot{\theta}$ with respect to centre, resulting in a total horizontal speed of $2a\dot{\theta}$.

To compute the speed of a general point P, it's best to think about the hoop as rotating about A. From the argument above, we know that the angular speed about A is also $\dot{\theta}$. But the distance $AP = 2a\cos(\theta/2)$, which means that the speed v of the point P relative to A (which is the same as relative to the ground) is

$$v = 2a\dot{\theta}\cos(\theta/2)$$

We check that this gives the right answer when P is at the top and bottom of the hoop: $\theta = 0$ and $\theta = \pi$ gives $v = 2a\dot{\theta}$ and $v = 0$ respectively, as it should.

Note that the velocity of the point P does not lie tangent to the circle. That would only be the case if the hoop was rotating while staying fixed. Instead the velocity of point P is at right-angles to the line AP. This reflects the fact that the point P is rotating about the origin, but also moving forwards as the hoop moves.

Finally, a quick comment: despite the presence of friction, this is one example where we can still use energy conservation. This is because the point of the wheel that is in contact with the ground is at rest, which means that friction acting on this point does no work. Instead, the only role of friction is to impose the no-slip condition. We'll see an example of this motion which can be solved using energy conservation shortly.

Another Example: A Swinging Rod

Until now, a "pendulum" has always consisted of a mass sitting at the end of a light rod, where light means effectively massless. Let's now look at an example where the rod itself has mass m.

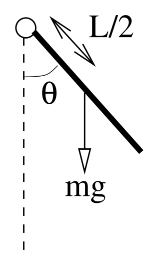

Figure 29:

This is a case where the most natural description of the rotation is around the pivot, rather than around the centre of mass. We already calculated the moment of inertia I for a rod of length L which pivots about its end point (5.13): $I = \frac{1}{3}mL^2$. With the angular speed $\omega = \dot{\theta}$, the kinetic energy can be written as

$$T = \frac{1}{2}I\dot{\theta}^2$$

Alternatively, we could also look at this as motion of the centre of mass, together with rotation around the centre of mass. As we saw above, the angular speed about the centre of mass remains $\dot{\theta}$: it is the same as the angular speed about the pivot. The speed of the centre of mass is $v = (L/2)\dot{\theta}$ and the kinetic energy splits in the form (5.15)

$$T = \text{Translational K.E.} + \text{Rotational K.E.} = \frac{1}{2}m\left(\frac{L}{2}\dot{\theta}\right)^2 + \frac{1}{2}I_{\text{CoM}}\dot{\theta}^2$$

But, by the parallel axis theorem, we know that $I = I_{\text{CoM}} + m(L/2)^2$ which happily means that the kinetic energies computed in these two different ways coincide.

To derive the equation of motion of the pendulum, it's perhaps easiest to first get the energy. The centre of mass of the pendulum sits at a distance $-(L/2)\cos\theta$ below the pivot. So combining the kinetic and gravitational energies, we have

$$E = \frac{1}{2}I\dot{\theta}^2 - mg\frac{L}{2}\cos\theta$$

Differentiating with respect to time, we get the equation of motion

$$I\ddot{\theta} = -mg\frac{L}{2}\sin\theta$$

We can compare this with our earlier treatment of a pendulum where all the mass sits at the end of the length l. In that case, the equation is (2.9). We see that the equations of motion agree if set $l = 2I/Lm = 2L/3$.

Yet Another Example: A Rolling Disc

A disc of mass M and radius a rolls down a slope without slipping. The plane of the disc is vertical. The moment of inertia of the disc about an axis which passes through the centre, perpendicular to the plane of the disc, is I. (We already know from our earlier calculation that $I = \frac{1}{2}Ma^2$, but we'll leave it general for now).

We'll denote the speed of the disc down the slope as v and the angular speed of the disc as ω. (From the picture and the right-hand rule, we see that the angular velocity $\boldsymbol{\omega}$ is a vector point out of the page). As in (5.16), the no-slip condition gives us the relation

$$v = a\omega$$

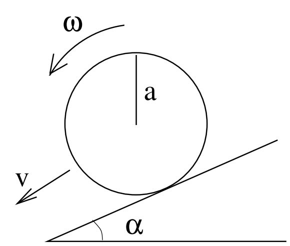

Figure 30:

To understand the motion of the disc, it is simplest to work with the energy. This is allowed since, as we mentioned before, when friction imposes the no-slip condition it does no work. We've seen a number of times — e.g. in (5.15) — that the kinetic energy splits into the translational kinetic energy of the centre of mass, together with the rotational kinetic energy about the centre of mass. In the present case, this means

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{I}{a^2} + M\right)v^2$$

Including the gravitational potential energy, we have

$$E = \frac{1}{2}\left(\frac{I}{a^2} + M\right)\dot{x}^2 - Mgx\sin\alpha$$

where x measures the progress of the disc down the slope, so $\dot{x} = v$. From this we can derive the equation of motion simply by taking the time derivative. We have

$$\left(\frac{I}{a^2} + M\right)\ddot{x} = Mg\sin\alpha$$

We learn that while the overall mass M drops out of the calculation (recall that I is proportional to M), the moment of inertia I does not. The larger the moment of inertia I of an object, the slower its progress down the slope. This is because the gravitational potential energy is converted into both translational and rotational kinetic energy. But only the former affects how fast the object makes it down. The upshot of this is that if you take a hollow cylinder and a solid cylinder with equal diameter, the solid one – with smaller moment of inertia – will make it down the slope more quickly.