Dynamics and Relativity: Example Sheet 2

Professor David Tong, February 2013

1. A particle moves in a fixed plane and its position vector at time t is $\mathbf{x}$. Let (r, θ) be plane polar coordinates and let $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ be unit vectors in the direction of increasing r and increasing θ respectively. Show that

$$\dot{\mathbf{x}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

The particle moves outwards with speed v on the equiangular spiral $r = a\exp(\theta\cot\alpha)$, where a and α are constants, with $0 < \alpha < \frac{1}{2}\pi$. Show that

$$v\sin\alpha = r\dot{\theta}$$

and hence that

$$\dot{\mathbf{x}} = v\cos\alpha\,\hat{\mathbf{r}} + v\sin\alpha\,\hat{\boldsymbol{\theta}}$$

Give an expression for $\ddot{\mathbf{x}}$ and show that $|\ddot{\mathbf{x}}|^2 = \dot{v}^2 + v^2\dot{\theta}^2$.

(*) If $\dot{\theta}$ takes a constant value ω, show that the acceleration has magnitude v^2/r and is directed at an angle 2α to the position vector.

2. In these orbital questions, the particles move in a gravitational potential

$$V = -\frac{km}{r} \quad \text{with } k > 0$$

In what follows, you should answer all questions using *only* energy and angular momentum conservation, and (for the circular orbits) the radial component of the equation of motion.

Since no one ever listens to that last sentence, I'll say it again. You should answer the following questions using *only* the conservation of energy and angular momentum together with (for circular orbits) the radial component of the equation of motion. Ok?

(a) Show that the radius, R, of the orbit of a satellite in geostationary orbit (in the equatorial plane) is approximately $(28)^{-2/3}R_m$, where R_m is the radius of the moon's orbit round the Earth.

(b) A particle moves in a parabolic orbit and another particle moves in a circular orbit. Show that if they pass through the same point then the ratio of their speeds at this point is $\sqrt{2}$. For a satellite orbiting the Earth in a circular orbit, what is the relationship between its orbiting speed and its escape velocity?

If, instead of passing through the same point, the particles have the same angular momentum per unit mass, show that the perihelion distance of the parabola is half the radius of the circle.

(c) A particle moves with angular momentum l per unit mass in an ellipse, for which the distances from the focus to the periapsis (closest point to focus) and apoapsis (furthest point) are p and q, respectively. Show that

$$l^2\left(\frac{1}{p}+\frac{1}{q}\right)=2k$$

Show also that the speed V of the particle at the periapsis is related to the speed v of a particle moving in a circular orbit of radius p by $V^2 = 2v^2(1+p/q)^{-1}$.

(d) A particle P is initially at a very large distance from the origin moving with speed v on a trajectory that, in the absence of any force, would be a straight line for which the shortest distance from the origin is b. The shortest distance between P's actual trajectory and the origin is d. Show that $2kd = v^2(b^2 - d^2)$

3. For a particle subject to an inverse square force given by $\mathbf{F} = -mk\hat{\mathbf{r}}/r^2$, the vectors $\mathbf{h}$ and $\mathbf{e}$ are defined by

$$\mathbf{h} = \mathbf{x}\times\dot{\mathbf{x}} \quad \text{and} \quad \mathbf{e} = \frac{\dot{\mathbf{x}}\times\mathbf{h}}{k} - \frac{\mathbf{x}}{r}$$

Show that $\mathbf{h}$ is constant and deduce that the particle moves in a plane through the origin. (Note that in the lectures, the vector $\mathbf{h}$ was called $\mathbf{l}$).

The vector $\mathbf{e}$ is known as the *Laplace-Runge-Lenz* vector. Show that it too is is constant and that

$$er\cos\theta = h^2/k - r$$

where $e = |\mathbf{e}|$, $h = |\mathbf{h}|$ and θ is the angle between $\mathbf{x}$ and $\mathbf{e}$. Deduce that the orbit is a conic section.

