Dynamics and Relativity: Example Sheet 1

Professor David Tong, January 2013

1. In one spatial dimension, two frames of reference S and S' have coordinates (x, t) and (x', t') respectively. The coordinates are related by $t' = t$ and

$$x' = f(x, t)$$

Viewed from frame S, a particle follows a trajectory $x = x(t)$. It has velocity $v = \dot{x}$ and acceleration $a = \ddot{x}$. Viewed from S', the trajectory is $x' = f(x(t), t)$. Using the chain rule, show that the speed and acceleration of the particle in S' are given by

$$\begin{aligned} \frac{dx'}{dt'} &= v\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} \\ \frac{d^2x'}{dt'^2} &= a\frac{\partial f}{\partial x} + v^2\frac{\partial^2 f}{\partial x^2} + 2v\frac{\partial^2 f}{\partial x\,\partial t} + \frac{\partial^2 f}{\partial t^2}. \end{aligned}$$

Suppose now that both S and S' are inertial frames. Explain why the function f must obey $\partial^2 f/\partial x^2 = \partial^2 f/\partial x \partial t = \partial^2 f/\partial t^2 = 0$. What is the most general form of f with these properties? Interpret this result.

2. A particle at position $\mathbf{r}$ experiences a force

$$\mathbf{F} = \left(-\frac{a}{r^2} + \frac{2b}{r^3}\right)\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction and a and b are positive constants. Show, by finding a potential $V(r)$ such that $\mathbf{F} = -\nabla V$, that $\mathbf{F}$ is conservative. (Hint: you will need the result $\nabla r = \hat{\mathbf{r}}$).

Sketch the potential $V(r)$ and describe qualitatively the possible motions of the particle moving in the radial direction, considering different starting positions and speeds. If the particle starts at the point $r = 2b/a$, what is the minimum speed that the particle must have in order to escape to infinity?

3. A satellite falls freely towards the Earth starting from rest at a distance R, much larger than the Earth's radius. Treating the Earth as a point of mass M, use dimensional analysis to show that the time T taken by the satellite to reach the Earth is given by

$$T = C\left(\frac{R^3}{GM}\right)^{\frac{1}{2}},$$

where G is the gravitational constant and C is a dimensionless constant. (You will need the fact that the acceleration due to the Earth's gravitational field at a distance r from the centre of the Earth is GM/r^2).

What is the conserved energy of the satellite? By integrating this equation, show that $C = \pi/2\sqrt{2}$.

4. A long time ago, in a galaxy far far away, a Death Star was constructed. Its surrounding force field caused a particle at distance $\mathbf{r}$ relative to the Death Star to experience an acceleration

$$\ddot{\mathbf{r}} = \lambda\,\mathbf{r} \times \dot{\mathbf{r}}$$

where λ is a constant. Show that particles move in this field with constant speed. Show, moreover, that the magnitude of acceleration is also constant.

(a) A particle is projected *radially* with speed v from a point $\mathbf{r} = R\hat{\mathbf{r}}$ on the surface of the Death Star. Show that its trajectory is given by

$$\mathbf{r} = (vt + R)\hat{\mathbf{r}}$$

(b) By considering the second derivative of $\mathbf{r} \cdot \mathbf{r}$ show that, for any particle moving in the force field, the distance r to the centre of the Death Star is given by

$$r^2 = v^2(t - t_0)^2 + r_0^2$$

where t_0 and r_0 are constants and v is the speed of the particle. Obtain an expression for $\mathbf{r} \cdot \dot{\mathbf{r}}$ and show that $|\ddot{\mathbf{r}}| = \lambda v r_0$.

5. A particle of mass m, charge q and position $\mathbf{x}$ moves in a constant, uniform magnetic field $\mathbf{B}$ which points in a horizontal direction. The particle is also under the influence of gravity, $\mathbf{g}$, acting vertically downwards. Write down the equation of motion and show that it is invariant under translations $\mathbf{x} \to \mathbf{x} + \mathbf{x}_0$. Obtain

$$\dot{\mathbf{x}} = \alpha \mathbf{x} \times \mathbf{n} + \mathbf{g}t + \mathbf{a}$$

where $\alpha = qB/m$, $\mathbf{n}$ is a unit vector in the direction of $\mathbf{B}$ and $\mathbf{a}$ is a constant vector. Show that, with a suitable choice of origin, $\mathbf{a}$ can be written in the form $\mathbf{a} = a\mathbf{n}$.

By choosing suitable axes, show that the particle undergoes a helical motion with a constant horizontal drift.

Suppose that you now wish to eliminate the drift by imposing a uniform electric field $\mathbf{E}$. Determine the direction and magnitude of $\mathbf{E}$.

6. At time $t = 0$, an insect of mass m jumps from a point O on the ground with velocity $\mathbf{v}$, while a wind blows with velocity $\mathbf{u}$. The gravitational acceleration is $\mathbf{g}$ and the air exerts a retarding force on the insect equal to mk times the velocity of the wind *relative to the insect*.

(a) Show that the path of the insect is given by

$$\mathbf{x} = (\mathbf{u} + \mathbf{g}/k)t + \frac{1 - e^{-kt}}{k}(\mathbf{v} - \mathbf{u} - \mathbf{g}/k)$$

(b) In the case where the insect jumps vertically in a horizontal wind, show that the time T that elapses before it returns to earth satisfies

$$(1 - e^{-kT}) = \frac{kT}{1+\gamma}$$

where $\gamma = kv/g$. Find an expression for the range R in terms of γ, u and T. (Here $v = |\mathbf{v}|$, $g = |\mathbf{g}|$, and $u = |\mathbf{u}|$.)

7. A ball of mass m moves in a resisting medium that produces a friction force of magnitude kv^2, where v is the ball's speed. If the ball is projected vertically upwards with initial speed u, show by dimensional analysis that when the ball returns to its point of projection, its speed w can be written in the form

$$w = uf(\lambda)\,,$$

where $\lambda = ku^2/mg$.

Integrate the equations of motion to show that $f(\lambda) = (1+\lambda)^{-1/2}$. Discuss what happens in the two extremes $\lambda \gg 1$, and $\lambda \ll 1$.

8*. The temperature $\theta(x,t)$ in a very long rod is governed by the one-dimensional diffusion equation

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial x^2}$$

where D is a constant (the *thermal diffusivity* of the rod). At time $t = 0$, the point $x = 0$ is heated to a high temperature. At all later times, the conservation of heat energy implies that

$$Q = \int_{-\infty}^{\infty} \theta(x,t)dx$$

is constant. Use dimensional analysis to show that $\theta(x,t)$ can be written in the form

$$\theta(x,t) = \frac{Q}{\sqrt{Dt}}F(z)$$

where $z = x/\sqrt{Dt}$ and show further that

$$\frac{d^2F}{dz^2} + \frac{z}{2}\frac{dF}{dz} + \frac{1}{2}F = 0$$

Integrate this equation once directly to obtain a first order differential equation. Evaluate the constant of integration by considering either the symmetry of the problem or the behaviour of the solution as $z \to \pm\infty$. Hence show that

$$\theta(x,t) = \frac{Q}{\sqrt{4\pi Dt}}\, e^{-x^2/4Dt}$$