4. A particle of unit mass moves with speed v in the gravitational field of the Sun and is influenced by radiation pressure. The forces acting on the particle are μ/r^2 towards the sun and kv opposing the motion, where μ and k are constants. Write down the vector equation of motion and show that the vector **H**, defined by

$$\mathbf{H} = e^{kt}\,\mathbf{x} \times \dot{\mathbf{x}}$$

is constant. Deduce that the particle moves in a plane through the origin. Establish the equations

$$r^2\dot{\theta} = h\mathrm{e}^{-kt} \qquad \text{and} \qquad \mu r = h^2\mathrm{e}^{-2kt} - r^3(\ddot{r} + k\dot{r})$$

where r and θ are plane polar coordinates centred on the Sun and h is a constant. Show that, when $k = 0$, a circular orbit of radius a exists for any value of a, and find its angular frequency ω in terms of a and μ.

When $k/\omega \ll 1$, r varies so slowly that $\dot{r}$ and $\ddot{r}$ may be neglected in the above equations. Verify that in this case an approximate solution is

$$r = a\mathrm{e}^{-2kt}, \qquad \dot{\theta} = \omega\mathrm{e}^{3kt}$$

Give a brief qualitative description of the behaviour of this solution for $t > 0$. Does the speed of the particle increase or decrease?

5. A particle P of unit mass moves in a plane under a central force

$$F(r) = -\frac{\lambda}{r^3} - \frac{\mu}{r^2}$$

where λ and μ are positive constants. Write down the differential equation satisified by $u(\theta)$, where $u = 1/r$.

Given that P is projected with speed V from the point $r = r_0$, $\theta = 0$ in the direction perpendicular to OP, find the equation of the orbit under the assumptions

$$\lambda < V^2 r_0^2 < 2\mu r_0 + \lambda$$

Explain the significance of these inequalities. Show that between consecutive apsides (points of greatest or least distance) the radius vector turns through an angle

$$\pi\left(1 - \frac{\lambda}{V^2 r_0^2}\right)^{-1/2}$$

Under what condition is the orbit a closed curve?

6. A particle P of mass m moves under the influence of a central force of magnitude mk/r^3 directed towards a fixed point O. Initially $r = a$ and P has velocity v perpendicular to OP, where $v^2 < k/a^2$. Use the orbit equation to prove that P spirals in towards O (you should give the geometric equation of the spiral). Show also that it reaches O in a time

$$T = \frac{a^2}{\sqrt{k - a^2 v^2}}$$

7*. A particle of mass m moves in a circular orbit of radius R under the influence of an attractive central force of magnitude $F(r)$. Obtain an equation relating R, $F(R)$, m and the orbital angular momentum per unit mass l.

The particle experiences a very small radial perturbation of the form $u(\theta) = U + \epsilon(\theta)$, where $u = 1/r$ and $U = 1/R$. The orbital angular momentum is not affected. Obtain the equation for $\epsilon''(\theta)$. Given that the subsequent orbit is both stable and closed, show that

$$\frac{RF'(R)}{F(R)} = \beta^2 - 3$$

where β is a rational number. Deduce that, if β is independent of R, then $F(r)$ is of the form Ar^α, where α is rational and greater than -3.

8. Two particles of masses m_1 and m_2 move under their mutual gravitational attraction. Show from first principles that the quantity

$$\frac{1}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - \frac{GM}{r}$$

is constant, where $\mathbf{r}$ is the position vector of one particle relative to the other and $M = m_1 + m_2$.

The particles are released from rest a long way apart, and fall towards each other. Show that the position of their centre of gravity is fixed, and that when they are a distance r apart their relative speed is $\sqrt{2GM/r}$.

(*) When the particles are a distance a apart, they are given equal and opposite impulses (change of momentum) each of magnitude I, and each perpendicular to the direction of motion. Show that subsequently $r^2\omega = aI/\mu$, where ω is the angular speed of either particle relative to the centre of mass and μ is the reduced mass of the system.

Show further that the minimum separation, d, of the two particles in the subsequent motion satisfies

$$(a^2 - d^2)I^2 = 2GM\mu^2 d$$